G-STRUCTURE ON THE COHOMOLOGY OF HOPF ALGEBRAS

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Abstract. We prove that \( \text{Ext}^\bullet_{A}(k,k) \) is a Gerstenhaber algebra, where \( A \) is a Hopf algebra. In case \( A = D(H) \) is the Drinfeld double of a finite-dimensional Hopf algebra \( H \), our results imply the existence of a Gerstenhaber bracket on \( H^\bullet_{GS}(H,H) \). This fact was conjectured by R. Taillefer. The method consists of identifying \( H^\bullet_{GS}(H,H) = \text{Ext}^\bullet_{A}(k,k) \) as a Gerstenhaber subalgebra of \( H^\bullet(A,A) \) (the Hochschild cohomology of \( A \)).

Introduction

The motivation of this paper is to prove that \( H^\bullet_{GS}(H,H) \) has a structure of a G-algebra. The G-algebra structure is, roughly speaking, the existence of two products with compatibilities between them: one is associative graded commutative, and the other is a graded Lie bracket. We prove this result when \( H \) is a finite-dimensional Hopf algebra (see Theorem 2.1 and Corollary 2.5). \( H^\bullet_{GS} \) is the cohomology theory for Hopf algebras defined by Gerstenhaber and Schack in [4]. In order to obtain commutativity of the cup product we prove a general statement on Ext groups over Hopf algebras (without any finiteness assumption).

When \( H \) is finite dimensional, the category of Hopf bimodules is isomorphic to a module category, over an algebra \( X \) (also finite dimensional) defined by Cibils and Rosso (see [2]), and this category is also equivalent to the category of Yetter-Drinfeld modules, which is isomorphic to the category of modules over the Hopf algebra \( D(H) \) (the Drinfeld double of \( H \)). In [10], Taillefer has defined a natural cup product in \( H^\bullet_{GS}(H,H) = H^\bullet_{b}(H,H) \) (see [3] for the definition of \( H^\bullet_{b} \)). When \( H \) is finite dimensional, she proved that \( H^\bullet_{b}(H,H) \cong \text{Ext}^\bullet_{X}(H,H) \), and using this isomorphism she showed that it is (graded) commutative. In a later work [11] she extended the result of commutativity of the cup product to arbitrary-dimensional Hopf algebras, and she conjectured the existence (and a formula) of a Gerstenhaber bracket.

Our method for giving a Gerstenhaber bracket is the following: under the equivalence of categories \( \text{X-mod} \cong D(H)^\text{-mod} \), the object \( H \) corresponds to \( H^{\text{co}H} = k \). So \( \text{Ext}^\bullet_{X}(H,H) \cong \text{Ext}^\bullet_{D(H)}(k,k) \) (isomorphism of graded algebras); according to Ştefan [8] one knows that \( \text{Ext}^\bullet_{D(H)}(k,k) \cong H^\bullet(D(H),k) \). In Theorem 1.8 we prove

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that, if $A$ is an arbitrary Hopf algebra, then $H^\bullet(A, k)$ is isomorphic to a subalgebra of $H^\bullet(A, A)$—in particular, it is graded commutative—and the morphisms are defined at the complex level. In Theorem 2.1 we prove that the image of $C^\bullet(A, k)$ in $C^\bullet(A, A)$ is stable under the brace operation (if $M$ is an $A$-bimodule, $C^\bullet(A, M)$ denotes the standard Hochschild complex whose homology is $H^\bullet(A, M)$); in particular, the image of $H^\bullet(A, k)$ is closed under the Gerstenhaber bracket of $H^\bullet(A, A)$. So, the existence of the Gerstenhaber bracket on $H^\bullet(H; H)$ follows, at least in the finite-dimensional case, by taking $A = D(H)$. We did not know if this bracket coincides with the formula proposed in [11], but Taillefer, in a personal communication, told us that, using arguments as in [7], one can actually prove that the bracket given by us, in the finite-dimensional case, must agree with the bracket proposed by her. Nevertheless, the argument does not give a proof of existence in the infinite-dimensional case. So the problem, in that generality, remains open.

