SHIMURA CORRESPONDENCE FOR LEVEL p^2 AND THE CENTRAL VALUES OF L-SERIES

ARIEL PACETTI AND GONZALO TORNARÍA

ABSTRACT. Given a weight 2 and level p^2 modular form f, we construct two weight $3/2$ modular forms (possibly zero) of level $4p^2$ and non trivial character mapping to f via the Shimura correspondence. Then we relate the coefficients of the constructed forms to the central value of the L-series of certain imaginary quadratic twists of f. Furthermore, we give a general framework for our construction that applies to any order in definite quaternion algebras, with which one can, in principle, construct weight $3/2$ modular forms of any level, provided one knows how to compute ideal classes representatives.

INTRODUCTION

The theory of modular forms of half integral weight was developed by Shimura in [Sh]. There he defined a map known as the "Shimura correspondence" that associates to a half integral weight modular form (eigenvalue of the Hecke operators) an integral weight modular form, and raised the question of the meaning of their Fourier coefficients. Later Waldspurger related these Fourier coefficients to the central values of twisted L-series for the integral weight modular form (see [Wa]). In [Gr] Gross gave (under some restrictions) an explicit method to construct, given a weight 2 and level p modular form f, a weight $3/2$ modular form (of level $4p$ and trivial character) mapping to f via the Shimura correspondence.

In the first section of this paper we revise the ideal theory of quaternion algebras and Hecke operators acting on them. We then generalize the correspondence of Gross to any order in a definite quaternion algebra, obtaining a Hecke linear

2000 *Mathematics Subject Classification.* Primary: 11F37; Secondary: 11F67.

Key words and phrases. Shimura Correspondence, L-series, Real Twists.

The first author was supported by a CONICET grant.

The authors would like to thank the "Institut Henri Poincaré" and the organizers of the trimestre "Méthodes explicites en théorie des nombres," where part of this work was written.

correspondence from the quaternary theta series associated to ideals to the ternary theta series associated to their right orders.

In the second section we construct certain orders "of level p^2" for which the correspondence yields modular forms of level $4p^2$ and show how to construct ideal classes representatives from representatives for the maximal order. In this way we improve the speed of the algorithm; moreover, the matrices that we need to diagonalize are much smaller. For instance, we have computed some of the weight $3/2$ modular forms for p up to 500, i.e. corresponding to modular forms of weight 2 and level p^2 up to 250000 (see [To]). In the third section we give an example of how to construct these ideals for the case $p = 7$.

In the fourth section we explain the relation between the Fourier coefficients of weight $3/2$ modular forms that are obtained using the methods of §1 and §2, and the central values of the L-series of the corresponding modular form f of weight 2. We conjecture a precise formula for this relation, similar to Gross' formula for the level p case. Furthermore, we show how our conjecture implies an easy criteria to decide when the constructed modular forms are zero: essentially when $L(f, 1) = 0$.

Examples of our method, as well as an application to computing the central values for *real* quadratic twists, were presented at the workshop "Special Week on Ranks of Elliptic Curves and Random Matrix Theory" held at the Isaac Newton Institute, and can be found at [Pa-To].

1. Orders in quaternion algebras and Shimura correspondence

Let B be a definite quaternion algebra over $\mathbb{Q}$. For $x \in B$ we denote $\mathcal{N}\,x$ and $\operatorname{Tr} x$ the *reduced norm* and *reduced trace* of x, respectively. The *norm* of a lattice $\mathfrak{a}$ is defined as $\mathcal{N}\,\mathfrak{a} := \gcd\{\mathcal{N}\,x \,:\, x \in \mathfrak{a}\}$. We equip $\mathfrak{a}$ with the quadratic form $\mathcal{N}_{\mathfrak{a}}(x) := \mathcal{N}\,x / \mathcal{N}\,\mathfrak{a}$, which is primitive; its determinant is a square, and we denote its positive square root by $\mathrm{D}(\mathfrak{a})$. In particular, when $R \subseteq B$ is an order, $\mathrm{D}(R)$ is its *reduced discriminant*. The subscript p will denote localization at p, namely $\mathfrak{a}_p := \mathfrak{a} \otimes \mathbb{Z}_p$.

If R is an order in B, we let $\widetilde{\mathcal{I}}(R)$ be the set of *left R-ideals*, i.e. the set of lattices $\mathfrak{a} \subseteq B$ such that $\mathfrak{a}_p = R_p x_p$ for every prime p, with $x_p \in B_p^\times$. Two left ideals $\mathfrak{a}, \mathfrak{b} \in \widetilde{\mathcal{I}}(R)$ are in the same class if $\mathfrak{a} = \mathfrak{b}x$, with $x \in B^\times$; we write $[\mathfrak{a}]$ for the class of $\mathfrak{a}$. The set of all left R-ideal classes, which we denote by $\mathcal{I}(R)$, is known to be finite.

1.1. **The height pairing.** Let $\mathcal{M}(R)$ be the free $\mathbb{Z}$-module with basis $\mathcal{I}(R)$. We define *the height paring* by

$$\langle [\mathfrak{a}], [\mathfrak{b}] \rangle := \tfrac{1}{2}\#\{x \in B^\times \ : \ \mathfrak{a}x = \mathfrak{b}\} = \begin{cases} \tfrac{1}{2}\#R_r(\mathfrak{a})^\times & \text{if } [\mathfrak{a}] = [\mathfrak{b}], \\ 0 & \text{otherwise,} \end{cases}$$

where $R_r(\mathfrak{a})$ is the right order of $\mathfrak{a}$, namely

$$R_r(\mathfrak{a}) := \{x \in B \ : \ \mathfrak{a}x \subseteq \mathfrak{a}\}.$$

The height pairing induces an inner product on $\mathcal{M}_\mathbb{R}(R) := \mathcal{M}(R) \otimes \mathbb{R}$; note that $\mathcal{I}(R)$ is an orthogonal basis of this space.

The *dual lattice* $\mathcal{M}^\vee(R) := \{\mathbf{v} \in \mathcal{M}_\mathbb{R}(R) \ : \ \langle \mathbf{v}, \mathcal{M}(R)\rangle \subseteq \mathbb{Z}\}$ is spanned by the *dual basis*

$$\mathcal{I}^\vee(R) := \{[\mathfrak{a}]^\vee \ : \ [\mathfrak{a}] \in \mathcal{I}(R)\},$$

where $[\mathfrak{a}]^\vee := \frac{1}{\langle \mathfrak{a}, \mathfrak{a}\rangle}[\mathfrak{a}]$. We will identify $\mathcal{M}^\vee(R)$ with $\mathrm{Hom}(\mathcal{M}(R), \mathbb{Z})$; indeed, a vector $\mathbf{v} \in \mathcal{M}^\vee(R)$ defines a map $\langle \mathbf{v}, \cdot \rangle : \mathcal{M}(R) \to \mathbb{Z}$, and conversely, a map $f : \mathcal{M}(R) \to \mathbb{Z}$ determines a vector

$$\sum_{[\mathfrak{a}] \in \mathcal{I}(R)} f([\mathfrak{a}])[\mathfrak{a}]^\vee \in \mathcal{M}^\vee(R).$$

For instance, the map $\deg : \mathcal{M}(R) \to \mathbb{Z}$ defined by $\deg([\mathfrak{a}]) := 1$ for $\mathfrak{a} \in \widetilde{\mathcal{I}}(R)$, corresponds to the vector

$$\mathbf{e}_0 := \sum_{[\mathfrak{a}] \in \mathcal{I}(R)} [\mathfrak{a}]^\vee \in \mathcal{M}^\vee(R).$$

1.2. **Hecke operators.** Let $\mathfrak{a} \in \widetilde{\mathcal{I}}(R)$, and $m \geq 1$ an integer. We set

$$\mathcal{T}_m(\mathfrak{a}) := \Big\{\mathfrak{b} \in \widetilde{\mathcal{I}}(R) \ : \ \mathfrak{b} \subseteq \mathfrak{a}, \quad \mathcal{N}\,\mathfrak{b} = m\, \mathcal{N}\,\mathfrak{a}\Big\}.$$

The *Hecke operators* $t_m : \mathcal{M}(R) \to \mathcal{M}(R)$ are then defined by

$$t_m[\mathfrak{a}] := \sum_{\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})} [\mathfrak{b}]$$

for $m \geq 1$ and $[\mathfrak{a}] \in \mathcal{I}(R)$. In addition, we define $t_0 : \mathcal{M}(R) \to \mathcal{M}(R)$ by $t_0[\mathfrak{a}] := \frac{1}{2}\mathbf{e}_0$.

Lemma 1.1.

$$\mathcal{T}_m(\mathfrak{a}) = \Big\{ \mathfrak{b} \in \widetilde{\mathcal{I}}(R) \;:\; \mathfrak{b} \subseteq \mathfrak{a}, \quad [\mathfrak{a} : \mathfrak{b}] = m^2 \Big\}.$$

Moreover, $m\mathfrak{a} \subseteq \mathfrak{b}$ *for every* $\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})$*; in particular,* $\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})$ *if and only if* $m\mathfrak{a} \in \mathcal{T}_m(\mathfrak{b})$.

