Abstract

Let $A$ be an associative $k$-algebra with involution, where $k$ is a commutative ring of characteristic not equal to two. Then the dihedral groups act on the Hochschild complex and, following Loday, a new homological theory, called dihedral homology, can be defined generalizing the notion of cyclic homology defined by Connes. Here we give a model to compute dihedral homology of a commutative algebra over a characteristic zero field. As for an involutive algebra, we have a decomposition of Hochschild homology into a direct sum of two $k$-modules: $\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies. We give smoothness criteria in terms of vanishing of some $\mathbb{Z}_2$-equivariant Hochschild homology groups.

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0. Introduction

Let $k$ be a unital commutative ring, where 2 is invertible, and let $A$ be an associative $k$-algebra with involution. We recall, [5,14], that the definition of cyclic homology uses explicitly the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ over the Hochschild complex.

If $A$ is involutive, an action of the dihedral groups $D_8$ on the Hochschild complex can be defined as in [14], and it gives two new homological theories called dihedral homology and skew dihedral homology, see also [6] and [15].

The Hochschild homology of an involutive algebra decomposes into two parts ($\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies), and the Connes’ long exact sequence splits into two long exact sequences relating dihedral homology, skew dihedral homology, $\mathbb{Z}_2$-equivariant and skew $\mathbb{Z}_2$-equivariant Hochschild homologies.
Theorem 2.1. Let \((A, d_A)\) be a commutative differential graded algebra endowed with an involution \(\omega\). Then there exists a free commutative differential graded algebra \((AV, \partial)\) and a morphism \(\rho: (AV, \partial) \to (A, d_A)\) inducing an isomorphism in homology such that

1. \(V = \bigoplus_{n \in \mathbb{N}} V_n\), on each \(V_n\), there exists an involution \(\omega\), which induces a morphism of commutative differential graded algebras,
2. \(\rho \omega = \omega \rho\).

Such an algebra \((AV, \partial)\) is called an equivariant model of \((A, d_A)\).

Remark. Let \(A\) be an involutive commutative algebra of finite type, then \(A\) is isomorphic to \(k[x_1, \ldots, x_p]/I\), where the involution \(\omega\) of \(A\) is the image of \(\omega'\) on \(k[x_1, \ldots, x_p]\) satisfying \(\omega'(x_i) = \pm x_i\) for all \(i\), and \(I\) contains \(\omega'(I)\). So we can construct an equivariant model of \((A, \partial)\), with \(V_0 = \bigoplus_{1 \leq i \leq p} k x_i\) and \(\text{dim } V_n < \infty\), for all \(n\).

Proposition III. 2.9 of [8] can be transposed in this context:

Proposition 2.2. Let \(f: (A, d_A) \to (B, d_B)\) be an equivariant morphism of involutive commutative differential graded algebras over a field. If \(f_*\) is an isomorphism from \(H_*(A, d_A)\) to \(H_*(B, d_B)\), then \(f\) induces isomorphisms between \(\mathbb{Z}_2\)-equivariant (resp. skew \(\mathbb{Z}_2\)-equivariant) Hochschild homology and dihedral homology.

From now on, we will assume that \(k\) is a field of characteristic zero, and using Proposition 2.2, we will work with the equivariant model \((AV, \partial)\).

In the appendix of [12], we define the module of differential forms \(\Omega^1\) of a commutative graded algebra \((A, \partial)\), extending the classical definition, so that \(\Omega^1\) is an \((\partial, \cdot)\)-differential module with a differential \(\delta\) satisfying \(\delta c + c \delta = 0\).

If \((A, \partial)\) is endowed with an involution \(\omega\), we define an involution still denoted \(\omega\) on \(\Omega^1\) satisfying \(\omega d + d \omega = 0\), \(\omega \delta = \delta \omega\).

By definition, \((\Omega^*_{(\partial, \cdot)}, \delta)\) is the \((A, \partial)\)-commutative differential graded algebra on \(\Omega^1\). So the formula:

\[
\omega^\delta(a_0 \land da_1 \land \cdots \land da_n) = (-1)^n \omega(a_0) d\omega(a_1) \land \cdots \land d\omega(a_n)
\]

defines an involution \(\omega\) on \((\Omega^*_{(\partial, \cdot)}, \delta)\) which is a morphism of commutative differential graded algebras satisfying \(\omega d + d \omega = 0\).