We also provide a proof that the algebra $\text{Ext}^\bullet_C(k, k)$ is graded commutative when $C$ is a braided monoidal category satisfying certain exactness hypotheses (see Theorem 1.4). This gives an alternative proof of the commutativity of the cup product in the arbitrary-dimensional case by taking $C = H_\text{YD}$, the category of Yetter-Drinfeld modules.

In this paper $A$ will denote a Hopf algebra over a field $k$.

1. Cup products

This section has two parts. First we prove a generalization of the fact that the cup product on group cohomology $H^\bullet(G, k)$ is graded commutative. The general abstract setting is that of a braided (abelian) category with enough injectives satisfying an exactness condition (see Definition 1.2 below). The other part will concern the relation between self extensions of $k$ and Hochschild cohomology of $A$ with coefficients in $k$.

Let us recall the definition of a braided category:

Definition 1.1. The data $(\mathcal{C}, \otimes, k, c)$ is called a braided category with unit element $k$ if

1. $\mathcal{C}$ is an abelian category.
2. $\otimes$ is a bifunctor, bilinear, associative, and there are natural isomorphisms $k \otimes X \cong X \otimes k$ for all objects $X$ in $\mathcal{C}$.
3. For all pair of objects $X$ and $Y$, $c_{X,Y} : X \otimes Y \to Y \otimes X$ is a natural isomorphism. The isomorphisms $c_{X,k} : X \otimes k \cong k \otimes X$ agree with the isomorphism of the unit axiom, and for all triples $X, Y, Z$ of objects in $\mathcal{C}$, the Yang-Baxter equation is satisfied:

$$\text{id}_Z \otimes c_{X,Y} \circ (c_{X,Z} \otimes \text{id}_Y) \circ (\text{id}_X \otimes c_{Y,Z}) = (c_{Y,Z} \otimes \text{id}_X) \circ (\text{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \text{id}_Z).$$

A data $(\mathcal{C}, \otimes, k)$ satisfying axioms 1 and 2, but not necessarily axiom 3 is called a monoidal category.

We will use the notion of exact functor for a monoidal structure.

Definition 1.2. Let $(\mathcal{C}, \otimes, k)$ be an abelian monoidal category. We say that $\otimes$ is exact if and only if the canonical morphism

$$H_\ast(X \ast d_X) \otimes H_\ast(Y \ast d_Y) \to H_\ast(X \otimes Y \ast d_X \otimes d_Y)$$

is an isomorphism for all pairs of complexes in $\mathcal{C}$. 
Example 1.3. Let $H$ be a Hopf algebra over a field $k$. Then $\mathcal{C} = \mathcal{H} \text{-mod}$ is a monoidal category with $\otimes = \otimes_k$, and this functor is clearly exact.

**Theorem 1.4.** Let $(\mathcal{C}, \otimes, k, c)$ be a braided category with enough injectives and exact tensor product. Then $\text{Ext}^{\bullet}_{\mathcal{C}}(k, k)$ is graded commutative.

*Proof.* We proceed as in the proof that $H^\bullet(G, k)$ is graded commutative (see for example [1], page 51, Vol. 1). The proof is based on two points: first a definition of a cup product using the bifunctor $\otimes$, and second a lemma relating this construction and the Yoneda product of extensions.

Let $0 \to M \to X_p \to \cdots X_1 \to N \to 0$ and $0 \to M' \to X'_q \to \cdots X'_1 \to N' \to 0$ be two extensions in $\mathcal{C}$. Then $N_* := (0 \to M \to X_p \to \cdots X_1 \to 0)$ and $N'_* := (0 \to M' \to X'_q \to \cdots X'_1 \to 0)$ are two complexes, quasi-isomorphic to $N$ and $N'$ respectively. By the K"{u}nneth formula, $N_* \otimes N'_*$ is a complex quasi-isomorphic to $N \otimes N'$. So "completing" this complex with $N \otimes N'$ (more precisely considering the mapping cone of the chain map $N_* \otimes N'_* \to N \otimes N'$) one has an extension in $\mathcal{C}$, beginning with $M \otimes M'$ and ending with $N \otimes N'$.