Proof. Let $\mathfrak{b} \in \widetilde{\mathcal{I}}(R)$ such that $\mathfrak{b} \subseteq \mathfrak{a}$. Locally, since $\mathfrak{a}_p$ and $\mathfrak{b}_p$ are principal, there is some $x_p \in R_r(\mathfrak{a}_p)$ such that $\mathfrak{b}_p = \mathfrak{a}_p x_p$. Then

$$[\mathfrak{a}_p : \mathfrak{b}_p] = (\mathcal{N}\, x_p)^2 = (\mathcal{N}\, \mathfrak{b}_p / \mathcal{N}\, \mathfrak{a}_p)^2,$$

which proves the first statement. For the second, note that $\overline{x_p} \in R_r(\mathfrak{a}_p)$, and therefore $m\mathfrak{a}_p = \mathfrak{a}_p \overline{x_p} x_p \subseteq \mathfrak{a}_p x_p = \mathfrak{b}_p$. □

Lemma 1.2. *Fix a prime* p *and let* $x_p \in M_{2\times 2}(\mathbb{Z}_p)$*. The set*

$$\big\{ y_p \in M_{2\times 2}(\mathbb{Z}_p) \;:\; \det y_p = p, \quad x_p y_p^{-1} \in M_{2\times 2}(\mathbb{Z}_p) \big\}$$

is invariant under left multiplication by unimodular matrices, and the number of orbits for this action is 0 *if* $p \nmid \det x_p$*,* $1 + p$ *if* $x_p \in pM_{2\times 2}(\mathbb{Z}_p)$*, and* 1 *otherwise.*

Proof. A well known set of representatives of $\{y_p \in M_{2\times 2}(\mathbb{Z}_p) \;:\; \det y_p = p\}$ modulo left multiplication by unimodular matrices is

$$\big\{ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & p \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} 1 & p-1 \\ 0 & p \end{smallmatrix}\right) \big\}.$$

An easy calculation shows that

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \in M_{2\times 2}(\mathbb{Z}_p) \iff a \equiv c \equiv 0 \pmod{p},$$
$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right)^{-1} \in M_{2\times 2}(\mathbb{Z}_p) \iff b - ai \equiv d - ci \equiv 0 \pmod{p},$$

from which the statement follows. □

Proposition 1.3. *the Hecke operators have the following properties:*

1. t_m *is self-adjoint.*
2. *If* $(m, m') = 1$*, then* $t_{mm'} = t_m t'_m$.
3. *If* $p \nmid \mathrm{D}(R)$ *is a prime, then* $t_{p^{k+2}} = t_{p^{k+1}} t_p - p\, t_{p^k}$.

Proof. (1) Note that

$$\begin{aligned}\langle[\mathfrak{a}], t_m[\mathfrak{b}]\rangle &= \sum_{\mathfrak{c}\in\mathcal{T}_m(\mathfrak{b})} \langle[\mathfrak{a}],[\mathfrak{c}]\rangle \\ &= \sum_{\mathfrak{c}\in\mathcal{T}_m(\mathfrak{b})} \tfrac{1}{2}\#\big\{x\in B^\times \ :\ \mathfrak{a}x=\mathfrak{c}\big\} \\ &= \tfrac{1}{2}\#\big\{x\in B^\times \ :\ \mathfrak{a}x\in\mathcal{T}_m(\mathfrak{b})\big\}.\end{aligned}$$

By the last part of Lemma 1.1, this equals

$$\begin{aligned}&= \tfrac{1}{2}\#\big\{x\in B^\times \ :\ m\mathfrak{b}\in\mathcal{T}_m(\mathfrak{a}x)\big\} \\ &= \tfrac{1}{2}\#\big\{x\in B^\times \ :\ m\mathfrak{b}x^{-1}\in\mathcal{T}_m(\mathfrak{a})\big\},\end{aligned}$$

and as before the latter is $\langle t_m[\mathfrak{a}],[\mathfrak{b}]\rangle$.

(2) For any $\mathfrak{c}\in\mathcal{T}_{mm'}(\mathfrak{a})$, there is a unique $\mathfrak{b}\in\mathcal{T}_{m'}(\mathfrak{a})$ such that $\mathfrak{c}\in\mathcal{T}_m(\mathfrak{b})$, namely $\mathfrak{b}_p=\mathfrak{a}_p$ for $p\nmid m'$, and $\mathfrak{b}_p=\mathfrak{c}_p$ for $p\nmid m$.

(3) Let $\mathfrak{c}\in\mathcal{T}_{p^{k+2}}(\mathfrak{a})$. Locally, $\mathfrak{c}_p=\mathfrak{a}_p x_p$ for some $x_p\in R_r(\mathfrak{a}_p)$ with $\mathcal{N}\,x_p=p^{k+2}$. Since $p\nmid D$, we can identify $R_r(\mathfrak{a}_p)$ with $M_{2\times2}(\mathbb{Z}_p)$, and use Lemma 1.2 to count the number of $\mathfrak{b}\in\mathcal{T}_p(\mathfrak{a})$ such that $\mathfrak{c}\in\mathcal{T}_{p^{k+1}}(\mathfrak{b})$. Indeed, any $\mathfrak{b}\in\mathcal{T}_p(\mathfrak{a})$ will be given by $\mathfrak{b}_p=\mathfrak{a}_p y_p$, where $y_p\in R_r(\mathfrak{a}_p)$ is such that $\mathcal{N}\,y_p=p$, and the condition $\mathfrak{c}\in\mathcal{T}_{p^{k+1}}(\mathfrak{b})$ is equivalent to $x_p y_p^{-1}\in R_r(\mathfrak{a}_p)$.

Thus, if $x_p\notin pR_r(\mathfrak{a}_p)$ there is a unique such $\mathfrak{b}$, while for $x_p\in pR_r(\mathfrak{a}_p)$ there are $1+p$ of them. But $x_p\in pR_r(\mathfrak{a}_p)$ if and only if $\mathfrak{c}=p\mathfrak{c}'$ for some $\mathfrak{c}'\in\mathcal{T}_{p^k}(\mathfrak{a})$, and the formula follows since $[\mathfrak{c}]=[\mathfrak{c}']$.

□

It follows from this Proposition that the Hecke operators t_m with $(m,\mathrm{D}(R))=1$ generate a commutative ring $\mathbb{T}^0$ of self-adjoint operators; by the spectral theorem $\mathcal{M}_{\mathbb{R}}(R)$ has an orthogonal basis of eigenvectors for $\mathbb{T}^0$.

1.3. **Modular forms of weight 2.** The following construction will show that there is a correspondence between $\mathcal{M}(R)$ and modular forms of weight 2 and level

$$\mathrm{L}(R) := \mathcal{N}(R^*)^{-1},$$

where $R^* := \{x\in B\ :\ \mathrm{Tr}(xR)\subseteq\mathbb{Z}\}$. Indeed, we will exhibit a $\mathbb{T}^0$-linear map

$$\phi : \mathcal{M}^\vee(R)\otimes_{\mathbb{T}^0}\mathcal{M}(R)\longrightarrow M_2(\mathrm{L}(R)),$$

where $M_2(\mathrm{L})$ is the space of modular forms of weight 2, level $\mathrm{L} = \mathrm{L}(R)$ and trivial character, with t_n acting on this space by the Hecke operator $T(n)$. We remark that this map is *not* in general surjective.

Definition. Let $\mathbf{v} \in \mathcal{M}^\vee(R)$ and $\mathbf{w} \in \mathcal{M}(R)$. We set

$$\phi(\mathbf{v}, \mathbf{w}) := \sum_{m \geq 0} \langle \mathbf{v}, t_m \mathbf{w} \rangle q^m = \frac{\deg \mathbf{v} \cdot \deg \mathbf{w}}{2} + \sum_{m \geq 1} \langle \mathbf{v}, t_m \mathbf{w} \rangle q^m.$$

Proposition 1.4. *$\phi(\mathbf{v}, \mathbf{w})$ is a weight 2 modular form of level $\mathrm{L}(R)$ and trivial character. Moreover,*

$$\phi(\mathbf{v}, \mathbf{w})_{|T(n)} = \phi(t_n \mathbf{v}, \mathbf{w}) = \phi(\mathbf{v}, t_n \mathbf{w}),$$

for any $n \geq 1$ such that $(n, \mathrm{D}(R)) = 1$. In particular, for any eigenvector $\mathbf{v} \in \mathcal{M}(R)$ for $\mathbb{T}^0$, the modular form $\phi(\mathbf{v}, \mathbf{v})$ is an eigenform for $\mathbb{T}^0$.

Proof. Since

$$\langle [\mathfrak{a}], t_m[\mathfrak{b}] \rangle = \tfrac{1}{2} \#\{x \in B \ : \ \mathfrak{a}x \in \mathcal{T}_m(\mathfrak{b})\} = \tfrac{1}{2} \#\big\{x \in \mathfrak{a}^{-1}\mathfrak{b} \ : \ \mathcal{N} x = m \, \mathcal{N}(\mathfrak{a}^{-1}\mathfrak{b})\big\},$$

and $\langle [\mathfrak{a}], t_0[\mathfrak{b}] \rangle = \frac{1}{2}$, it follows that

$$\phi([\mathfrak{a}], [\mathfrak{b}]) = \frac{1}{2} \sum_{x \in \mathfrak{c}} q^{\mathcal{N}_\mathfrak{c}(x)},$$

is the theta series of the lattice $\mathfrak{c} = \mathfrak{a}^{-1}\mathfrak{b}$, with the quadratic form $\mathcal{N}_\mathfrak{c}$. Its discriminant is a square; we claim that its level is $\mathrm{L}(R)$. Indeed, locally $\mathfrak{c}_p = x_p R_p y_p$ for some $x_p, y_p \in B_p^\times$ (where $\mathfrak{a}_p = R_p x_p^{-1}$, $\mathfrak{b}_p = R_p y_p$,) and $\mathcal{N}_\mathfrak{c}(x_p a_p y_p) = u_p \, \mathcal{N}(a_p)$ for $a_p \in R_p$, where $u_p = \mathcal{N}_\mathfrak{c}(x_p y_p)$ is a p-adic unit. Therefore, the level of $\mathcal{N}_\mathfrak{c}$ in $\mathfrak{c}$ is equal to the level of $\mathcal{N}$ in R. The latter is, by definition, the smallest positive integer u such that $u \, \mathcal{N}(R^*) \subseteq \mathbb{Z}$, since the matrix of $\mathcal{N}$ on a given basis of R is the inverse of the matrix of $\mathcal{N}$ on the dual basis of R^*; but this is just $\mathrm{L}(R) = \mathcal{N}(R^*)^{-1}$, as claimed.

