ON THE EMBEDDING PROBLEM FOR 2^+S_4 REPRESENTATIONS

ARIEL PACETTI

ABSTRACT. Let 2^+S_4 denote the double cover of S_4 corresponding to the element in $\mathrm{H}^2(S_4, \mathbb{Z}/2\mathbb{Z})$ where transpositions lift to elements of order 2 and the product of two disjoint transpositions to elements of order 4 (denoted $\tilde{S}_4$ in [15]). Given an elliptic curve E, let $E[2]$ denote its 2-torsion points. Under some conditions on E (as in [1]) elements in $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2])\backslash\{0\}$ correspond to Galois extensions N of $\mathbb{Q}$ with Galois group (isomorphic to) S_4. On this work we give an interpretation of the addition law on such fields, and prove that the obstruction for N having a Galois extension $\tilde{N}$ with $\mathrm{Gal}(\tilde{N}/\mathbb{Q}) \simeq 2^+S_4$ gives an homomorphism $s_4^+ : \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2]) \to \mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$. As a Corollary we can prove (if E has conductor divisible by few primes and high rank) the existence of 1-dimensional representations attached to E and use them in some examples to construct 3/2 modular forms mapping via the Shimura map to (the modular form attached to) E.

INTRODUCTION

The study of modular forms of weight 1 is equivalent to that of two dimensional continuous faithful irreducible complex representations of $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (see [6]). Looking at their projectivization we have five different kinds: cyclic, dihedral, A_4, S_4, A_5 (see [7]). If the image is cyclic then the original representation is abelian, hence reducible. The dihedral case corresponds to weight 1 modular forms which are linear combination of theta series attached to binary quadratic forms. The "special ones" are the last three cases. They are constructed using different approaches (see [9] for algorithms to construct the A_4 and A_5 cases, and [10] for the S_4 case). To study the S_4 case, in [1] the next method is proposed: let E be an elliptic curve over $\mathbb{Q}$ with negative discriminant (if the discriminant is positive the same method gives Maas forms), no 2-torsion points over $\mathbb{Q}$ and non-trivial Selmer 2-group. The set $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2])\backslash\{0\}$ is in one to one correspondence with fields N with Galois group S_4 over $\mathbb{Q}$ containing $\mathbb{Q}(E[2])$. The obstruction for N having a field extension $\tilde{N}$ with Galois group over $\mathbb{Q}$ isomorphic to 2^+S_4 is an element in $\mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$ (see [15] for a formula of the obstruction and [4],[5] for a method to compute a solution to the embedding problem when the obstruction is trivial). This induces a map $s_4^+ : \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2])\backslash\{0\} \to \mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$. The main result of this work is that if we define $s_4^+(0) = 0$, then s_4^+ is a group homomorphism. As a corollary all elliptic curves (in the above conditions) of conductor $2^r p^s$ with $r, s \in \mathbb{N}_0$ and p a prime number such that the 2-Selmer group has rank at least two

2000 *Mathematics Subject Classification.* Primary: 11F80; Secondary: 11F37.
Key words and phrases. Galois Representations, Shimura Correspondence.
The author was supported by a CONICET grant.
The author would like to thank the "Universitat de Barcelona" where this work was done.

have a 1-dimensional representations with Galois group 2^+S_4 attached to them. We end this work with some examples of how using these weight 1 modular forms one can construct weight 3/2 modular forms mapping via Shimura (see [16], Main Theorem) to the modular form (attached to) E.

I would like to thank the University of Barcelona for its hospitality and specially to Professor Pilar Bayer for useful conversations and contributions to improve this work. I would also like to thank Professor Jordi Quer for explaining me how he computed the explicit solution to the embedding problems of the examples in [1] and some other useful conversations as well.

1. Correspondence between $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2])$ and fields

Let E be an elliptic curve over $\mathbb{Q}$ with negative discriminant and no 2-rational points. The field $L = \mathbb{Q}(E[2])$ is a Galois extension of $\mathbb{Q}$ with Galois group S_3. Let S be a finite set of primes containing 2 and the primes dividing the conductor of E and denote $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)$ the cocycles unramified outside S. Abusing notation we will denote $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)^{\times}$ the set $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)\backslash\{0\}$. By $\mathcal{D}(K)$ we denote the discriminant of the field K.

Proposition 1.1. *The elements in* $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)^{\times}$ *are in one to one correspondence with fields* N *such that* $L \subset N$, $\mathrm{Gal}(N/\mathbb{Q}) = S_4$ *and* $N/\mathbb{Q}$ *is unramified outside* S. *Furthermore, if* $K \subset N$ *is a degree* 4 *extension of* $\mathbb{Q}$ *whose normal closure is* N *then* $\mathcal{D}(K) = \mathcal{D}(L)$ *in* $\mathbb{Q}^{\times}/(\mathbb{Q}^{\times})^2$.

Proof. The correspondence is Proposition 1.1 of [1] (although they do not state the ramification condition). The main idea (that we will need latter) is that if ϕ is a non-trivial cocycle, then $\phi|_{\mathrm{Gal}_L}$ is a group homomorphism. If we denote N_ϕ the fixed field of $\ker(\phi)$, then $L \subset N_\phi$ and $\mathrm{Gal}(N_\phi/L) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (the Klein group). The ramification condition follows from this isomorphism. For the last statement, note that since S_4 has a unique normal subgroup of index 2 (namely A_4), there is a unique quadratic Galois subextension, namely $\mathbb{Q}(\sqrt{\mathcal{D}(K)})$. Since $\mathbb{Q}(\sqrt{\mathcal{D}(L)})$ is another quadratic Galois subextension they must be equal. $\square$

1.1. **Field Addition Interpretation.** The group structure of $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2])^{\times}$ induces a group structure on fields N satisfying the above condition. Using elementary field theory we will show a natural construction of such addition. It is an easy group theory exercise to check that $S_4 \simeq S_3 \ltimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ where, if we denote $\{P_1, P_2, P_3\}$ the three nonzero elements of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the action is given by $\sigma.P_i = P_{\sigma(i)}$.