If \((A, \partial) = (AV, \partial)\), the algebra \((\Omega^*_{(\partial, \cdot)}, \delta)\) of differential forms has the form \((AV \otimes AV, \partial)\) with \(\partial = dV\), and \(\delta d + d \delta = 0\).

Now, we recall the main result of [3] (Theorem 2.4).

Proposition 2.3 (Burghelea and Vigué-Poirrier [3]). The map

\[
\theta_{p, n-p}: C_{p, n-p}(AV, \partial) \to (\Omega^p_{(AV, \partial)})_n
\]

defined by

\[
\theta_p(a_0 \otimes \cdots \otimes a_p) = [(-1)^{p(n-p)}]^{p!} (a_0 \land da_1 \land \cdots \land da_p),
\]

is an isomorphism.
where $a_0 \in AV, a_i \in AV/k$ if $i \geq 1$, $\varepsilon_p(a) = |a_1| + |a_3| + \ldots$ satisfies

1. $\theta_0 b = 0, \theta_0 \delta = \delta_0 \theta, \theta_0 B = d_0 \theta$;
2. $\theta$ induces isomorphisms: $HH_n(AV, \delta) \cong H_n(\Omega^*, \delta)$ for all $n \geq 0$ and $HC_n(AV, \delta) \cong HC_n(\Omega^*, \delta, \delta)$, where $HC_*(\Omega^*, \delta, \delta)$ is the total homology of the bicomplex

\[ \begin{array}{cccccccccccc}
\cdots & \searrow & \searrow & \cdots \\
(\Omega^*)_n & \xleftarrow{d} & (\Omega^*)_{n-1} & \xleftarrow{d} & (\Omega^*)_{n-2} & \xleftarrow{d} & \cdots \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta \\
(\Omega^*)_{n-1} & \xleftarrow{d} & (\Omega^*)_{n-2} & \xleftarrow{d} & (\Omega^*)_{n-3} & \xleftarrow{d} & \cdots \\
\end{array} \]

Lemma 2.4. The following diagram commutes

\[ \begin{array}{ccc}
C_p(AV, \delta) & \xrightarrow{\theta_p} & \Omega^p_{(AV, \delta)} \\
\downarrow u & & \downarrow \omega_p \\
C_p(AV, \delta) & \xrightarrow{\theta_p} & \Omega^p_{(AV, \delta)} \\
\end{array} \]

Proof. Left to the reader.

We have a decomposition $\Omega^*_{(AV, \delta)} = (\Omega^*)^+ \oplus (\Omega^*)^-$ where $(\Omega^*)^+ = \{x/\omega(x) = x\}$ and $(\Omega^*)^- = \{x/\omega(x) = -x\}$.

From Proposition 2.3 and Lemma 2.4, we have directly:

Theorem 2.5. We have explicit isomorphisms, induced by $\theta$, for each $n \geq 0$.

\[ \begin{align*}
HH_n(A) & \cong HH_n^*(AV, \delta) \cong H_n((\Omega^*)^+, \delta) = \bigoplus_i H_n^{(i)}((\Omega^*)^+, \delta) \\
HD_n(A) & \cong HD_n(AV, \delta) \cong HC_n((\Omega^*)^+, \delta, d) = \bigoplus_i HC_n^{(i)}((\Omega^*)^+, \delta, d),
\end{align*} \]

where $H_n^{(i)}((\Omega^*)^+, \delta) = H_n((\Omega^*)^+ \cap (\Omega^i, \delta))$.

$HC_n^{(i)}((\Omega^*)^+, \delta, d)$ is the total homology of the bicomplex

\[ \begin{array}{cccccccccccc}
\cdots & \searrow & \searrow & \cdots \\
(\Omega^i_n)^+ & \xleftarrow{d} & (\Omega^i_{n-1})^- & \xleftarrow{d} & (\Omega^i_{n-2})^- & \xleftarrow{d} & \cdots \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \delta \\
(\Omega^i_{n-1})^+ & \xleftarrow{d} & (\Omega^i_{n-2})^- & \xleftarrow{d} & (\Omega^i_{n-3})^- & \xleftarrow{d} & \cdots \\
\end{array} \]
Since \( H_n(\Omega^+_n, d) = (\Omega^+_n \cap \text{Ker } d)/d(\Omega^+_n) \) and
\[
H_n(\Omega^-_n, d) = (\Omega^-_n \cap \text{Ker } d)/d(\Omega^-_n)
\]
for all \( n > 0 \), we have a similar result to Theorem 2.1 of [3].