For the second statement, in view of Proposition 1.3, it is enough to prove the identity for $n = p \nmid \mathrm{D}(R)$ a prime. But

$$\phi(\mathbf{v}, \mathbf{w})_{|T(p)} = \sum_{m \geq 0} \Big(\langle \mathbf{v}, t_{mp} \mathbf{w} \rangle + p \big\langle \mathbf{v}, t_{m/p} \mathbf{w} \big\rangle \Big) \, q^m,$$

and

$$\phi(\mathbf{v}, t_p\mathbf{w}) = \sum_{m\geq 0} \langle \mathbf{v}, t_m t_p \mathbf{w}\rangle\, q^m,$$

and the result follows since Proposition 1.3 implies that $t_{mp} + p\, t_{m/p} = t_m\, t_p$. $\square$

1.4. **Special points.** Let $\mathfrak{a} \in \widetilde{\mathfrak{I}}(R)$, and $-d \leq 0$ an integer, $-d \equiv 0, 1 \pmod 4$. The *special points* of discriminant $-d$ for $\mathfrak{a}$ are

$$\widetilde{\mathcal{A}}_d(\mathfrak{a}) := \{x \in R_r(\mathfrak{a}) \;:\; \Delta\, x = -d\},$$

where $\Delta\, x := (\operatorname{Tr} x)^2 - 4\,\mathcal{N}\, x$ is the discriminant of the characteristic polynomial of x. These sets are stable under translations by integers. For each d the set of orbits, which will be denoted by $\mathcal{A}_d(\mathfrak{a})$, is finite, and it is in bijection with any of the sets

$$\widetilde{\mathcal{A}}_{d,s}(\mathfrak{a}) := \Big\{x \in \widetilde{\mathcal{A}}_d(\mathfrak{a}) \;:\; \operatorname{Tr} x = s\Big\} = \Big\{x \in R_r(\mathfrak{a}) \;:\; \operatorname{Tr} x = s, \quad \mathcal{N}\, x = \frac{s^2+d}{4}\Big\},$$

where s is an arbitrary integer subject to the condition $s \equiv d \pmod 2$.

The maps $a_d : \mathcal{M}(R) \to \mathbb{Z}$ are then defined by $a_d([\mathfrak{a}]) = \#\mathcal{A}_d(\mathfrak{a})$. When $-d \not\equiv 0, 1 \pmod 4$ we set $a_d([\mathfrak{a}]) = 0$. As before, we identify this maps with vectors

$$\mathbf{e}_d := \sum_{[\mathfrak{a}]\in\mathfrak{I}(R)} a_d([\mathfrak{a}])[\mathfrak{a}]^\vee \in \mathcal{M}^\vee(R).$$

This is consistent with the previous definition of $\mathbf{e}_0$.

Lemma 1.5. *Fix a prime p and let $x_p \in M_{2\times 2}(\mathbb{Z}_p)$. The set*

$$\big\{y_p \in M_{2\times 2}(\mathbb{Z}_p) \;:\; \det y_p = p, \quad y_p x_p y_p^{-1} \in M_{2\times 2}(\mathbb{Z}_p)\big\}$$

is invariant under left multiplication by unimodular matrices, and the number of orbits for this action is $1+p$ if $x_p \in \mathbb{Z}_p + pM_{2\times 2}(\mathbb{Z}_p)$, and $1 + \left(\frac{\Delta\, x_p}{p}\right)$ otherwise.

Proof. As in the proof of Lemma 1.2, the statement follows from

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)^{-1} \in M_{2\times 2}(\mathbb{Z}_p) \iff c \equiv 0 \pmod p,$$

$$\left(\begin{smallmatrix} 1 & i \\ 0 & p\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & i \\ 0 & p\end{smallmatrix}\right)^{-1} \in M_{2\times 2}(\mathbb{Z}_p) \iff ci^2 + (a-d)i - b \equiv 0 \pmod p,$$

since the discriminant of the quadratic equation above is $\Delta\, x_p$. $\square$

Lemma 1.6. *Let $x \in \widetilde{\mathcal{A}}_d(\mathfrak{a})$. Then*

$$\#\Big\{\mathfrak{b} \in \mathcal{T}_p(\mathfrak{a}) \;:\; x \in \widetilde{\mathcal{A}}_d(\mathfrak{b})\Big\} = \begin{cases} 1+p & \text{if } x \in \mathbb{Z} + pR_r(\mathfrak{a}), \\ 1 + \left(\frac{-d}{p}\right) & \text{if } x \notin \mathbb{Z} + pR_r(\mathfrak{a}), \end{cases}$$

for any prime $p \nmid \mathrm{D}(R)$.

Proof. If $b \in \mathcal{T}_p(\mathfrak{a})$ we have $\mathfrak{b}$ and $\mathfrak{a}$ equal outside p, and $\mathfrak{b}_p = \mathfrak{a}_p y_p$ for some $y_p \in R_r(\mathfrak{a}_p)$ with $\mathcal{N}\, y_p = p$; two such y_p give the same $\mathfrak{b}$ if and only if they are in the same orbit under left multiplication by units of $R_r(\mathfrak{a}_p)$. Since $p \nmid \mathrm{D}(R)$, we can identify $R_r(\mathfrak{a}_p)$ with $M_{2\times 2}(\mathbb{Z}_p)$, and Lemma 1.5 proves the claim. □

Proposition 1.7. *For any prime $p \nmid \mathrm{D}(R)$ we have*

$$t_p\, \mathbf{e}_d = \mathbf{e}_{dp^2} + \left(\frac{-d}{p}\right) \mathbf{e}_d + p\, \mathbf{e}_{d/p^2}.$$

Remark. Compare this with the formula for the action of the Hecke operators for weight $^3\!/_2$ in terms of Fourier coefficients [Sh, Theorem 1.7].

Proof. For an arbitrary $[\mathfrak{a}] \in \mathcal{I}(R)$, the left hand side evaluated at $[\mathfrak{a}]$ is

$$\langle t_p\, \mathbf{e}_d, [\mathfrak{a}]\rangle = \langle \mathbf{e}_d, t_p\, [\mathfrak{a}]\rangle = \sum_{b \in \mathcal{T}_p(\mathfrak{a})} \langle \mathbf{e}_d, [\mathfrak{b}]\rangle = \sum_{b \in \mathcal{T}_p(\mathfrak{a})} a_d([\mathfrak{b}]),$$

which just counts the number of pairs $(\mathfrak{b}, x)$ such that $\mathfrak{b} \in \mathcal{T}_p(\mathfrak{a})$ and $x \in \mathcal{A}_d(\mathfrak{b})$. Since $p\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, it is clear that $x \in R_r(\mathfrak{b})$ implies that $px \in R_r(\mathfrak{a})$. We count the number of possible pairs in each of three disjoint cases for x:

(1) $\#\{(\mathfrak{b}, x) \;:\; x \in \mathbb{Z} + pR_r(\mathfrak{a})\} = (1+p)\, a_{d/p^2}([\mathfrak{a}])$.
There are $a_{d/p^2}([\mathfrak{a}])$ such x, and the count follows from Lemma 1.6.

(2) $\#\{(\mathfrak{b}, x) \;:\; x \in R_r(\mathfrak{a}), \quad x \notin \mathbb{Z} + pR_r(\mathfrak{a})\}$
$$= \left(1 + \left(\frac{-d}{p}\right)\right) \left(a_d([\mathfrak{a}]) - a_{d/p^2}([\mathfrak{a}])\right).$$
There are $a_d([\mathfrak{a}]) - a_{d/p^2}([\mathfrak{a}])$ such x, and the count follows from Lemma 1.6.

(3) $\#\{(\mathfrak{b}, x) \;:\; x \notin R_r(\mathfrak{a})\} = a_{dp^2}([\mathfrak{a}]) - a_d([\mathfrak{a}])$.
There are $a_{dp^2}([\mathfrak{a}]) - a_d([\mathfrak{a}])$ such x; the count follows now from Lemma 1.6 applied to px, since $\Delta(px) = -dp^2$ and $\left(\frac{-dp^2}{p}\right) = 0$.

Adding up these expressions we get

$$\langle t_p\, \mathbf{e}_d, [\mathfrak{a}]\rangle = a_{dp^2}([\mathfrak{a}])) + \left(\frac{-d}{p}\right) a_d([\mathfrak{a}]) + \left(p - \left(\frac{-d}{p}\right)\right) a_{d/p^2}([\mathfrak{a}]),$$

and the statement now follows using the fact that $\left(\frac{-d}{p}\right) a_{d/p^2}([\mathfrak{a}]) = 0$. □

1.5. **Modular forms of weight $^3\!/_2$.** We will use the special points to construct modular forms of weight $^3\!/_2$. Let

$$\Omega = \Omega(R) := \gcd\{\Delta\, x \;:\; x \in R\}.$$

Note that $\Delta\, x \leq 0$, and $\Delta\, x = 0$ if and only if $x \in \mathbb{Q}$; thus

$$\Delta_R(x) := -\,\Delta\, x / \,\Omega(R)$$

defines a *primitive* positive definite ternary quadratic form on the lattice $R/\mathbb{Z}$, which we denote Q_R. More generally, if $\mathfrak{a} \in \widetilde{\mathfrak{I}}(R)$, it defines a positive definite ternary quadratic form on the lattice $R_r(\mathfrak{a})/\mathbb{Z}$, which we denote $Q_{\mathfrak{a}}$.