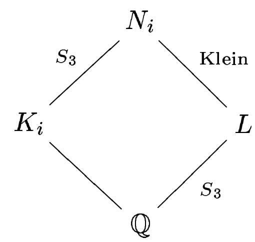
Lemma 1.2. *Let* N_1, N_2 *be fields corresponding to elements in* $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)^{\times}$, *then* $N_1 \cap N_2 = L$.

Proof. Clearly $L \subset N_1 \cap N_2$. Since $N_1 \cap N_2$ is Galois over $\mathbb{Q}$, $\mathrm{Gal}(N_1/N_1 \cap N_2)$ corresponds to a non-zero normal subgroup of S_4 contained in $\mathrm{Gal}(N_1/L)$ (isomorphic to the Klein group), hence $\mathrm{Gal}(N_1/N_1 \cap N_2) \simeq \mathrm{Gal}(N_1/L)$. $\square$

In particular N_1N_2 is a Galois extension of $\mathbb{Q}$ with Galois group of order 96 and $\mathrm{Gal}(N_1N_2/L) \simeq \mathrm{Gal}(N_1/L) \oplus \mathrm{Gal}(N_2/L) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proposition 1.3. $\mathrm{Gal}(N_1N_2/\mathbb{Q}) \simeq S_3 \ltimes \oplus_{i=1}^{2}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$, *where if* $u, v \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *and* $\sigma \in S_3$, $\sigma.(u, v) = (\sigma.u, \sigma.v)$.

Proof. Let $K_i \subset N_i$ be the fixed field of the subgroup S_3 looked as a subgroup of $S_4 = \mathrm{Gal}(N_i/\mathbb{Q})$ (fixing the fourth element). We have the diagram:



Claim: $\mathrm{Gal}(N_1N_2/K_1K_2) \simeq \mathrm{Gal}(L/\mathbb{Q})$. To see this consider the sequence:

$$\mathrm{Gal}(N_i/K_i) \hookrightarrow \mathrm{Gal}(N_i/\mathbb{Q}) \stackrel{\pi_i}{\to} \mathrm{Gal}(L/\mathbb{Q})$$

where $\pi_i : \mathrm{Gal}(N_i/\mathbb{Q}) \to \mathrm{Gal}(L/\mathbb{Q})$ is the restriction map (which is the same as the quotient by $\mathrm{Gal}(N_i/L)$). Since $\mathrm{Gal}(N_i/K_i) \cap \mathrm{Gal}(N_i/L) = \{id\}$, the composition is an isomorphism.

Since $N_1K_2 = N_1N_2$, $\mathrm{Gal}(N_1N_2/K_1K_2) \simeq \mathrm{Gal}(N_1/K_1) \simeq \mathrm{Gal}(L/\mathbb{Q})$ by the restriction map (respectively $\mathrm{Gal}(N_1N_2/K_1K_2) \simeq \mathrm{Gal}(N_2/K_2)$). From the sequence

$$\mathrm{Gal}(N_1N_2/K_1K_2) \hookrightarrow \mathrm{Gal}(N_1N_2/\mathbb{Q}) \stackrel{\Pi}{\to} \mathrm{Gal}(L/\mathbb{Q})$$

given by restriction (where the composition is an isomorphism) we conclude that $\mathrm{Gal}(N_1N_2/\mathbb{Q}) \simeq \mathrm{Gal}(L/\mathbb{Q}) \ltimes \mathrm{Gal}(N_1N_2/L)$ and comparing the action with that of $\mathrm{Gal}(N_i/\mathbb{Q}) \simeq \mathrm{Gal}(L/\mathbb{Q}) \ltimes \mathrm{Gal}(N_i/L)$, the result follows. □

Proposition 1.4. *The group* $G := S_3 \ltimes \sum_{i=1}^{4}(\mathbb{Z}/2\mathbb{Z})$ *with the previous action has three normal subgroups of order* 4.

Proof. Clearly the subgroups $\{0\} \ltimes (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0)$, $\{0\} \ltimes (0, 0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and $\{0\} \ltimes \{(a, b, a, b) \ : \ a, b \in \mathbb{Z}/2\mathbb{Z}\}$ are normal. Let $\mathrm{H} \triangleleft G$ be any normal subgroup of order 4. Then $\Pi(H) \triangleleft S_3$ with order 1 or 2. Since S_3 has no normal subgroups of order 2, $\Pi(H) = \{0\}$. The orbits of S_3 acting on $\{0\} \ltimes \sum_{i=1}^{4}(\mathbb{Z}/2\mathbb{Z})$ are:

- $\{(0, 0, 0, 0)\}$
- $\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 1, 0, 0)\}$
- $\{(0, 0, 1, 0), (0, 0, 0, 1), (0, 0, 1, 1)\}$
- $\{(1, 0, 1, 0), (0, 1, 0, 1), (1, 1, 1, 1)\}$
- $\{(1, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\}$

□

Corollary 1.5. *Let* $\psi_i \in \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)^{\times}$ *and* N_i *the corresponding field. The cocycle* $\psi_1 + \psi_2$*, if non-trivial, corresponds to the field fixed by the third normal subgroup of order* 4 *in* $\mathrm{Gal}(N_1N_2/\mathbb{Q})$.