So, we have defined a cup product:

$$\text{Ext}^p_{\mathcal{C}}(N, M) \times \text{Ext}^q_{\mathcal{C}}(N', M') \to \text{Ext}^{p+q}_{\mathcal{C}}(N \otimes N', M \otimes M').$$

We will denote this product by $\otimes$ and the Yoneda product by $\triangleright$. The lemma relating this product and the Yoneda one is the following:

**Lemma 1.5.** If $\eta \in \text{Ext}^p_{\mathcal{C}}(M, N)$ and $\xi \in \text{Ext}^q_{\mathcal{C}}(M', N')$, then

$$\eta \otimes \xi = (\eta \otimes \text{id}_{N'}) \triangleright (\text{id}_M \otimes \xi).$$

*Proof of the Lemma.* Interpreting the elements $\eta$ and $\xi$ as extensions, it is clear how to define a morphism of complexes $(\eta \otimes \text{id}_{N'}) \triangleright (\text{id}_M \otimes \xi) \otimes \eta \otimes \xi$, and by the K"{u}nneth formula, it is a quasi-isomorphism.

In the particular case that $M = M' = N = N' = k$, the lemma implies that $\eta \otimes \xi = \eta \triangleright \xi$ for all $\eta$ and $\xi$ in $\text{Ext}^p_{\mathcal{C}}(k, k)$. Now the theorem is a consequence of the isomorphism $(X_\ast \otimes Y_\ast, d_{X \otimes Y}) \cong (Y_\ast \otimes X_\ast, d_{Y \otimes X})$, valid for every pair of complexes in $\mathcal{C}$, defined by

$$(-1)^{pq}c_{X,Y} : X_p \otimes Y_q \to Y_q \otimes X_p.$$

Note that the differentials are morphisms in the category $\mathcal{C}$. So the map defined above commutes with the differentials because of the bifunctoriality of the braiding.

**Example 1.6.** Let $H$ be a cocommutative Hopf algebra. Then $\mathcal{H} \text{-mod}$ is braided with $c$ the usual flip. When $H = k[G]$ we recover that $H^\bullet(G, k)$ is graded commutative. The other typical example is $H = U(g)$, the enveloping algebra of a Lie algebra $g$. It is known that $\text{Ext}_{U(g)}^{\bullet}(k, k) = \Lambda^\bullet(g)$, is graded commutative.

**Example 1.7.** Let $H$ be an arbitrary Hopf algebra with bijective antipode and $\mathcal{C} = \mathcal{H} \mathcal{YD}$ the category of Yetter-Drinfeld modules over $H$. It is well known (see [6], p. 214) that the map $M \otimes N \to N \otimes M$ defined by $m \otimes n \mapsto m_{-1} n \otimes m_0$ is a braiding on $\mathcal{H} \mathcal{YD}$. So $\text{Ext}_{\mathcal{H} \mathcal{YD}}^\bullet(k, k)$ is graded commutative.

**Theorem 1.8.** If $A$ is a Hopf algebra, then $\text{Ext}^\bullet_{\mathcal{H} \mathcal{YD}}(k, k) \cong H^\bullet(A, k)$. Moreover, $H^\bullet(A, k)$ is isomorphic to a subalgebra of $H^\bullet(A, A)$. 

Proof. After Stefan [8], since $A$ is an $A$-Hopf Galois extension of $k$, $H^\bullet(A, M) \cong \text{Ext}^\bullet_A(k, M_{ad})$ for all $A$-bimodules $M$.

Here, $M_{ad}$ denotes the left $H$-module with underlying vector space $M$, but with structure $h_{-ad}m := h_1 m S(h_2)$. The notation $(S$ for the antipode, and the Sweedler-type summation) is the standard one.