Proposition 1.8. *$Q_{\mathfrak{a}}$ is in the same genus as Q_R. In particular, $Q_{\mathfrak{a}}$ is integral and primitive. Conversely, any ternary quadratic form in the genus of Q_R will be equivalent to $Q_{\mathfrak{a}}$ for some $\mathfrak{a} \in \widetilde{\mathfrak{I}}(R)$.*

Proof. The claim is that $R_r(\mathfrak{a})/\mathbb{Z}$ is locally isometric to $R/\mathbb{Z}$. Indeed, $\mathfrak{a}_p = R_p x_p$ for some $x_p \in B_p^\times$, and thus $R_r(\mathfrak{a}_p) = x_p^{-1} R_p x_p$, inducing an isometry between $R_p/\mathbb{Z}_p$ and $R_r(\mathfrak{a}_p)/\mathbb{Z}_p$.

Conversely, let Q be a quadratic form in the genus of Q_R. By the correspondence between ternary quadratic forms and orders in quaternion algebras (see [Ll]), there is an order R' in B such that $Q_{R'} \sim Q$. Moreover, since Q and Q_R are in the same genus, it follows that R' and R are locally conjugate, i.e. $R'_p = x_p^{-1} R_p x_p$. Thus the right order of the R-ideal $\mathfrak{a}$ given by $\mathfrak{a}_p = R_p x_p$ will be R', and so $Q_{\mathfrak{a}} \sim Q$. □

Corollary 1.9. $\mathbf{e}_d = 0$ *unless* $d \equiv 0 \pmod{\Omega}$.

Proof. Note that $a_d([\mathfrak{a}])$ equals the number of representations of $d/\,\Omega$ by $Q_{\mathfrak{a}}$ which, being integral, represents only integers. Thus $\Omega \nmid d$ implies $a_d([\mathfrak{a}]) = 0$ for any $\mathfrak{a}$, so that $\mathbf{e}_d = 0$. □

Proposition 1.10. *The level of $Q_{\mathfrak{a}}$ is $4\,\mathrm{L}(R)/\,\Omega(R)$, and its discriminant is, up to squares, $\Omega(R)$.*

Proof. It is enough to find the level and discriminant for Q_R. Consider the map $\rho : R/\mathbb{Z} \to R$ given by $\rho(u) = u - \overline{u}$. Note that $\mathcal{N}\rho(u) = \Omega \cdot \Delta_R\, u$. Now, if $u \in R$ we have $\rho(u) = 2u - \operatorname{Tr} u \in \mathbb{Z} + 2R$, and conversely for $t \in \mathbb{Z}$ we can write,

$t+2u = t+\operatorname{Tr} u+\rho(u) \in \mathbb{Z}+\rho(R/\mathbb{Z})$. Thus $\mathbb{Z}+\rho(R/\mathbb{Z}) = \mathbb{Z}+2R$, where the first sum is orthogonal (with respect to the quadratic form $\mathcal{N}$). It follows that the quadratic form $\mathcal{N}$ in the lattice $\mathbb{Z}+2R$ is equivalent to the quadratic form $1+\Omega\cdot Q_R$. The assertion follows now from a straightforward calculation, since the determinant of $\mathbb{Z}+2R$ is a square, and its level is $\mathrm{L}(\mathbb{Z}+2R) = 4\,\mathrm{L}(R)$. □

Accordingly, we define the *level* and the *character* of an order R to be

$$\mathrm{N}(R) := 4\,\mathrm{L}(R)/\,\Omega(R), \qquad \text{and} \qquad \chi_R(n) := \left(\frac{\Omega(R)}{n}\right).$$

We will be constructing a $\mathbb{T}^0$-linear map

$$\Theta : \mathcal{M}(R) \to M_{3/2}\left(\mathrm{N}(R), \chi_R\right),$$

where $M_{3/2}(\mathrm{N}(R), \chi_R)$ is the space of modular forms of weight $3/2$, level $\mathrm{N} = \mathrm{N}(R)$ and character $\chi = \chi_R$, with t_n acting on this space by the Hecke operator $T(n^2)$ (see [Sh] for the definition of modular forms of half integral weight and the Hecke operators acting on them.)

Definition. Let $\mathbf{v} \in \mathcal{M}(R)$. We set

$$\Theta(\mathbf{v}) := \frac{1}{2}\sum_{d\geq 0}\langle \mathbf{e}_d, \mathbf{v}\rangle q^{d/\,\Omega} = \frac{\deg \mathbf{v}}{2} + \frac{1}{2}\sum_{d\geq 1} a_d(\mathbf{v}) q^{d/\,\Omega}$$

Proposition 1.11. *$\Theta(\mathbf{v})$ is a weight $3/2$ modular form of level $\mathrm{N}(R)$ and character χ_R. It is a cusp form if and only if $\deg \mathbf{v} = 0$. Moreover,*

$$\Theta(\mathbf{v})_{|T(n^2)} = \Theta(t_n\mathbf{v}),$$

for any $n \geq 1$ such that $(n, \mathrm{D}(R)) = 1$.

Proof. We have

$$\Theta(\mathfrak{a}) = \frac{1}{2}\sum_{x\in R_r(\mathfrak{a})/\mathbb{Z}} q^{\Delta_R(x)},$$

is the theta series of the quadratic form $Q_{\mathfrak{a}}$, and the claim on the level and character follows from Proposition 1.10.

For the second claim, note that $\Theta(\mathbf{v})$ is a linear combination of theta series corresponding to quadratic forms in the same genus; thus it vanishes at all the cusps if and only if it vanishes at the ∞ cusp.

The last statement is exactly Proposition 1.7. □

Proposition 1.12. *If* $R' = \mathbb{Z} + bR$ *for some* $b \in \mathbb{Z}$, *then* $\mathrm{D}(R') = b^3\,\mathrm{D}(R)$, $\mathrm{L}(R') = b^2\,\mathrm{L}(R)$, *and* $\Omega(R') = b^2\,\Omega(R)$, *hence*

$$\mathrm{N}(R') = \mathrm{N}(R), \qquad \textit{and} \qquad \chi_{R'} = \chi_R.$$

Moreover, $\Theta(\mathcal{M}(R')) = \Theta(\mathcal{M}(R))$.

Proof. Indeed since $R'/\mathbb{Z} = b(R/\mathbb{Z})$ it is obvious that $\Omega(R') = b^2\,\Omega(R)$ and that indeed $Q_{R'} = Q_R$. Everything else follows easily as in the proof of Proposition 1.10. □

Definition. An order R is called *primitive* if it is not of the form $R = \mathbb{Z} + bR'$ with $b \in \mathbb{Z}$, $b \neq \pm 1$ and R' an order.

Corollary 1.13. *For the purpose of constructing modular forms of weight* $^3/_2$*, it is enough to consider primitive orders.*

1.6. **Subideals.** Let R, R' be orders and $\mathfrak{a}$ be a left R-ideal. We define

$$\Psi^R_{R'}(\mathfrak{a}) := \Big\{ \mathfrak{b} \in \widetilde{\mathcal{I}}(R') \;:\; \mathfrak{b} \subseteq \mathfrak{a}, \quad \mathcal{N}\,\mathfrak{b} = \mathcal{N}\,\mathfrak{a} \Big\}.$$

This induces a map $\psi^R_{R'} : \mathcal{M}(R) \mapsto \mathcal{M}(R')$ by $\psi^R_{R'}([\mathfrak{a}]) := \sum_{\mathfrak{b} \in \Psi^R_{R'}(\mathfrak{a})} [\mathfrak{b}]$.

Proposition 1.14. *Let* R, R' *orders in* B *then*

$$t_m\,\psi^R_{R'} = \psi^R_{R'}\,t_m$$

provided $(R)_p = (R')_p$ *for all primes* p *dividing* m.

Proof. Let $\mathfrak{a} \in \widetilde{\mathcal{I}}(R)$. Given $\mathfrak{b} \in \Psi^R_{R'}(\mathfrak{a})$ and $\mathfrak{c} \in \mathcal{T}_m(\mathfrak{b})$ there is a unique $\mathfrak{b}' \in \mathcal{T}_m(\mathfrak{a})$ such that $\mathfrak{c} \in \Psi^R_{R'}(\mathfrak{b}')$. Namely, if $p \nmid m$, $\mathfrak{b}_p = \mathfrak{c}_p$ and $\mathfrak{b}'_p = \mathfrak{a}_p$. Otherwise, $\mathfrak{a}_p = \mathfrak{b}_p$ and $\mathfrak{b}'_p = \mathfrak{c}_p$. Similarly, given $\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})$ and $\mathfrak{c} \in \Psi^R_{R'}(\mathfrak{b})$ there is a unique $\mathfrak{b}'$ such that $\mathfrak{b}' \in \Psi^R_{R'}(\mathfrak{a})$ and $\mathfrak{c} \in \mathcal{T}_m(\mathfrak{b}')$. □

In the particular case where $R' \subset R$ the hypothesis of the Proposition is equivalent to $\gcd(m, [R' : R]) = 1$.