Proof. The morphisms $\psi_i|_{\mathrm{Gal}_L} : \mathrm{Gal}_L \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, satisfy $N_i = \ker(\psi_i|_{\mathrm{Gal}_L})$. Clearly $\psi_1 + \psi_2$ is zero on $\mathrm{Gal}_{N_1N_2}$, hence its kernel is a normal subgroup of order 4 in $\mathrm{Gal}(N_1N_2/L)$ and the result follows from Proposition 1.4. □

Remark. all normal subgroups of G have pairwise trivial intersection (corresponding to normal subfields K_1, K_2 and K_3). If we define the subgroups:

- $\mathrm{H}_4 = \{0\} \ltimes \{(0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0)\}$
- $\mathrm{H}_5 = \{0\} \ltimes \{(0, 0, 0, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\}$

then all these five subgroups have trivial pairwise intersection. They correspond to the other two (unique) subfields N_4 and N_5 of index 4 of N_1N_2 with the property that $N_i \cap N_j = L$ for all $i \neq j$. Furthermore N_4 and N_5 are Galois conjugates.

It is a nice exercise to prove that given any three order 4 subgroups of $\oplus_{i=1}^4 \mathbb{Z}/2\mathbb{Z}$ having trivial pairwise intersection, there exists another two order 4 subgroups such that all of them have the same property.

1.2. **Two coverings of S_4.** We will consider cohomology groups with the trivial action. The central 2-extensions of S_4 correspond to elements in the group $\mathrm{H}^2(S_4, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the four groups are:

- $S_4 \oplus \mathbb{Z}/2\mathbb{Z}$
- $2^{\det}S_4$, corresponding to the cup product of the signature with itself.
- 2^+S_4 and 2^-S_4.

The group 2^+S_4 is isomorphic to $Gl_2(\mathbb{F}_3)$. A complete character table of the group 2^+S_4 can be found in [7], Lemma 28.2.

Lemma 1.6. *The group 2^+S_4 has a subgroup isomorphic to S_3.*

Proof. The subgroup of $Gl_2(\mathbb{F}_3)$ spanned by $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\rangle$ is isomorphic to S_3, an isomorphism given by sending the generators to the elements in S_3: id, $(1,2)$ and $(1,3)$, respectively. $\square$

Let s_4^+ denote the element in $\mathrm{H}^2(S_4, \mathbb{Z}/2\mathbb{Z})$ corresponding to the group 2^+S_4. Consider the projections from G to S_4:

- $\Pi_1(\sigma, (x, y, z, w)) = (\sigma, (x, y))$.
- $\Pi_2(\sigma, (x, y, z, w)) = (\sigma, (z, w))$.
- $\Pi_3(\sigma, (x, y, z, w)) = (\sigma, (x + z, y + w))$.

where Π_i maps G to $\mathrm{Gal}(N_i/\mathbb{Q})$ with kernel H_i, for $i = 1, 2, 3$ (the three normal subgroups of Proposition 1.4). The obstruction for the existence of $\tilde{K}_i$, a field containing K_i and Galois group 2^+S_4 is the element in the 2-Brauer group $\Gamma^*(\Pi_i^*(s_4^+))$ where $\Gamma : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Gal}(K_1K_2/\mathbb{Q})$ is the restriction map. Our main theorem can be stated as

Theorem 1.7. $\Pi_1^*(s_4^+) + \Pi_2^*(s_4^+) + \Pi_3^*(s_4^+) = 0$.

From class field theory we know that the 2-Brauer group injects into the sum of its local components, i.e. $\mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \oplus_l \mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}_l}, \mathbb{Z}/2\mathbb{Z})$ and the local components are isomorphic to $\{\pm 1\}$. Let $\mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}, S)$ be the subgroup of the 2-Brauer group of elements with trivial image at the primes outside S. If we extend s_4^+ to $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S)$ by setting $s_4^+(0) = 0$, we get

Corollary 1.8. *The map $s_4^+ : \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, E[2], S) \to \mathrm{H}^2(\mathrm{Gal}_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}, S)$ is a group homomorphism.*

From Serre's formula for the obstruction ([15], Theorem 1) it is clear that the image of s_4^+ is on this subgroup of the 2-Brauer group. The Corollary is an immediate consequence of Theorem 1.7 noting that the case when two fields are equal is trivial from the fact that the cohomology groups are 2-groups.

Proof. (Theorem) Let Z_2 denote the group $\oplus_{i=1}^4 \mathbb{Z}/2\mathbb{Z}$ and consider the exact sequence

$$0 \to Z_2 \to G \to S_3 \to 0$$

using the inflation-restriction map we get an exact sequence

$$0 \to \mathrm{H}^2(S_3, \mathbb{Z}/2\mathbb{Z}) \stackrel{Inf}{\to} \mathrm{H}^2(G, \mathbb{Z}/2\mathbb{Z}) \stackrel{Res}{\to} \mathrm{H}^2(Z_2, \mathbb{Z}/2\mathbb{Z})$$

Let $\psi := \Pi_1^*(s_4^+) + \Pi_2^*(s_4^+) + \Pi_3^*(s_4^+)$. Restricted to Z_2, $\Pi_1 + \Pi_2 + \Pi_3 = 0$, hence $\mathrm{Res}(\psi) = 0$, i.e. ψ is in the image of the inflation map. This implies that ψ does not depend on representatives of the quotient map, in particular it is determined by its values on $S_3 \ltimes \{(0,0,0,0)\} \times S_3 \ltimes \{(0,0,0,0)\}$. By Lemma 1.6, S_3 is a subgroup of 2^+S_4, then $s_4^+((S_3,(0,0)),(S_3,(0,0))) = 0$ and the result follows. □

Remark. The fact that the group 2^+S_4 has a subgroup isomorphic to S_3 is crucial for the map being a homomorphism. The same statement is false (in general) considering the maps between $\mathrm{H}^1(\mathrm{Gal}_\mathbb{Q}, E[2], S) \to \mathrm{H}^2(\mathrm{Gal}_\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}, S)$ coming from the other groups $2^{\det}S_4$ and 2^-S_4.