**Theorem 2.6.** The map \( \phi: \Omega^-_n \oplus \Omega^-_{n-2} \oplus \cdots \to (\Omega^-_{n+1} \cap d(\Omega^+_n), \delta) \) defined by \( \phi(c_n, c_{n-2}, \ldots) = (-1)^n dc_n \) for \( c_{n-2i} \in \Omega^-_{n-2i} \), is a morphism of complexes and induces an isomorphism between \( H\overline{D}_n(AV, \delta) = H\overline{D}_n(AV, \delta)/H\overline{D}_n(k) \) and \( H_{\delta+1}(\Omega^-_n \cap d(\Omega^+_n), \delta) \). Analogously, we have an isomorphism between \( H\overline{D}_n(AV, \delta) = H\overline{D}_n(AV, \delta)/H\overline{D}_n(k) \) and \( H_{\delta+1}(\Omega^-_n \cap d(\Omega^+_n), \delta) \).

The famous Hochschild–Kostant–Rosenberg theorem implies that if \( A \) is smooth, then the \( \mathbb{Z}_2 \)-equivariant Hochschild homology groups \( HH^+_n(A) \) are zero for \( n \) sufficiently large.

For graded algebras, we can prove a converse of this result, using the theory developed in the present paragraph. This is, in fact, the proof of a refinement of a conjecture by Rodicio [16].

**Theorem 2.7.** Let \( A \) be a graded algebra over a characteristic zero field, and \( \omega \) an involution on \( A \). If there exist three integers \( i, j, k \) such that \( i - j, j - k \) and \( i - k \) are not divisible by 4, and
\[
HH^+_i(A) = HH^+_j(A) = HH^+_k(A) = 0
\]
then \( A \) is a polynomial algebra.

**Proof.** The proof relies on Theorem 2.6 and the existence of a minimal model for a graded algebra. Then, we proceed as in the proofs of theorems 1 and 2 of [18]. If \( A \) is not a polynomial algebra, we write \( A = k[x_1, \ldots, x_m]/I \), with \( I \neq 0 \), and we consider the elements \( Z_{m+2n} = (dx_1 \ldots dx_m)(dy)^n \), and their images by the involution \( \omega \). Since \( A \) is graded, we have short exact sequences:
\[
0 \to H\overline{D}_n(A, d_A) \to HH^+_n(A, d_A) \to H\overline{D}_n(A, d_A) \to 0
\]
\[
0 \to H\overline{D}_n(A, d_A) \to HH^-_n(A, d_A) \to H\overline{D}_n(A, d_A) \to 0
\]
The elements \( Z_{m+2n} \) define nonzero classes in \( H\overline{D}_m+2n-1 \) or \( H\overline{D}_{m+2n-1} \), depending on the actions of \( \omega \). This allows us to determine when the groups \( HH^+_n(A) \) are not zero.

**Remark.** In [18], it is proven that if \( A \) is not a polynomial algebra, then \( HC_n(A) \neq 0 \) for infinitely many \( n \). Here we cannot prove the same result for dihedral homology or skew dihedral homology, but instead, it is valid for \( \mathbb{Z}_2 \)-equivariant Hochschild homology.
3. Localization of \( \mathbb{Z}_2 \)-equivariant Hochschild homology. Applications

Let \( A \) be a commutative algebra. One of the most important properties of Hochschild homology, specially for geometrical applications, is that it is well-behaved with respect to localization. Explicitly, if \( S \) is a multiplicatively closed subset of \( A \), and \( A_S = S^{-1}A \), then by a result of Brylinski [2]

\[
HH_*(A_S) = HH_*(A) \otimes_A A_S
\]

If \( A \) is provided with an involution \( \omega \), and \( A^+ \) is the subalgebra of the elements of \( A \) fixed by \( \omega \), then \( HH_*(A) \) is no more an \( A \)-algebra but an \( A^+ \)-algebra.

Let \( S \) be a multiplicatively closed subset of \( A \), stable by the involution (i.e. \( \omega(S) \) is included in \( S \)), and let \( S^+ = \{ s \in S/\omega(s) = s \} \).

Then \( 1 \in S^+ \), and \( S^+ \) is also a multiplicatively closed subset of \( A \).

If \( a, a' \in A, s, s' \in S \), then \( a/s = a'/s' \) in \( A_S \) if and only if \( \exists ! s \in S \) such that \( t.(as' - a's) = 0 \). In this case, \( \omega(a)/\omega(s) = \omega(a')/\omega(s') \) in \( A_S \).

So, the formula \( \omega(a/s) = \omega(a)/\omega(s) \) makes sense and defines an involution on \( A_S \).