2. Orders of level p^2

Fix a prime $p > 2$, and let B be the quaternion algebra over $\mathbb{Q}$ which is ramified at p and ∞. Let $\mathcal{O}$ be a maximal order in B and let $\tilde{\mathcal{O}} := \{x \in \mathcal{O} : p \mid \Delta\, x\}$ be the unique order of index p in $\mathcal{O}$. Note $\mathrm{L}(\tilde{\mathcal{O}}) = \mathrm{D}(\tilde{\mathcal{O}}) = p^2$, but $\mathrm{N}(\tilde{\mathcal{O}}) = 4p$ since $\Omega(\tilde{\mathcal{O}}) = p$.

Lemma 2.1. Let R be an order, and let $x \in B$. A necessary and sufficient condition for $R' = \mathbb{Z}x + R$ to be an order is that x be integral and $xR \subseteq R'$.

Proof. Clearly if R' is an order, x has to be integral, and since $R'R' = R'$, we have $xR \subseteq R'$. The other implication follows from the fact that x being integral implies $x^2 = \mathrm{Tr}(x)x - \mathcal{N}(x) \in R'$, hence R' is closed by multiplication. □

Proposition 2.2. *Let $L \subset \tilde{\mathcal{O}}$ be a lattice such that $[\tilde{\mathcal{O}} : L] = p$. Then L is an order if and only if $\mathbb{Z} + p\mathcal{O} \subset L$.*

Proof. Let $\mathcal{O}'$ be an order of index p in $\tilde{\mathcal{O}}$. Locally, $\mathcal{O}'$ is already maximal outside p hence all orders are the same. At p there is a unique maximal order since B is ramified there. Let $\{u_0 = 1, u_1, u_2, u_3\}$ be an orthogonal basis for $\mathcal{O}'_p$. If $\mathbb{Z}+p\mathcal{O} \not\subset \mathcal{O}'$ then there exists $v \in \mathcal{O}'$ such that $\mathcal{O} = \mathbb{Z}\frac{v}{p^2} + \mathcal{O}'$. Since $p^4 \mid \mathcal{N}(v)$ we claim that one of the basis elements of $\mathcal{O}'_p$ has norm divisible by p^4. Note that $v/p \notin \mathcal{O}'$ hence v can be written as $v = \sum a_i u_i$ with some a_i not divisible by p. If $p^4 \nmid \mathcal{N}(u_i)$ for $i = 0, .., 3$ we would have a non zero solution of $\mathcal{N}(v) = 0 \bmod (p^4)$ which by Hensel Lemma would lift to w_p, a non zero element in B_p with $\mathcal{N}(w_p) = 0$. This cannot happen since the quadratic form norm is anisotropic on B_p. Hence $p^4 \mid \mathcal{N}(u_i)$ for some i (say $p^4 \mid \mathcal{N}(u_3)$). Then $\mathbb{Z}_p\frac{u_3}{p^2} + \mathcal{O}'_p = \mathcal{O}_p$. $\mathcal{O}'_p$ being an order and the chosen basis being orthogonal implies $u_1u_2 = ku_3$ with $k \in \mathbb{Z}_p$, i.e. $\mathcal{O} = \left\{1, u_1, u_2, \frac{u_1u_2}{k}\right\}$. But $p^2 \nmid N(u_i)$ for $i = 1, 2$ and $\frac{u_1u_2}{p^2k} \in \mathcal{O}$ then $\frac{\mathcal{N}(u_1)\,\mathcal{N}(u_2)}{p^4k^2} \in \mathbb{Z}_p$ therefore $k \notin \mathbb{Z}_p$ which is a contradiction.

Conversely, if L is such a lattice, let $x \in L$ such that $x \notin \mathbb{Z} + p\mathcal{O}$, so that $L = \mathbb{Z}x + (\mathbb{Z} + p\mathcal{O})$. But $x(\mathbb{Z} + p\mathcal{O}) = \mathbb{Z}x + px\mathcal{O} \subseteq \mathbb{Z}x + (\mathbb{Z} + p\mathcal{O})$, and Lemma 2.1 implies that L is an order. □

The orders as in the above Proposition will be said to be the *orders of level* p^2, and will be denoted by $\mathcal{O}'$. Note that $\mathrm{L}(\mathcal{O}') = \mathrm{D}(\mathcal{O}') = p^3$ and $\mathrm{N}(\mathcal{O}') = 4p^2$.

Remark. There are no orders R with $\mathrm{L}(R) = p^2$ and $\mathrm{N}(R) = 4p^2$.

Lemma 2.3. *Let $x \in \mathcal{O}'$. If $x \notin \mathbb{Z} + p\mathcal{O}$ then $p \parallel \Delta x$ and $\left(\frac{\Delta x/p}{p}\right) = \pm 1$ is independent of x.*

Proof. Let $x_0 \in \mathcal{O}'$ be such that $\mathcal{O}' = \mathbb{Z}x_0 + (\mathbb{Z} + p\mathcal{O})$. Any element $x \in \mathcal{O}'$ is of the form $ax_0 + v$ where $v \in \mathbb{Z} + p\mathcal{O}$, hence $x \notin \mathbb{Z} + p\mathcal{O}$ if and only if $p \nmid a$. But $\Delta x = a^2 \Delta x_0 + 2a(\mathcal{N}(x_0 v) - \mathcal{N}(x_0 \bar{v})) + \Delta x \equiv a^2 \Delta x_0 \pmod{p^2}$, hence the Kronecker symbol is independent of x. □

We define the character of $\mathcal{O}'$ to be

$$\sigma(\mathcal{O}') := \left(\frac{\Delta x/p}{p}\right)$$

where x is in the conditions of the Lemma.

Proposition 2.4. *Two orders $\mathcal{O}'_1$ and $\mathcal{O}'_2$ of level p^2 are locally conjugate if and only if $\sigma(\mathcal{O}'_1) = \sigma(\mathcal{O}'_2)$.*

Proof. It is clear that $\sigma(\mathcal{O}')$ is an invariant by conjugation, since $\Delta(\alpha x \alpha^{-1}) = \Delta x$ for any $\alpha \in B_p^\times$.

For the converse, let $x_i \in (\mathcal{O}'_i)_p$ $(i = 1, 2)$ such that $\mathrm{Tr}(x_i) = 0$ and $(\mathcal{O}'_i)_p = (\mathbb{Z}_p + p\mathcal{O}_p) + \mathbb{Z}_p x_i$. Since $\Delta x_i = 4\,\mathcal{N}(x_i)$, $\mathcal{N}(x_1/x_2) \in (\mathbb{Z}_p^\times)^2$, thus we can assume that $\mathcal{N}(x_1) = \mathcal{N}(x_2)$. In this case there is an element $\alpha \in B_p$ that sends x_1 to x_2 via conjugation. Clearly conjugation by α sends $(\mathcal{O}'_1)_p$ onto $(\mathcal{O}'_2)_p$ since the maximal order being unique at p, any conjugation has to send $\mathcal{O}_p$ onto $\mathcal{O}_p$, and thus $\mathbb{Z}_p + p\mathcal{O}_p$ onto $\mathbb{Z}_p + p\mathcal{O}_p$. □

Let $\mathcal{H} := \mathbb{Z}_p^\times(1 + p\mathcal{O}_p)\backslash\mathcal{O}_p^\times$, $\mathcal{G} := \mathbb{Z}_p^\times(1 + p\mathcal{O}_p)\backslash\tilde{\mathcal{O}}_p^\times$ and $\mathcal{S} := \mathbb{Z}_p^\times(1 + p\mathcal{O}_p)\backslash(\mathcal{O}'_p)^\times$.

Lemma 2.5. *The groups defined above satisfy:*

(1) $\#\mathcal{H} = p^2(p+1)$, $\#\mathcal{G} = p^2$ *and* $\#\mathcal{S} = p$.
(2) $\mathcal{S} \triangleleft \mathcal{G} \triangleleft \mathcal{H}$.

Proof. Let π be a generator of the unique order of norm p in $\mathcal{O}_p$ (in particular $\pi^2 = pu$ with $u \in \mathbb{Z}_p^\times$), then :

$$\mathcal{O}_p^\times = \left\{a_0 + a_1\pi + a_2 p + \ldots \ : \ a_i \in \mathcal{O}_p/\langle\pi\rangle \text{ and } a_0 \in (\mathcal{O}_p/\langle\pi\rangle)^\times\right\}$$

$$\tilde{\mathcal{O}}_p^\times = \left\{\mathbb{Z}_p^\times + a_1\pi + a_2 p + \ldots \;:\; a_i \in \mathcal{O}_p/\langle\pi\rangle\right\}$$

and

$$\mathbb{Z}_p^\times(1+pO_p) = \left\{\mathbb{Z}_p^\times + a_2 p + \ldots \;:\; a_i \in \mathcal{O}_p/\langle\pi\rangle \text{ and } a_0 \in (\mathcal{O}_p/\langle\pi\rangle)^\times\right\}$$

These equations and the fact that $\mathcal{O}'_p \neq \mathbb{Z}_p + p\mathcal{O}_p$ imply the first statement. Also it is clear that $\tilde{\mathcal{O}}_p^\times \triangleleft \mathcal{O}_p^\times$ and $(\mathcal{O}'_p)^\times \triangleleft \tilde{\mathcal{O}}_p^\times$ for any order $\mathcal{O}'$ which proves the second statement. □

Given an order R and $\mathcal{A} \subset \widetilde{\mathcal{I}}(R)$, we denote $[\mathcal{A}] := \{[\mathfrak{a}] \;:\; \mathfrak{a} \in \mathcal{A}\}$.