Let K_i be a degree 4 extensions of $\mathbb{Q}$ with normal closure N_i , Q_{N_i} be the quadratic form $\mathrm{Tr}_{K_i/\mathbb{Q}}(x^2)$ and $W(Q_{N_i})$ its Witt invariant,

Corollary 1.9. *$W(Q_{N_3}) = W(Q_{N_1}) + W(Q_{N_2}) + (2, \mathfrak{D}(L))$ on $Br_2(\mathbb{Q})$.*

Proof. This follows from Serre's Formula for the obstruction, see [15], Theorem 1. □

Corollary 1.10. *Let E be an elliptic curve with conductor $2^r p^s$ with $r, s \in \mathbb{N}_0$ and 2-Selmer group of rank at least two, then there exists a 2^+S_4 representation attached to E.*

Proof. Since the Selmer group has rank at least 2, let N_1 and N_2 be two different fields corresponding to elements in $\mathrm{H}^1(\mathrm{Gal}_\mathbb{Q}, E[2])^\times$, and $N_3 := N_1 + N_2$. Let $s_4^+(N_i)$ denote their obstruction. From the injection of the 2-Brauer group into its local components it is clear that the obstruction is characterized by the (finite set of) primes with -1 sign. Such set has an even number of primes and is contained in the set $\{2, p\}$, hence if two elements have non-zero obstruction in the Brauer group, the third one does. □

2. Applications and Examples

We give a brief summary of how to construct the weight 1 modular forms attached to the Galois group 2^+S_4. Let $K = \mathbb{Q}(x_1)$ be a degree four extension of $\mathbb{Q}$ with normal closure N, an extension with S_4 Galois group. By Theorem 1 of [15] the obstruction (for a lift with Galois group 2^+S_4) is trivial if and only if the quadratic form $\mathrm{Tr}_{K/\mathbb{Q}}(x^2)$ is isomorphic (over $\mathbb{Q}$) to the form $x_1^2 + x_2^2 + 2x_3^2 + 2\mathfrak{D}(K)x_4^2$. Furthermore if P is a transformation matrix sending one form to the other one, let

$$\gamma := \det\left[\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2\sqrt{\mathfrak{D}(K)}} & \frac{-1}{2\sqrt{\mathfrak{D}(K)}} \end{pmatrix} + id\right]$$

If $\gamma \neq 0$, all the solutions to the embedding problem are $\tilde{N} = N(\sqrt{c\gamma})$ with $c \in \mathbb{Q}^\times$ (see [5], Theorem 5). Furthermore, c can be chosen such that $\tilde{N}$ is unramified outside S and has minimum ramification at the primes in S. This choice of c gives a weight 1 modular, with character $\left(\frac{\mathfrak{D}(K)}{\cdot}\right)$ and minimum level. See [1], Proposition

2.3, Proposition 2.4 and Proposition 2.5 to compute the Fourier coefficients of the weight 1 modular form once γ is known.

We will give some examples of how this weight 1 modular forms can be used to construct some "special" 3/2 modular forms. Given a weight 2 and level p modular form f (attached to an elliptic curve E), in [8] Gross gave a method to construct a weight 3/2 modular form (as linear combination of theta series) in the Kohnen space mapping to f via the Shimura map. If E has positive rank, the constructed weight 3/2 modular form is the zero form. For this elliptic curves we will show (in some examples) how using the weight 1 modular form coming from the solution of the obstruction problem (by Corollary 1.10 we know that such form exists if E has rank greater than 1) one can construct a non-zero weight 3/2 modular form mapping to f via the Shimura map. This construction has some limitations (from a computation point of view) as we will see latter, but works on many cases (our approach is similar to that on [2]).

For $n \in \mathbb{N}$, let

$$\Theta_n(z) := \sum_{j=-\infty}^{+\infty} q^{nj^2}$$

then the theta function $\Theta_n(z)$ is a weight 1/2 modular form of level $4n$ and character $\left(\frac{n}{\cdot}\right)$ (see [16] for the definition of modular forms of half integral weight).

Lemma 2.1. *Let $g(z)$ be a weight 1, level n and character $\left(\frac{-d}{\cdot}\right)$ modular form (with $d > 0$ and $d \mid n$), then $g(z)\Theta_d(z)$ is a modular form of weight 3/2, level $\operatorname{lcm}(n, 4d)$ and trivial character.*

Proof. Let $M \in \Gamma_0(\operatorname{lcm}(n, 4d))$, say $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then:

- $g(Mz) = (\gamma z + \delta) \left(\frac{-d}{\delta}\right) g(z)$.
- $\Theta_d(Mz)\Theta(z) = \left(\frac{d}{\delta}\right) \Theta_d(z)\Theta(Mz)$.
- $\Theta(Mz)^2 = (\gamma z + \delta) \left(\frac{-1}{\delta}\right) \Theta(z)^2$.