In particular, $H^\bullet(A, k) = \text{Ext}^\bullet_A(k, k)$. But one can give, for this particular case, an explicit morphism at the complex level. In order to do this, we will choose a specific resolution of $k$ as a left $A$-module. Notice that, in particular, our argument will give an alternative proof of Stefan’s result for this case.

Let $C_n(A, b')$ be the standard resolution of $A$ as an $A$-bimodule, namely $C_n(A, b') = A \otimes A^n \otimes A$ and $b'(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n}(1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}$ ($a_i \in A$). This resolution splits on the right. So $(C_n(A) \otimes A k, b' \otimes id_k)$ is a resolution of $A \otimes_A k = k$ as a left $A$-module. Using this resolution, $\text{Ext}^\bullet_A(k, k)$ is the cohomology of the complex $(\text{Hom}_A(C_n(A) \otimes_A k), (b' \otimes id_k)^*) \cong (\text{Hom}(A^{op}, k), \partial)$. Under this isomorphism, the differential $\partial$ is given by

$$(\partial f)(a_1 \otimes \ldots \otimes a_n) = \epsilon(a_1) f(a_2 \otimes \ldots \otimes a_n) + \sum_{i=1}^{n-1}(1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) + (-1)^n f(a_1 \otimes \ldots \otimes a_{n-1}) \epsilon(a_n),$$

which is precisely the formula of the differential of the standard Hochschild complex computing $H^\bullet(A, k)$.

One can easily check that the cup product on $\text{Ext}^\bullet_A(k, k)$ which, by Lemma 1.5 equals the Yoneda product, corresponds to the cup product on $H^\bullet(A, k)$. So this isomorphism is an algebra isomorphism.

Now we will give two multiplicative maps $H^\bullet(A, k) \to H^\bullet(A, A)$ and $H^\bullet(A, A) \to H^\bullet(A, k)$. Consider the counit $\epsilon : A \to k$. It is an algebra map, and so the induced map $\epsilon_* : H^\bullet(A, A) \to H^\bullet(A, k)$ is multiplicative. We will define a multiplicative section of this map.

Let $f : A^{op} \to k$ be a Hochschild cocycle, and define $\hat{f} : A^{op} \to A$ by the formula

$$\hat{f}(a^1 \otimes \ldots \otimes a^p) := a_1^1 \ldots a_1^p f(a_2^1 \otimes \ldots \otimes a_2^p)$$

where we have used the Sweedler-type notation with summation symbol omitted: $a_1^1 \otimes a_2^1 = \Delta(a^1)$, for $a^1 \in A$.

Let us check that $\hat{f}$ is a Hochschild cocycle with values in $A$.

$$\partial(\hat{f})(a^0 \otimes \ldots \otimes a^p) = a^0 \hat{f}(a^1 \otimes \ldots \otimes a^p) + \sum_{i=0}^{p-1}(1)^{i+1} \hat{f}(a^0 \otimes \ldots \otimes a^i a^{i+1} \otimes \ldots \otimes a^p) + (-1)^{p+1} \hat{f}(a^0 \otimes \ldots \otimes a^{p-1}) a^p$$

$$= a^0 a_1^1 \ldots a^p_1 f(a_2^1 \otimes \ldots \otimes a_2^p) + (-1)^{p+1} a^0_1 \ldots a^{p-1}_1 f(a_2^0 \otimes \ldots \otimes a_2^{p-1}) a^p$$

$$+ \sum_{i=0}^{p-1}(1)^{i+1} a^0_1 \ldots a^i_1 a^{i+1}_1 \ldots a^p_1 f(a_2^0 \otimes \ldots \otimes a_2^i a_2^{i+1} \otimes \ldots \otimes a_2^p).$$
Using that \( f \) is a Hochschild cocycle with values in \( k \), we know that

\[
0 = \epsilon(a^0)f(a^1 \otimes \ldots \otimes a^p) + \sum_{i=0}^{p-1} (-1)^{i+1}f(a^0 \otimes \ldots \otimes a^i.a^{i+1} \otimes \ldots \otimes a^p) \\
+ (-1)^{p+1}f(a^0 \otimes \ldots \otimes a^{p-1})\epsilon(a^p).
\]