Theorem 2.6. *Let $\mathcal{O}$, $\tilde{\mathcal{O}}$ and $\mathcal{O}'$ as before. Then:*

(1) $\mathcal{I}(\tilde{\mathcal{O}}) = \bigsqcup_{[\mathfrak{a}]\in\mathcal{I}(\mathcal{O})} \left[\Psi_{\tilde{\mathcal{O}}}^{\mathcal{O}}(\mathfrak{a})\right]$ *(disjoint union).*

(2) *The set $\Psi_{\tilde{\mathcal{O}}}^{\mathcal{O}}(\mathfrak{a})$ is a principal homogeneous space for the cyclic group $\mathcal{G}\backslash\mathcal{H}$.*

(3) $\mathcal{I}(\mathcal{O}') = \bigsqcup_{[\mathfrak{b}]\in\mathcal{I}(\tilde{\mathcal{O}})} \left[\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})\right]$. *Furthermore* $\#\left[\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})\right] = \#\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b}) = p$.

(4) *All elements in $\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$ have the same right order for any $[\mathfrak{b}] \in \mathcal{I}(\tilde{\mathcal{O}})$.*

(5) *$\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$ is a principal homogeneous space for $\mathcal{S}\backslash\mathcal{G}$.*

Proof. Points (1) and (2) are subsection 3.3 of [Pa-Vi], where the action of $x_p \in B_p^\times$ on an ideal $\mathfrak{b}$ is given by right multiplication by the adele

$$(x_q) = \begin{cases} x_p & \text{if } q = p, \\ 1 & \text{if } q \neq p. \end{cases}$$

Note that $\mathcal{G}\backslash\mathcal{H}$ might not act freely on $\left[\Psi_{\tilde{\mathcal{O}}}^{\mathcal{O}}(\mathfrak{a})\right]$. Indeed, one can see that the stabilizer of this action is a subgroup of order $\langle[\mathfrak{a}], [\mathfrak{a}]\rangle$.

In (3) the union is clearly disjoint: if $\mathfrak{c}_i \in \Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b}_i)$ with $\mathfrak{b}_i \in \mathcal{I}(\tilde{\mathcal{O}})$ for $i = 1, 2$ and $\mathfrak{c}_1 = \mathfrak{c}_2 x$ for some $x \in B$, then $\mathfrak{b}_1 x = \tilde{\mathcal{O}}\mathfrak{c}_1 x = \tilde{\mathcal{O}}\mathfrak{c}_2 = \mathfrak{b}_2$. Now let $\mathfrak{c}$ be any left $\mathcal{O}'$-ideal. Then $\tilde{\mathcal{O}}\mathfrak{c}$ is a left $\tilde{\mathcal{O}}$-ideal, and $\mathcal{N}\,\mathfrak{c} = \mathcal{N}(\tilde{\mathcal{O}}\mathfrak{c})$, hence $\mathfrak{c} \in \Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\tilde{\mathcal{O}}\mathfrak{c})$.

Hence we are led to prove that $\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$ has exactly p elements, all non equivalent. Clearly if $\mathfrak{c} \in \Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$, $[\mathfrak{b} : \mathfrak{c}] = p$.

Lemma 2.7. *The set $\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$ is non empty.*

Proof. Let $\mathfrak{b}$ be a left $\tilde{\mathcal{O}}$ ideal, say $\mathfrak{b}_q = \tilde{\mathcal{O}}_q x_q$, then the lattice $\mathfrak{c}$ given locally by $\mathfrak{c}_q = \mathcal{O}'_q x_q$ is in $\Psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$. □

Consider the left action of $\mathcal{G}$ on $\Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$ defined by $g \in \mathcal{G}$ on an ideal $\mathfrak{c} \in \Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$, say $\mathfrak{c}_q = \mathcal{O}'_q x_q$ locally, by

$$\left(g \cdot \mathfrak{c}\right)_q = \begin{cases} \mathcal{O}'_p \, g x_p & \text{if } q = p, \\ \mathcal{O}'_q x_q = \mathfrak{c}_q & \text{if } q \neq p. \end{cases}$$

Since $(\mathcal{O}'_p)^\times \lhd \tilde{\mathcal{O}}_p^\times$ it is easy to check that this action is well defined and that the stabilizer of this action is $\mathcal{S}$. Furthermore if $\mathfrak{c}, \mathfrak{d} \in \Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$, then $\tilde{\mathcal{O}}\mathfrak{c} = \tilde{\mathcal{O}}\mathfrak{d}$. Hence, if $\mathfrak{c}_p = \mathcal{O}'_p x_p$ and $\mathfrak{d}_p = \mathcal{O}'_p y_p$, there exists $g \in \tilde{\mathcal{O}}_p^\times$ such that $x_p = g y_p$, thus $g \cdot \mathfrak{d} = \mathfrak{c}$, i.e. $\Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$ is a principal homogeneous space for $\mathcal{S}\backslash\mathcal{G}$. In particular the number of elements in $\Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$ is p and they all have the same right order. Two elements cannot be equivalent since there are no units in $\tilde{\mathcal{O}}$ other than ± 1 and all elements in $\Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$ have the same norm. □

A left $\tilde{\mathcal{O}}$-ideal $\mathfrak{b} \in \Psi^{\mathcal{O}}_{\tilde{\mathcal{O}}}(\mathfrak{a})$ can be computed using Lemma 2 in [Pa-Vi]. Acting on the right of $\mathfrak{b}$ by representatives of $\mathcal{G}\backslash\mathcal{H}$ we obtain all of $\Psi^{\mathcal{O}}_{\tilde{\mathcal{O}}}(\mathfrak{a})$. Repeating for all $\mathfrak{a}$ in a set of representatives of $\mathcal{I}(\mathcal{O})$, we can obtain a set of representatives for $\mathcal{I}(\tilde{\mathcal{O}})$.

Lemma 2.8. *Let* $\mathfrak{m} = \mathfrak{m}_{\mathcal{O}'} := \left\{x : x\tilde{\mathcal{O}} \subset \mathcal{O}'\right\}$. *Then* $\mathfrak{m}_{\mathcal{O}'}$ *is a bilateral* $\tilde{\mathcal{O}}$*-ideal of index* p^2 *in* $\tilde{\mathcal{O}}$.

Proof. Since $\mathfrak{m}_q = \tilde{\mathcal{O}}_q$ for all primes $q \neq p$ it is clear that $\mathfrak{m}$ is bilateral. Also from the chain $R_p \supsetneq \tilde{\mathcal{O}}_p \supsetneq \mathcal{O}'_p \supsetneq \mathfrak{m}_p \supsetneq pR_p$ we see that $\mathfrak{m}$ has index p^2 in $\tilde{\mathcal{O}}$. □

Proposition 2.9. $\mathfrak{m}_{\mathcal{O}'}\mathfrak{b} \subset \mathfrak{b}$ *with index* p^2 *for any left* $\tilde{\mathcal{O}}$*-ideal* $\mathfrak{b}$. *Furthermore,*

$$\Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b}) = \left\{\mathfrak{c} \ : \ \mathfrak{m}_{\tilde{\mathcal{O}}}\mathfrak{b} \subsetneq \mathfrak{c} \subsetneq \mathfrak{b} \text{ and } \mathcal{N}\mathfrak{c} = \mathcal{N}\mathfrak{b}\right\}.$$

Proof. The first claim follows directly from the Lemma. Thus, the number of lattices $\mathfrak{c}$ such that $\mathfrak{m}_{\tilde{\mathcal{O}}}\mathfrak{b} \subsetneq \mathfrak{c} \subsetneq \mathfrak{b}$ is $p+1$. The left $\mathcal{O}$-ideal corresponding to the different of $\mathcal{O}$ times $\mathfrak{b}$ is among these $p+1$ lattices but has norm $p\,\mathcal{N}\mathfrak{b}$, then the set on the right is empty or has exactly p elements. Hence we are lead to prove that there exists a left $\mathcal{O}'$-ideal $\mathfrak{c} \in \Psi^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathfrak{b})$ such that $\mathfrak{m}_{\tilde{\mathcal{O}}}\mathfrak{b} \subsetneq \mathfrak{c} \subsetneq \mathfrak{b}$ and $\mathcal{N}\mathfrak{c} = \mathcal{N}\mathfrak{b}$. Let $\mathfrak{c}_q = \mathfrak{b}_q$ for all primes $q \neq p$ and if $\mathfrak{b}_p = \tilde{\mathcal{O}}_p x_p$, let $\mathfrak{c}_p = \mathcal{O}'_p x_p$, then $\mathfrak{c}$ has the required properties. □

Remark. We can obtain representatives for $\mathcal{I}(\mathcal{O}')$ by applying this Proposition to each one of a set representatives for $\mathcal{I}(\tilde{\mathcal{O}})$.

3. Example: prime $p = 7$

Let $B = B(-1,-7)$, the quaternion algebra ramified precisely at 7 and ∞. This quaternion algebra has class number 1 (hence type number 1 also). A maximal order is given by

$$\mathcal{O} = \left\langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \right\rangle.$$

Its index p suborder is given by

$$\tilde{\mathcal{O}} = \left\langle 1, 7i, \frac{1+j}{2}, \frac{7i+k}{2} \right\rangle;$$

A generator of $\mathcal{G}\backslash\mathcal{H}$ is the element $\frac{1+2i+j}{2} \in \mathcal{O}_p^\times$. Hence, we get all the left $\tilde{\mathcal{O}}$-ideals by repeatedly acting on $\tilde{\mathcal{O}}$ by this element.

Let $\mathfrak{b}$ be a lattice in B, and let $x_p \in B_p^\times$. We denote $\mathfrak{b} \star x_p$ the action, by right multiplication, of the adele

$$(x_q) = \begin{cases} x_p & \text{if } p = q, \\ 1 & \text{if } p \neq q, \end{cases}$$

on the lattice $\mathfrak{b}$. Let R be the right order of $\mathfrak{b}$. We can compute this action as follows:

(1) Let k be the smallest integer such that $p^k R_p \subseteq R_p x_p$ (so that $p^k \mathfrak{b}_p \subseteq \mathfrak{b}_p x_p$.)
For instance, let $k = t + s$, where t is the valuation at p of $\mathcal{N}\, x_p$, and s is the smallest integer such that $p^s x_p \in R_p$.