Then $g(Mz)\Theta_d(Mz)\Theta(z)^3 = g(z)\Theta_d(z)\Theta(Mz)^3$ which is the definition of a weight 3/2 modular form of trivial character. □

What we do is to compute the space of ternary quadratic forms whose theta series are modular forms with trivial character (this is equivalent to the forms have square discriminant), and add the product of the weight 1 modular form with the corresponding theta series. Applying the Hecke operators on this set we look for the eigenform corresponding to E. In practice the difference between the dimension of the space of weight 3/2 modular forms and that of the subspace spanned by theta series increases with the level (see [12] for some tables). Use this method in general implies knowing too many Fourier coefficients of the constructed modular forms, which is impracticable. For this reason we use this method for the case $n = 43$, where the dimension of the whole space is almost the same as the subspace of theta series. If the weight 1 modular form g has level p and character $\left(\frac{-p}{\cdot}\right)$, a prime congruent to $3 \pmod 4$, $g(4z)$ has level $4p$ and the same character, hence the product $g(4z)\Theta_p(z)$ is in the Kohnen space of level $4p$ and trivial character, which (by [11] Theorem 1) is isomorphic to $S_2(\Gamma_0(p))$. In these cases we can construct the weight 3/2 modular forms for big values of p.

	a_1	a_2	a_3	a_{23}	a_{13}	a_{12}		a_1	a_2	a_3	a_{23}	a_{13}	a_{12}
Q_1	1	11	43	0	0	1	Q_8	5	18	26	18	2	4
Q_2	4	11	14	−10	3	2	Q_9	6	15	23	2	6	4
Q_3	6	9	10	4	5	1	Q_{10}	9	10	24	10	2	4
Q_4	1	43	43	0	0	0	Q_{11}	11	14	16	−6	4	10
Q_5	2	22	43	0	0	2	Q_{12}	4	43	44	0	4	0
Q_6	3	29	29	−28	2	2	Q_{13}	11	16	47	16	2	4
Q_7	4	11	43	0	0	2	Q_{14}	15	23	24	12	8	2

TABLE 2.1. Coefficients of ternary quadratic forms, level $2^2 43$.

	a_1	a_2	a_3	a_{23}	a_{13}	a_{12}		a_1	a_2	a_3	a_{23}	a_{13}	a_{12}
Q_1	3	115	115	−114	−2	−2	Q_{10}	3	29	86	0	0	−2
Q_2	8	43	88	0	−8	0	Q_{11}	5	18	86	0	0	−4
Q_3	19	20	91	20	6	12	Q_{12}	8	22	43	0	0	−4
Q_4	4	87	87	2	4	4	Q_{13}	19	20	26	−4	−16	−12
Q_5	15	24	92	24	4	8	Q_{14}	1	86	86	0	0	0
Q_6	15	23	95	−22	−14	−2	Q_{15}	6	15	86	0	0	−4
Q_7	16	44	47	4	16	8	Q_{16}	9	10	86	0	0	−4
Q_8	23	31	47	18	14	10	Q_{17}	13	16	47	16	6	12
Q_9	2	43	86	0	0	0	Q_{18}	14	21	31	14	4	12

TABLE 2.2. Coefficients of ternary quadratic forms, level $2^3 43$.

2.1. **Examples.** We use the method described above in some particular examples. The computations were done with the PARI/GP system [13].
Notation: the ternary quadratic forms will be denoted a_1, a_2, a_3, a_{23}, a_{13}, a_{12}, to express the form:

$$(1)\quad Q(X_1, X_2, X_3) = a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_{23} X_2 X_3 + a_{13} X_1 X_3 + a_{12} X_1 X_2.$$

2.1.1. *Case* 43. The elliptic curve $43A$ in Cremona's table with equation $y^2 + y = x^3 + x^2$ has rank 1. A generator is $P = (0, 0)$ and it corresponds to the field K with equation $P = x^4 - 2x - 1$. By [1] we know that the embedding problem for this case is solvable, and a solution with minimal level is given by the element

$$\gamma = 3(x_1^3 x_2^2 - x_2^2 - x_1^2 x_2 + x_1 x_2 + x_2) + x_1^3 - 2x_1^2 + 4x_1,$$

where x_1, x_2 are roots of P. The corresponding modular form has level $2^3 43$ and character $\left(\frac{-43}{\cdot}\right)$. All ternary quadratic forms of level $2^2 43$ and $2^3 43$ with trivial character are given in tables 2.1 and 2.2, where we denote Q_{43_i} (respectively Q_{86_i}) the i-th form on the table of ternary quadratic forms of level $2^2 43$ (respectively of level $2^3 43$). See [17] for interactive tables of ternary quadratic forms of a given level (in some specific genera) and [12], Theorem 4 and Theorem 5, for the bijection between different genera.

The theta functions of these quadratic forms are not linearly independent. A basis is given by the theta functions of the ternary quadratic forms: $\{Q_{43\_1}, Q_{43\_2}, Q_{43\_3}, Q_{43\_4}, Q_{43\_5}, Q_{43\_6}, Q_{43\_7}, Q_{43\_8}, Q_{43\_9}, Q_{43\_10}, Q_{43\_11}, Q_{86\_1}, Q_{86\_2}, Q_{86\_3}, Q_{86\_4}, Q_{86\_5}, Q_{86\_6}, Q_{86\_7}, Q_{86\_8}, Q_{86\_9}, Q_{86\_11}\}$ (just by looking at enough Fourier coefficients). The space $S_{3/2}(2^3 43)$ has dimension 25 (see [3] Theorem 2), hence there are 4 modular forms missing.