Lemma 3.1. The inclusion \( i: A_S^+ \to A_S; i(a/s) = a/s \) is an isomorphism of algebras, such that \( i\omega = \omega i \).

Proof. It is clear that \( i \) is a morphism of algebras which is injective. It is also surjective because if \( a/s \in A_S \), then \( a/s = a.\omega(s)/s.\omega(s) \) in \( A_S \), and \( s.\omega(s) \in S^+ \).

As a consequence of this lemma, from now on we can suppose \( S = S^+ \).

Consider now an \( A \)-bimodule \( M \), which is \( A^+ \)-symmetric (i.e. \( rm = mr \), for \( r \in A^+, m \in M \)), provided with an involution \( \omega_M \) compatible with \( \omega \).

More explicitly, \( \omega_M \) is \( k \)-linear, \( \omega^2 = id_M \), and if \( a, b \in A, m \in M \), then \( \omega_M(a.m.b) = \omega(b).\omega_M(m).\omega(a) \). We denote by \( M^+ = \{ m \in M/\omega_M(m) = m \} \).

As in the previous sections, the Hochschild complex \( C_*(A,M) \) can be decomposed into \( C^+_*(A,M) \) and \( C^-_*(A,M) \), whose homologies are, respectively, \( H^+_*(A,M) \) and \( H^-_*(A,M) \) [14].

\( H_*(A,M) \) (resp. \( H^+_*(A,M) \)) has a natural structure of symmetric \( A \)-bimodule (resp. \( A^+ \)-bimodule).

If \( S \) is a multiplicatively closed subset of \( A \), suppose \( S = S^+ \), and define \( M_S = A^+_S \otimes_A^+ M \otimes_A^+ A_S^+ \).

Remark. \( (M^+)_S \cong (M_S)^+ \) as \( A_S^+ \)-bimodule.

Theorem 3.2. In the above conditions,

\[
H^+_*(A_S, M_S) \cong [H^+_*(A,M)]_S \quad (and \ analogously \ for \ H^-_*)
\]

Proof. First observe that the functor \( X \to X^+ \) from the category of symmetric \( A \)-bimodules to the category of \( A^+ \)-bimodules is well-defined and exact.
Also, let \( \eta_0: [H_\omega(A, M)]_S \to H_0(A_S, M_S) \) be the natural isomorphism induced by \( \hat{f}: M/[A, M] \to M_S/[A_S, M_S] \); \( \hat{f}(\hat{m}) = c(1 \otimes m \otimes 1) \).

By a theorem of Grothendieck [9], as \( \eta_0 \) is an isomorphism and we also have natural functors \( \eta_n: [H_n(A, M)]_S \to H_n(A_S, M_S) \) for \( n \geq 0 \), then \( [H_\omega(A, M)]_S \) is isomorphic to \( H_\omega(A_S, M_S) \).

Also, \( \eta_0 \) commutes with the involution. So, \( [H_\omega(A, M)]_S \cong [(H_\omega(A, M))^+]_S \). By the previous remark, this last term is identical with \( ([H_\omega(A, M)]_S)^+ \), and by the result of Brylinski, this equals \( [H_\omega(A_S, M_S)]^+ = H_\omega(A_S, M_S) \).

Now, we apply Theorem 3.2 to the characterization of smoothness in terms of the nullity of some \( \mathbb{Z}_2 \)-equivariant Hochschild homology groups.

In [16], the author conjectures:

Let \( k \) be a field of characteristic zero and let \( A \) be a \( k \)-algebra of finite type. If \( HH_\omega(A) = 0 \) for \( n \) sufficiently large, then \( A \) is a smooth \( k \)-algebra.

In [4, 1], the authors prove the conjecture, under the less restrictive assumption that there exists two Hochschild homology groups \( HH_2 \) and \( HH_{2j+1} \) which vanish.

Here, we give a similar result for involutive commutative algebras.

**Theorem 3.3.** Let \( A = \mathbb{C}[x_1, \ldots, x_m]/I \) be a reduced commutative algebra of finite type. We assume that \( A \) is the coordinate ring of an algebraic subset \( V \) containing the origin and symmetric by the origin (so, the involution \( \omega(x_i) = -x_i \) for all \( i \), induces an involution on \( A \)).

Then, if \( V \) is not smooth at the origin, there exists an integer \( p \) such that \( HH_1^+(A) \neq 0 \) for all \( i < p \), and \( HH_{p+4n}^+(A) \neq 0 \) for all \( n \in \mathbb{N} \).