(2) Let $y \in R[p^{-1}]$ such that $y - x_p \in p^k R_p$.
For instance, write x_p in a basis of R with coefficients in $\mathbb{Q}_p$, and then reduce the coefficients modulo p^k.

(3) It now follows from a local computation that $\mathfrak{b} \star x_p = p^k \mathfrak{b} + \mathfrak{b} y$.
Indeed, at $q \neq p$ we have $y \in R_q$, and $\mathfrak{b}_q = p^k \mathfrak{b}_q$.
At p it follows from (2), since $p^k \mathfrak{b}_p \subseteq \mathfrak{b}_p x_p$.

As an example, consider the case $\mathfrak{b}_1 = \tilde{\mathcal{O}}$ and $x_p = \frac{1+2i+j}{2} \in B_p^\times$, with $p = 7$. The right order of $\mathfrak{b}_1$ is $R = \tilde{\mathcal{O}}$, and we can take $k = 1$. Note that $x_p \in R[1/7]$, hence we can use $y = \frac{1+2i+j}{2} \in B^\times$. It is now easy to compute $\mathfrak{b}_2 = 7\mathfrak{b}_1 + \mathfrak{b}_1 y$.

$\tilde{\mathcal{O}}$-subideals	χ
$\mathfrak{b}_1 = \langle 1, 7i, \frac{1+j}{2}, \frac{7i+k}{2} \rangle$	$+$
$\mathfrak{b}_2 = \langle 7, 4+i, \frac{7+j}{2}, \frac{4+i+k}{2} \rangle$	$-$
$\mathfrak{b}_3 = \langle 7, 1+i, \frac{7+j}{2}, \frac{8+i+k}{2} \rangle$	$+$
$\mathfrak{b}_4 = \langle 7, 2+i, \frac{7+j}{2}, \frac{2+i+k}{2} \rangle$	$-$
$\mathfrak{b}_5 = \langle 7, i, \frac{7+j}{2}, \frac{i+k}{2} \rangle$	$+$
$\mathfrak{b}_6 = \langle 7, 5+i, \frac{7+j}{2}, \frac{12+i+k}{2} \rangle$	$-$
$\mathfrak{b}_7 = \langle 7, 6+i, \frac{7+j}{2}, \frac{6+i+k}{2} \rangle$	$+$
$\mathfrak{b}_8 = \langle 7, 3+i, \frac{7+j}{2}, \frac{10+i+k}{2} \rangle$	$-$

TABLE 3.1. Table of left $\tilde{\mathcal{O}}$-subideals.

Repeating, we obtain $\Psi^{\mathcal{O}}_{\tilde{\mathcal{O}}}(\mathcal{O}) = \{\mathfrak{b}_i : 1 \le i \le 8\}$, where the left $\tilde{\mathcal{O}}$-ideals $\mathfrak{b}_i$ and their characters are shown in Table 3.1. The ideals $\mathfrak{b}_i$ and $\mathfrak{b}_{4+i}$ for $1 \le i \le 4$ are equivalent (since $\langle [\mathfrak{a}], [\mathfrak{a}] \rangle = 2$), hence

$$\mathcal{I}(\tilde{\mathcal{O}}) = \{\mathfrak{b}_i : 1 \le i \le 4\}.$$

We fix two index p suborders of $\tilde{\mathcal{O}}$

$$\mathcal{O}^+ = \left\langle 1, 7i, \frac{1+j}{2}, \frac{7i+7k}{2} \right\rangle,$$
$$\mathcal{O}^- = \left\langle 1, 7i, \frac{1+7j}{2}, \frac{1+7i+5j+k}{2} \right\rangle$$

in the $+$ and $-$ genus respectively. Table 3.2 shows the subideals under each $\mathfrak{b}_\mathfrak{i}$ for $\mathcal{O}^+$ (respectively $\mathcal{O}^-$), for $i = 1, \ldots, 4$.

4. GROSS FORMULA FOR LEVEL p^2

The aim here is to conjecture a formula that applies to modular forms of level p^2, similar to the one proved by Gross in [Gr, Proposition 13.5, p.179] for prime level. Keep the notation of the previous section, and let f be a newform of weight 2 and level p^2, such that f is an eigenform for the Hecke operators. We denote by $L(f, s)$ the Hecke L-series of f, and for D a fundamental discriminant we define its *twisted L-series* as

$$L(f, D, s) := L(f \otimes D, s),$$

$\tilde{\mathcal{O}}$-ideals	$\mathcal{O}^+$-subideals	$\mathcal{O}^-$-subideals
$\mathfrak{b}_1$	$\left\langle 1, 7i, \frac{1+j}{2}, \frac{7i+7k}{2}\right\rangle$	$\left\langle 1, 7i, \frac{1+7j}{2}, \frac{1+7i+5j+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{2+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{1+j}{2}, \frac{2+7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{4+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{3+j}{2}, \frac{6+7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{6+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{5+j}{2}, \frac{10+7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{8+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{9+j}{2}, \frac{4+7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{10+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{11+j}{2}, \frac{8+7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{7+j}{2}, \frac{12+7i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{13+j}{2}, \frac{12+7i+k}{2}\right\rangle$
$\mathfrak{b}_2$	$\left\langle 7, 4+i, \frac{7+7j}{2}, \frac{11+i+3j+k}{2}\right\rangle$	$\left\langle 7, 4+i, \frac{7+7j}{2}, \frac{4+i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{1+2i+j}{2}, \frac{4+i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{1+2i+j}{2}, \frac{7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{3+6i+j}{2}, \frac{12+3i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{3+6i+j}{2}, \frac{7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{5+10i+j}{2}, \frac{6+5i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{5+10i+j}{2}, \frac{7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{9+4i+j}{2}, \frac{8+9i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{9+4i+j}{2}, \frac{7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{11+8i+j}{2}, \frac{2+11i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{11+8i+j}{2}, \frac{7i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{13+12i+j}{2}, \frac{10+13i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{13+12i+j}{2}, \frac{7i+k}{2}\right\rangle$
$\mathfrak{b}_3$	$\left\langle 7, 1+i, \frac{7+7j}{2}, \frac{8+i+6j+k}{2}\right\rangle$	$\left\langle 7, 1+i, \frac{7+7j}{2}, \frac{8+i+2j+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{1+8i+j}{2}, \frac{8+i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{1+8i+j}{2}, \frac{12+5i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{3+10i+j}{2}, \frac{10+3i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{3+10i+j}{2}, \frac{8+i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{5+12i+j}{2}, \frac{12+5i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{5+12i+j}{2}, \frac{4+11i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{9+2i+j}{2}, \frac{2+9i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{9+2i+j}{2}, \frac{10+3i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{11+4i+j}{2}, \frac{4+11i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{11+4i+j}{2}, \frac{6+13i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{13+6i+j}{2}, \frac{6+13i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{13+6i+j}{2}, \frac{2+9i+k}{2}\right\rangle$
$\mathfrak{b}_4$	$\left\langle 7, 2+i, \frac{7+7j}{2}, \frac{9+i+5j+k}{2}\right\rangle$	$\left\langle 7, 2+i, \frac{7+7j}{2}, \frac{9+i+j+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{1+4i+j}{2}, \frac{2+i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{1+4i+j}{2}, \frac{6+3i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{3+12i+j}{2}, \frac{6+3i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{3+12i+j}{2}, \frac{4+9i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{5+6i+j}{2}, \frac{10+5i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{5+6i+j}{2}, \frac{2+i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{9+8i+j}{2}, \frac{4+9i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{9+8i+j}{2}, \frac{12+13i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{11+2i+j}{2}, \frac{8+11i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{11+2i+j}{2}, \frac{10+5i+k}{2}\right\rangle$
	$\left\langle 7, 7i, \frac{13+10i+j}{2}, \frac{12+13i+k}{2}\right\rangle$	$\left\langle 7, 7i, \frac{13+10i+j}{2}, \frac{8+11i+k}{2}\right\rangle$

TABLE 3.2. Table of left $\mathcal{O}^+$- and $\mathcal{O}^-$-subideals.

where $f \otimes D$ is (the newform corresponding to) the twist of f by the quadratic character $n \mapsto \left(\frac{D}{n}\right)$. Recall that $L(f, D, s)$ is an entire function of the complex plane with a functional equation relating its values at s and $2-s$, with central value $L(f, D, 1)$. The sign of the functional equation will be denoted by $\epsilon(f, D)$. This sign determines the parity of the order of vanishing of $L(f, D, s)$ at $s = 1$; in particular when $\epsilon(f, D) = -1$ it is trivial that $L(f, D, 1) = 0$.

We denote by $\mathcal{M}(\tilde{\mathcal{O}})^f$ the f-isotypical component of $\mathcal{M}(\tilde{\mathcal{O}})$. i.e. the eigenspace of $\mathbb{T}^0$ with the same eigenvalues as f. We have the following result due to Pizer ([Pi, Theorem 8.2, p.223]) :

$$\dim \mathcal{M}(\tilde{\mathcal{O}})^f = \begin{cases} 2 & \text{if } f \text{ is not the twist of a level } p \text{ form,} \\ 1 & \text{if } f \text{ is the quadratic twist of a level } p \text{ form,} \\ 0 & \text{otherwise.} \end{cases}$$

In what follows we will assume that f is not in the last case, i.e. that $\mathcal{M}(\tilde{\mathcal{O}})^f \neq 0$.