Let $f(z)$ denote the weight 1 modular form of weight 1, level $2^3 43$ and character $\left(\frac{-43}{\cdot}\right)$ associated to K. By Lemma 2.1, $f\,\Theta_{43}$ is a weight 3/2, level $2^3 43$ and trivial character cusp form. Since its coefficients are not rational, we consider the two modular forms $F_1(z) = \frac{1}{2}(f(z) + \bar{f}(z))\Theta_{43}$ and $F_2(z) = \frac{\sqrt{-2}}{2}(f(z) - \bar{f}(z))\Theta_{43}$. These two forms do have rational coefficients and are linearly independent from the ternary theta functions. Looking at the 23 modular forms together and computing the Hecke operators on them we get the two eigenforms (we denote by Θ_Q the Theta function of the ternary quadratic form Q):

(1) $G_{43A} = -\frac{3}{2}\Theta_{Q_{43\_6}} + \frac{4}{3}\Theta_{Q_{43\_7}} + \Theta_{Q_{43\_8}} - \frac{16}{3}\Theta_{Q_{86\_1}} + \Theta_{Q_{86\_2}} + 6\Theta_{Q_{8\_3}} + \frac{5}{2}\Theta_{Q_{86\_9}} - 5\Theta_{Q_{86\_11}} - 5F_2$.

(2) $G_{172A} = -\frac{1}{2}\Theta_{Q_{43\_1}} - \frac{1}{2}\Theta_{Q_{43\_2}} + \Theta_{Q_{43\_3}} + \Theta_{Q_{43\_4}} - \Theta_{Q_{43\_5}} + 3\Theta_{Q_{43\_9}} + 3\Theta_{Q_{43\_10}} - 6\Theta_{Q_{43\_11}} - 2\Theta_{Q_{86\_4}} + 4\Theta_{Q_{86\_5}} + 4\Theta_{Q_{86\_6}} - 2\Theta_{Q_{86\_7}} - 4\Theta_{Q_{86\_8}} + \frac{3}{2}F_1$.

They are Hecke eigenforms, and they map by the Shimura map to the weight two modular forms associated to the elliptic curves $43A$ and $172A$ on Cremona's table respectively. The curve $172A$ has rank 1 and is given by the equation $y^2 = x^3 + x^2 - 13x + 15$. The first 50 coefficients of their Fourier expansion are:

- $G_{43A} = q^2 + q^3 + q^5 - 5q^7 + 2q^8 - 4q^{12} - 3q^{18} + 2q^{19} + 2q^{20} - 3q^{22} - q^{26} - 2q^{27} + q^{29} + 2q^{32} - 3q^{33} + q^{34} + 4q^{37} + 5q^{39} + 2q^{42} + 2q^{43} - 3q^{45} + 3q^{46} + 6q^{48} - 3q^{50} + O(q^{51})$.

- $G_{172A} = q + q^6 - q^9 + q^{10} + q^{13} - q^{14} - q^{17} + 2q^{21} - q^{25} + q^{41} - 3q^{49} + O(q^{51})$.

2.1.2. *Case* 563*:* the elliptic curve $563A$ with equation $y^2 + xy + y = x^3 + x^2 - 15x + 16$ has rank 2. The points $[2, -1]$ and $[4, 4]$ are generators for the rational points. The field corresponding to the point $[2, -1]$ is given by the polynomial $P = x^4 - 8x^3 + 19x^2 - 14x - 1$. Its discriminant is -563 and the obstruction is trivial for this field. A solution to the embedding problem is given by

$$1126\gamma = (57426x_2^2 - 408738x_2 - 155984)x_1^3 + (-434073x_2^2 + 2329098x_2 + 1542884)x_1^2 + (342834x_2^2 - 1089141x_2 - 4555297)x_1 - 339522x_2^3 + 2651994x_2^2 - 6101295x_2 + 4078271,$$

where x_1 and x_2 are roots of P. Since E has discriminant -563, by Corollary 2.7 of [1] we know the attached weight 1 modular form has level $2^r 563$ and character $\left(\frac{-563}{\cdot}\right)$. By Theorem 2 of [14] we know that γ can be chosen such that the field $\tilde{N}$ above N is unramified at 2 over $\mathbb{Q}$, hence the weight 1 modular form has level exactly 563.

The field $\mathbb{Q}(\sqrt{\gamma})$ is given by $\mathbb{Q}(x_0)$ where x_0 is a root of the polynomial:
$x^{24} - 3x^{23} - 9x^{22} + 22x^{21} + 55x^{20} - 68x^{19} - 212x^{18} + 85x^{17} + 467x^{16} - 34x^{15} - 698x^{14} - 31x^{13} + 797x^{12} + 83x^{11} - 660x^{10} - 56x^9 + 420x^8 - 199x^6 + 32x^5 + 55x^4 - 20x^3 - 4x^2 + 3x + 1$.

Its discriminant is -563^{11} (confirming that our choice of γ gives an extension unramified at 2). All the Fourier coefficients of this weight 1 modular form can be computed as stated before except the one corresponding to the ramified prime. To compute a_{563} we look at the inertia degree of 563 in $\mathbb{Q}(x_0)$ and since it is 2 it follows that $a_{563} = -1$.