**Proof.** We recall that an algebraic subset \( V \) of the affine space \( A_m(\mathbb{C}) \) is defined by the data of a family of polynomials \( (P_i)_{i \in I} \in \mathbb{C}[x_1, \ldots, x_m] \) and

\[
V = \{(a_1, \ldots, a_m) \in \mathbb{C}^m/P_i(a_1, \ldots, a_m) = 0, \text{ for all } i\}.
\]

If we denote by \( I(V) \) the ideal generated by the polynomials \( Q \) such that \( Q(a_1, \ldots, a_m) = 0 \), for all \( (a_1, \ldots, a_m) \in V \), then \( I(V) \) is equal to the radical of the ideal generated by the family \( (P_i)_{i \in I} \). Then \( A = \mathbb{C}[x_1, \ldots, x_m]/I(V) \) is called the coordinate ring of \( V \). From the Nullstellensatz theorem, we have a one-to-one correspondence between reduced commutative algebras of finite type and coordinate rings of algebraic subsets.

Now consider an algebraic set \( V \) containing the origin \( O \). Let \( \sigma \) be the central symmetry of center \( O \) in \( A_m(\mathbb{C}) \), we assume that \( V \) contains \( \sigma(V) \). We denote by \( \omega \) the algebra morphism on \( \mathbb{C}[x_1, \ldots, x_m] \) defined by \( \omega(x_i) = -x_i \) for all \( i \). If \( \sigma(V) \) is a subset of \( V \) we can find generators \( P_1, \ldots, P_r \) of \( I(V) \) such that \( \omega(P_j) = \pm P_j \), for all \( j \in [1, \ldots, r] \). In the following, \( A = \mathbb{C}[x_1, \ldots, x_m]/I(V) \) will be endowed with the image of this involution \( \omega \). Let \( \mathcal{M} = (x_1, \ldots, x_m) \) and \( \mathcal{M} = \mathcal{M}/I \). From Theorem 3.2, we have \( HH_\omega^+(A_{\mathcal{M}}) \cong HH_\omega^+(A) \otimes_{A^+}(A^+)_{\mathcal{S}^+} \), with \( \mathcal{S}^+ = \{s \in \mathbb{C}[x_i] - \mathcal{M}/\omega(s) = s\} \).
So we work with the local ring $A_{3R}$ endowed with the induced involution. Since $\omega(x_i) = -x_i$ for all $i$, the ideal $3R'$ has a minimal set of generators on which $\omega$ operates as $-\text{Id}$. A classical argument [1], shows that we can write $A_{3R} = A_0/J$ with $A_0$ a local regular ring of maximal ideal $3R$, $J$ is contained in $3R^2$ and $A_0$ has an involution $\omega$ that operates as $-\text{Id}$ on a minimal set of generators $(f_1, \ldots, f_p)$ of $3R$. Furthermore, we have $A_0/3R \cong \mathbb{C}$.

Tate's construction [17], allows us to say that there exists a minimal commutative graded differential algebra $(A_0 \otimes AV, \partial), V = \bigoplus_{n \geq 1} V_n$, and a map from $(A_0 \otimes AV, \partial)$ onto $A_0/J$ which induces an isomorphism in homology.

On the other hand, since $A$ is involutive, we can build this model such that each $V_\sigma$ is endowed with an involution which is a morphism of commutative differential graded algebras, extending the involution of $A_0$.

In [7, 19], it is proved that the Hochschild homology of $A_0/J$ is isomorphic to the homology of $(\Omega^*_A \otimes \Omega^*_A, \partial)$ with $\delta d + d\delta = 0$.

A similar argument to those of Section 2 shows that

$$HH^*(A_{3R}) = H^*_*((\Omega^*_A \otimes \Omega^*_A)^+*, \partial)$$

Then the proof is the same as in [1]: if $A_{3R}$ is not local regular, then $J \neq 0$, so we have $V_1 \neq 0$, we can find an element $y \in V_1$ such that $\omega(y) = \pm y$. Since $\omega(f_i) = -f_i$ for all $i$, we have $\omega(df_i) = df_i$.

For $n \in \mathbb{N}$, we put $Z_{4n+p} = (df_1 \ldots df_p)(dy)^{2n}$, then $\omega(Z_{4n+p}) = Z_{4n+p}$, so $Z_{4n+p} \in (\Omega^*_A \otimes \Omega^*_A)^+$.

We conclude as in [1].

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References