So, the summation term in \( \partial(\hat{f}) \) can be replaced using the equality

\[
\sum_{i=0}^{p-1} (-1)^{i+1}a^0_i a_1^{i+1} \ldots a_p^i . f(a_2^0 \otimes \ldots \otimes a_2^i.a_2^{i+1} \otimes \ldots \otimes a_2^p) \\
= -a^0_1 \ldots a^p_1 \left( \epsilon(a^0_2)f(a^1_2 \otimes \ldots \otimes a^p_2) + (-1)^{p+1}f(a^0_2 \otimes \ldots \otimes a^p_2)\epsilon(a^1_2) \right) \\
= - a^0_1 \ldots a^p_1 . f(a^1_2 \otimes \ldots \otimes a^p_2) + (-1)^{p+1}a^0_1 \ldots a^{p-1}_1 \epsilon(a^p_2)f(a^0_2 \otimes \ldots \otimes a^{p-1}_2)
\]

and this finishes the computation of \( \partial(\hat{f}) \).

Clearly \( \hat{f} = f \); so \( \epsilon_* \) is a split epimorphism. To check that \( f \mapsto \hat{f} \) is multiplicative is straightforward:

Let \( g : A^\otimes q \to k \) be a cocycle and \( \hat{g} : A^\otimes q \to A \) the cocycle with values in \( A \) corresponding to \( g \). We can check the following:

\[
\hat{f} \circ \hat{g}(a^1 \otimes \ldots \otimes a^{p+q}) = a^1_1 \ldots a^{p+q}_1 . (f \circ g)(a^1_2 \otimes \ldots \otimes a^{p+q}_2) \\
= a^1_1 \ldots a^{p+q}_1 . f(a^1_2 \otimes \ldots \otimes a^{p+q}_2)g(a^{p+1}_2 \otimes \ldots \otimes a^{p+q}_2) \\
= (\hat{f} \circ \hat{g})(a^1 \otimes \ldots \otimes a^{p+q}).
\]

\[\square\]

2. **Brace operations**

In this section we prove our main theorem, stating that the map \( H^\bullet(A,k) \to H^\bullet(A,A) \) is “compatible” with the brace operations, and as a consequence with the Gerstenhaber bracket. Note that the map \( H^\bullet(A,k) \to H^\bullet(A,A) \) is defined at the standard complex level. Let us define \( C^p(A,M) := \text{Hom}(A^\otimes p, M) \).

**Theorem 2.1.** The image of the map \( C^\bullet(A,k) \to C^\bullet(A,A) \) is stable under the brace operation. Moreover, if \( \hat{f} \) and \( \hat{g} \) are the images in \( C^\bullet(A,A) \) of two elements \( f \) and \( g \) belonging to \( C^\bullet(A,k) \), then \( \hat{f} \circ \hat{g} = \hat{f} \circ \hat{g} \).

**Proof.** Let us recall the definition of the brace operations (see [3]). If \( F : A^\otimes p \to M \) and \( G : A^\otimes q \to A \) and \( 1 \leq i \leq p \), then \( F \circ_i G : A^\otimes p+q-1 \to M \) is defined by

\[
(F \circ_i G)(a^1 \otimes \ldots \otimes a^i \otimes b^1 \otimes \ldots \otimes b^q \otimes a^{i+1} \otimes \ldots \otimes a^p) \\
= F(a^1 \otimes \ldots \otimes a^i \otimes G(b^1 \otimes \ldots \otimes b^q) \otimes a^{i+1} \otimes \ldots \otimes a^p).
\]