	a_1	a_2	a_3	a_{23}	a_{13}	a_{12}		a_1	a_2	a_3	a_{23}	a_{13}	a_{12}
Q_1	4	563	564	0	−4	0	Q_{18}	36	68	563	0	0	−28
Q_2	3	751	751	−750	−2	−2	Q_{19}	23	196	299	−96	−22	−4
Q_3	39	59	584	−52	−32	−14	Q_{20}	40	119	284	8	20	32
Q_4	39	67	580	−48	−20	−38	Q_{21}	47	100	296	−36	−40	−28
Q_5	44	52	563	0	0	−12	Q_{22}	68	71	299	−62	−16	−36
Q_6	47	48	575	48	2	4	Q_{23}	44	155	207	106	20	16
Q_7	48	51	575	14	48	28	Q_{24}	47	155	192	−44	−8	−46
Q_8	51	52	576	52	20	40	Q_{25}	39	179	231	174	2	30
Q_9	7	323	644	−320	−4	−6	Q_{26}	59	120	191	40	6	36
Q_{10}	12	188	563	0	0	−4	Q_{27}	71	107	191	−26	−14	−58
Q_{11}	11	207	615	−202	−6	−10	Q_{28}	75	92	215	−84	−38	−24
Q_{12}	16	143	567	6	16	12	Q_{29}	63	143	144	36	16	2
Q_{13}	19	119	596	−116	−16	−6	Q_{30}	71	127	160	96	20	6
Q_{14}	23	99	591	−94	−18	−10	Q_{31}	76	119	160	64	60	12
Q_{15}	27	84	584	84	4	8	Q_{32}	64	143	176	−140	−4	−24
Q_{16}	28	84	563	0	0	−20	Q_{33}	103	108	171	104	86	92
Q_{17}	36	63	572	4	36	8							

TABLE 2.3. Coefficients of ternary quadratic forms, level 2^2563.

Let $F_{563}(z)$ denote the weight 1 and level 563 modular form attached to this representation. The form $F_{563}(4z)\Theta_{563}(z)$ is in the Kohnen space with trivial character, whose space has dimension 48 (while the whole space has dimension 143). The form $f_{563}(z) := \frac{1}{2}(F_{563}(4z) + \overline{F}_{563}(4z))\Theta_{563}(z)$, has rational coefficients. The space spanned by the 33 modular forms obtained from ternary quadratic forms with trivial character in the Kohnen space (see table 2.3) and $f_{563}(z)$ is closed for the Hecke operators, and the form

$$F_{563A} = -11\Theta_{Q_1} - 2\Theta_{Q_2} + 44\Theta_{Q_3} - 8\Theta_{Q_4} - 30\Theta_{Q_5} + 16\Theta_{Q_6} - 38\Theta_{Q_7} + 34\Theta_{Q_8} - 2\Theta_{Q_9} + 15\Theta_{Q_{10}} + 2\Theta_{Q_{11}} + 22\Theta_{Q_{12}} - 22\Theta_{Q_{14}} + 4\Theta_{Q_{15}} - 7\Theta_{Q_{16}} - 18\Theta_{Q_{17}} + 29\Theta_{Q_{18}} + 26\Theta_{Q_{19}} + 24\Theta_{Q_{20}} + 14\Theta_{Q_{21}} - 6\Theta_{Q_{22}} + 28\Theta_{Q_{23}} - 28\Theta_{Q_{24}} - 34\Theta_{Q_{25}} - 46\Theta_{Q_{26}} + 22\Theta_{Q_{27}} + 8\Theta_{Q_{28}} - 14\Theta_{Q_{29}} + 20\Theta_{Q_{30}} - 42\Theta_{Q_{31}} + 13f_{563}$$

is an eigenform for the Hecke operators mapping via Shimura to the modular form (attached to the elliptic curve) $563A$. The first 50 coefficients of its Fourier expansion are:

$$-2q^3+2q^4-2q^7+2q^{11}-2q^{16}+4q^{23}+2q^{27}+4q^{28}+2q^{39}-2q^{40}+2q^{47}+4q^{48}+O(q^{51}).$$

2.1.3. *Case* 643. This case is similar to the previous one. The elliptic curve $643A$ given by the equation $y^2 + xy = x^3 - 4x + 3$ has rank 2 and the points $[1, 0]$ and $[2, 1]$ generate the rational points. Their obstruction is non-trivial. They sum is the point $[-1, 3]$ which do have trivial obstruction. It corresponds to the polynomial $P = x^4 - x^3 - 2x + 1$, whose field has discriminant -643. A solution to the embedding problem is given by the element:

$$643\gamma = (-123456x_2^2 + 36008x_2 - 1376)x_1^3 + (79732x_2^2 - 70820x_2 + 21952)x_1^2 + (-9092x_2^2 - 101504x_2 + 51440)x_1 - 75964x_2^3 + 176272x_2^2 + 43724x_2 - 25540,$$

	a_1	a_2	a_3	a_{23}	a_{13}	a_{12}		a_1	a_2	a_3	a_{23}	a_{13}	a_{12}
Q_1	4	643	644	0	−4	0	Q_{16}	55	104	328	52	12	48
Q_2	7	368	735	368	2	4	Q_{17}	63	88	327	44	10	40
Q_3	16	163	647	6	16	12	Q_{18}	63	95	335	62	46	58
Q_4	23	112	671	112	2	4	Q_{19}	60	131	216	4	20	24
Q_5	28	92	643	0	0	−4	Q_{20}	51	152	256	−148	−28	−12
Q_6	31	83	671	−82	−30	−2	Q_{21}	83	95	220	36	32	26
Q_7	15	343	344	172	8	2	Q_{22}	92	95	231	74	52	64
Q_8	24	215	323	2	12	8	Q_{23}	64	167	204	−152	−28	−40
Q_9	23	228	339	−104	−18	−20	Q_{24}	96	136	139	60	44	28
Q_{10}	31	168	332	84	4	16	Q_{25}	96	135	156	−44	−92	−20
Q_{11}	40	135	324	8	20	32	Q_{26}	111	116	143	−68	−42	−8
Q_{12}	39	132	339	−64	−38	−4	Q_{27}	116	136	143	76	68	108
Q_{13}	39	135	331	−62	−14	−22	Q_{28}	104	124	167	104	88	12
Q_{14}	36	143	359	−142	−16	−4	Q_{29}	111	131	143	−54	−42	−82
Q_{15}	55	95	332	52	32	18	Q_{30}	104	144	167	−60	−88	−92