Assume now that \( f : A^\otimes p \to k \), \( g : A^\otimes q \to k \) and \( F = \hat{f} \) and \( G = \hat{g} \), namely

\[
F(a^1 \otimes \ldots \otimes a^p) = a^1_1 \ldots a^p_i . f(a^2_1 \otimes \ldots \otimes a^p_2)
\]
and similarly for $G$ and $g$. Then (denoting $(a \otimes b)$ by $(a, b)$),

\[
(F \circ_i G)(a, b) = F(a, G(b)) = F(a_1, \ldots, a_i, G(b_1, \ldots, b_{i+1})) = F(a_1, \ldots, a_i, b_1, \ldots, b_{i+1}, g(b_1, \ldots, b_{i+1}, a)) = a_1 \ldots a_i b_1 \ldots b_{i+1} a \cdot f(a_1, \ldots, a_i, b_1, \ldots, b_{i+1}, g(b_1, \ldots, b_{i+1}, a_{i+1}, \ldots, a_p))
\]

Recall that the brace operations define a “composition” operation $F \circ G = \sum_{i=1}^{p} (-1)^{i(i-1)} F \circ_i G$, where $F \in C^p(A, A)$ and $G \in C^i(A, A)$. The Gerstenhaber bracket is defined as the commutator of this composition. So we have the desired corollary:

**Corollary 2.2.** If $A$ is a Hopf algebra, then $H^\bullet(A, k)$ is a Gerstenhaber subalgebra of $H^\bullet(A, k)$.

**Example 2.3.** Let $A$ be a Hopf algebra. Then $\text{Ext}^1_A(k, k) \cong \text{Der}(A, k) = \text{Prim}(A^*)$, where $\text{Prim}(A^*) = \{x \in A^* \mid m^*(x) = x \otimes 1 + 1 \otimes x\}$. It is easy to check that the Lie bracket given in the above theorem coincides with the commutator of the convolution product, viewing $\text{Der}(A, k)$ as a subset of $A^*$.

**Example 2.4.** Let $G$ be a connected affine algebraic group and $\mathfrak{g} := \ker(\epsilon)/\ker(\epsilon)^2$ its tangent Lie algebra. One has that $HH^\bullet(O(G), \mathcal{O}(G)) = \Lambda^\bullet(\mathcal{O}(G)) = \text{Der}(\mathcal{O}(G)) \cong O(G) \otimes \Lambda^\bullet \mathfrak{g}$, where the Gerstenhaber structure here is the Schouten-Nijenhuis bracket. Also $\text{Ext}^\bullet_{\mathcal{O}(G)}(k, k) = \Lambda^\bullet \mathfrak{g}$, and it is generated (as an algebra) in degree one. So the bracket is determined by its values on $\text{Ext}^1_{\mathcal{O}(G)}(k, k) = \mathfrak{g}$, which is the bracket of $\mathfrak{g}$ as a Lie algebra. This $G$-algebra structure is also well known.

Consider $H$ a finite-dimensional Hopf algebra and $X = X(H)$ the algebra defined by Cibils and Rosso (see [2]). We can prove, at least in the finite-dimensional case, the conjecture of [11] that $H_{GS}^\bullet(H, H)$ is a Gerstenhaber algebra:

**Corollary 2.5.** Let $H$ be a finite-dimensional Hopf algebra. Then $H_{GS}^\bullet(H, H)$ is a Gerstenhaber algebra.

**Proof.** The isomorphism $H_{GS}^\bullet(H, H) \cong \text{Ext}^\bullet_X(H, H)$ was proved in [10].

Let $A$ denote $D(H)$, the Drinfeld double of $H$. One knows that $\chi$-mod $\cong A$-mod via $M \mapsto M^{coH}$. Then $\text{Ext}^\bullet_X(H, H) \cong \text{Ext}^\bullet_A(H^{coH}, H^{coH}) = \text{Ext}^\bullet_A(k, k)$, and this a Gerstenhaber subalgebra of $H^\bullet(A, A)$.

**References**


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