TABLE 2.4. Coefficients of ternary quadratic forms, level $2^2 643$.

where x_1 and x_2 are roots of P. The field $\mathbb{Q}(\sqrt{\gamma})$ is given by $\mathbb{Q}(x_0)$ where x_0 is a root of the polynomial:
$x^{24} - 5x^{23} + 11x^{22} - 8x^{21} - 10x^{20} + 23x^{19} + 9x^{18} - 86x^{17} + 171x^{16} - 121x^{15} - 212x^{14} + 636x^{13} - 504x^{12} - 156x^{11} + 766x^{10} - 1116x^9 + 1364x^8 - 1100x^7 + 697x^6 - 426x^5 + 227x^4 - 37x^3 + 25x^2 - 29x + 5$.

This field has discriminant -643^{11} hence the weight 1 modular form has level 643 and character $\left(\frac{-643}{\cdot}\right)$. Looking at the inertia, its Fourier coefficient $a_{643} = -1$.

Let $F_{643}(z)$ denote the weight 1 and level 643 modular form attached to this representation. The form $F_{643}(4z)\Theta_{643}(z)$ is in the Kohnen space with trivial character, whose space has dimension 54 (while the whole space has dimension 163). The form $f_{643}(z) := \frac{1}{2}(F_{643}(4z) + \overline{F}_{643}(4z))\Theta_{643}(z)$, has rational coefficients. The space spanned by the 30 modular forms obtained from ternary quadratic forms with trivial character in the Kohnen space (see table 2.4) and $f_{643}(z)$ is closed for the Hecke operators, and the form

$$F_{643A} = -3\Theta_{Q_1} - \Theta_{Q_2} + 6\Theta_{Q_3} - 2\Theta_{Q_4} - \Theta_{Q_5} + 8\Theta_{Q_6} + \Theta_{Q_7} + 7\Theta_{Q_8} + 3\Theta_{Q_9} - 7\Theta_{Q_{10}} + \Theta_{Q_{11}} - 4\Theta_{Q_{12}} + 4\Theta_{Q_{13}} + 6\Theta_{Q_{14}} + 12\Theta_{Q_{15}} - 11\Theta_{Q_{16}} + 6\Theta_{Q_{17}} - 4\Theta_{Q_{18}} - 3\Theta_{Q_{19}} - \Theta_{Q_{20}} - 9\Theta_{Q_{21}} + 2\Theta_{Q_{22}} + 2\Theta_{Q_{24}} - 8\Theta_{Q_{25}} - 12\Theta_{Q_{27}} + 2\Theta_{Q_{28}} + 6\Theta_{Q_{29}} + 4f_{643}$$

is an eigenform for the Hecke operators mapping via Shimura to the modular form (attached to the elliptic curve) 643A. The first 50 coefficients of its Fourier expansion are:

$$q^4 - q^7 + q^{15} - q^{16} + q^{23} - q^{24} + 2q^{28} + q^{31} - q^{36} + q^{40} + O(q^{51}).$$

REFERENCES

[1] Pilar Bayer and Gerhard Frey, *Galois representations of octahedral type and 2-coverings of elliptic curves*, Math. Z. **207** (1991), no. 3, 395–408.

[2] Michael Bungert, *Construction of a cuspform of weight* 3/2, Arch. Math. (Basel) **60** (1993), no. 6, 530–534.

[3] H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 69–78. Lecture Notes in Math., Vol. 627.
[4] Teresa Crespo, *Explicit construction of $\tilde{A}_n$ type fields*, J. Algebra **127** (1989), no. 2, 452–461.
[5] ———, *Explicit construction of $2S_n$ Galois extensions*, J. Algebra **129** (1990), no. 2, 312–319.
[6] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975).
[7] Larry Dornhoff, *Group representation theory. Part A: Ordinary representation theory*, Marcel Dekker Inc., New York, 1971.
[8] Benedict H. Gross, *Heights and the special values of L-series*, Number theory (Montreal, Que., 1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187.
[9] A. Jehanne, *Realization over Q of the groups $\tilde{A}_5$ and $\hat{A}_5$*, J. Number Theory **89** (2001), no. 2, 340–368.
[10] Arnaud Jehanne, *Sur les extensions de $\mathbf{Q}$ à groupe de Galois S_4 et $\widetilde{S}_4$*, Acta Arith. **69** (1995), no. 3, 259–276.
[11] Winfried Kohnen, *Newforms of half-integral weight*, J. Reine Angew. Math. **333** (1982), 32–72.
[12] J. Larry Lehman, *Levels of positive definite ternary quadratic forms*, Math. Comp. **58** (1992), no. 197, 399–417, S17–S22.
[13] *PARI/GP, version* 2.2.8, `http://pari.math.u-bordeaux.fr/`, 2004.
[14] Anna Rio, *Dyadic exercises for octahedral extensions ii*, Submitted (2005).
[15] Jean-Pierre Serre, *L'invariant de Witt de la forme* $\mathrm{Tr}(x^2)$, Comment. Math. Helv. **59** (1984), no. 4, 651–676.
[16] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.
[17] Gonzalo Tornaría, *Tables of ternary quadratic forms (part of computational number theory)*, `http://www.ma.utexas.edu/users/tornaria/cnt/`, 2004.

Departamento de Matemática, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria. C.P:1428, Buenos Aires, Argentina
E-mail address: `apacetti@dm.uba.ar`