Since $\mathrm{N}(\tilde{\mathcal{O}}) = 4p$, it follows that $\Theta\big(\mathcal{M}(\tilde{\mathcal{O}})^f\big) = 0$. In order to obtain modular forms of weight $3/2$ and level $4p^2$, we have to employ the orders of level p^2 introduced in §2. Let $\mathcal{O}'$ be such an order. The general construction of §1 gives a Hecke-linear map

$$\Theta : \mathcal{M}(\mathcal{O}') \to M_{3/2}(4p^2, \varkappa_p).$$

where $\varkappa_p(n) := \left(\frac{p}{n}\right)$ is the character of $\mathcal{O}'$. However, the space $\mathcal{M}(\mathcal{O}')$ represents weight 2 modular forms of level p^3, and hence it is too big. Indeed, by Theorem 2.6 (3) we know that $\dim \mathcal{M}(\mathcal{O}') = p \dim \mathcal{M}(\tilde{\mathcal{O}})$.

To overcome this problem, we will use the theory of §2 and define Hecke-linear maps

$$\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}} := \frac{1}{p}\Theta \circ \psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}} : \mathcal{M}(\tilde{\mathcal{O}}) \to M_{3/2}(4p^2, \varkappa_p),$$

which in view of Theorem 2.6 can also be defined, for $[\mathfrak{b}] \in \mathcal{I}(\tilde{\mathcal{O}})$, by

$$\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}}([\mathfrak{b}]) := \Theta([\mathfrak{c}]),$$

where $\mathfrak{c}$ is any ideal in $\psi_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathfrak{b})$. Note that, although the map $\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}}$ depends on $\tilde{\mathcal{O}}$ and $\mathcal{O}'$, its image depends only on $\sigma(\mathcal{O}')$, so that we really have only two different constructions.

Proposition 4.1. *Let* $p^* = \left(\frac{-1}{p}\right) p$ *be the prime discriminant associated to* p*. Note that the level of* $f \otimes p^*$ *can be* p *or* p^2*. If* $\epsilon(f,1) = +1$ *and either*

(A) $f \otimes p^*$ *has level* p *and* $\epsilon(f,p^*) = \left(\frac{-1}{p}\right) \sigma(\mathcal{O}')$*, or*

(B) $f \otimes p^*$ *has level* p^2 *and* $\epsilon(f,p^*) = \left(\frac{-1}{p}\right)$*,*

then $\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathcal{M}(\tilde{\mathcal{O}})^f) = 0$*.*

Proof. From $\epsilon(f,1) = +1$ it follows that $\epsilon(f,-d) = -1$ for any fundamental discriminant $-d < 0$ not divisible by p. Either of the conditions imply that for any fundamental discriminant $-pd < 0$ with $\left(\frac{d}{p}\right) = \sigma(\mathcal{O}')$, we have $\epsilon(f,-pd) = -1$ (see Lemma 30 and Theorem 6 of [At-Le]) , and in particular $L(f,-pd,1) = 0$. The result now follows from Waldspurger's formula [Wa, Théorème 1]. □

It is worth noting that the first examples of f satisfying condition (B) above (e.g. the modular form of level 13^2 and degree 2 over $\mathbb{Q}$, as well as the two rational modular forms of level 37^2 and rank 0, or one of the rational modular forms of level 43^2 and rank 0), we found that indeed $\mathcal{M}(\tilde{\mathcal{O}})^f = 0$. It is an interesting question whether condition (B) characterizes all modular forms of level p^2 and trivial character coming from level p ones by twisting by a non-quadratic character.

Conjecture 1. *Assume* $\mathcal{M}(\tilde{\mathcal{O}})^f \neq 0$

(1) *if* $L(f,1) = 0$*, then* $\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathcal{M}(\tilde{\mathcal{O}})^f) = 0$*.*

(2) *if* $L(f,1) \neq 0$*, then* $\Theta_{\mathcal{O}'}^{\tilde{\mathcal{O}}}(\mathcal{M}(\tilde{\mathcal{O}})^f) \neq 0$*, unless* f *and* $\mathcal{O}'$ *are in the conditions (A) or (B) of Proposition 4.1.*

Remark. In case $L(f,1) \neq 0$, the conjecture asserts that our construction yields two different weight $3/2$ modular forms (in Kohnen space) in Shimura correspondence with f, unless $f \otimes p^*$ has level p, where the construction yields only one modular form, or if f is in condition (B), where we obtain no modular forms. It follows from [Ue88, Prop 3] that this is exactly what is available in Kohnen space, even when $L(f,1) = 0$.

Note that the analogous of our conjecture for the case of level p is true, and follows from Gross's formula [Gr]. This has been extended to the case of odd square free level in [BSP90] and [BSP92]. To support our conjecture we will give an explicit version, namely a formula, which has been verified numerically for many cases.

The strong multiplicity one theorem of Ueda ([Ue93, Theorem 3.11, p.181]) implies that $\dim \Theta^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathcal{M}(\tilde{\mathcal{O}})^f) \leq 1$. The above conjecture gives conditions on $\mathcal{O}'$ and f so that $\Theta^{\tilde{\mathcal{O}}}_{\mathcal{O}'}\big(\mathcal{M}(\tilde{\mathcal{O}})^f\big) \neq 0$. In that case, $\ker \Theta^{\tilde{\mathcal{O}}}_{\mathcal{O}'}$ has codimension 1 in $\mathcal{M}(\tilde{\mathcal{O}})^f$, and thus a vector $\mathbf{e}_{f,\mathcal{O}'} \in \mathcal{M}(\tilde{\mathcal{O}})^f$ is well defined up to a constant by requiring it to be orthogonal to $\ker \Theta^{\tilde{\mathcal{O}}}_{\mathcal{O}'}$; we will write

$$\Theta_{f,\mathcal{O}'} := \Theta^{\tilde{\mathcal{O}}}_{\mathcal{O}'}(\mathbf{e}_{f,\mathcal{O}'}) = \sum_{d\geq 1} c_{f,\mathcal{O}'}(d)\, q^d.$$

Otherwise, although our method yields the zero modular form, we will use $\mathbf{e}_{f,\mathcal{O}'}$ to denote any nonzero vector in $\mathcal{M}(\tilde{\mathcal{O}})^f$, with the understanding that it doesn't really matter which one we choose since $\Theta_{f,\mathcal{O}'} = 0$ regardless of this choice, because our conjectural formula below is nontrivial even in this case.

Define the rational constant α_f by

$$\alpha_f := \frac{1}{2} \cdot \begin{cases} 1 & \text{if } f \text{ is not the twist of a level } p \text{ form,} \\ \frac{p}{p-1} & \text{if } f \text{ is the quadratic twist of a level } p \text{ form.} \end{cases}$$

Conjecture 2. *Let d be an integer such that $-pd < 0$ is a fundamental discriminant, and such that $\left(\frac{d}{p}\right) = \sigma(\mathcal{O}')$. Then*

$$L(f,-pd,1)\, L(f,1) = \alpha_f\, \frac{\langle f, f\rangle}{\sqrt{pd}}\, \frac{c_{f,\mathcal{O}'}(d)^2}{\langle \mathbf{e}_{f,\mathcal{O}'}, \mathbf{e}_{f,\mathcal{O}'}\rangle}.$$

Remark. The formula is true up to a constant (depending on f and $\mathcal{O}'$), by Waldspurger's theorem. Moreover, in case f is rational it is possible to prove that $L(f,-pd,1)L(f,1)\sqrt{pd}/\langle f,f\rangle$ is a rational number of bounded height, and a similar result holds for general f. Thus there is an effective procedure to determine if the formula is true for any particular f. See [Pa-To] for some numerical examples in the cases $p = 7, 11, 17, 19$, where the formula has been verified.

We note the following complement of Proposition 4.1

Lemma 4.2. *Assume f and $\mathcal{O}'$ are such that neither of the conditions (A) nor (B) of Proposition 4.1 hold. Then there is a fundamental discriminant $-pd < 0$ such that $\left(\frac{d}{p}\right) = \sigma(\mathcal{O}')$, and $L(f,-pd,1) \neq 0$.*

Proof. It is easy to see that if neither (A) nor (B) hold, then for any fundamental discriminant $-pd' < 0$ such that $\left(\frac{d'}{p}\right) = \sigma(\mathcal{O}')$, we have $\epsilon(f, -pd') = +1$. It now follows from [BFH] that there is a fundamental discriminant d'' prime to d' such that $d'' > 0$ and $\left(\frac{d''}{p}\right) = +1$, for which $L(f \otimes -pd', d'', 1) \neq 0$. It is clear that $d = d'd''$ will satisfy the claimed conditions. □

Proposition 4.3. *Conjecture 2 implies Conjecture 1.*

Proof. If $L(f, 1) = 0$, then clearly the formula implies that $\Theta_{f,\mathcal{O}'} = 0$. Conversely, if the conditions of Proposition 4.1 are not satisfied, the Lemma asserts the existence of some d in the hypothesis of Conjecture 2 for which $L(f, -pd, 1) \neq 0$. Thus, if $L(f, 1) \neq 0$, it follows that $c_{f,\mathcal{O}'}(d) \neq 0$, i.e. $\Theta_{f,\mathcal{O}'} \neq 0$. □

Remark. When $p \equiv 3 \pmod 4$, and f is a modular form of level p or p^2, this method can be used to compute a weight $3/2$ modular form whose Fourier coefficients are related to the central values of *real* quadratic twists of f. See [Pa-To] for an exposition of this method and examples.

References

[At-Le] A. Atkin and J. Lehner, *Hecke Operators on* $\Gamma_0(m)$, Math. Ann. 185 (1970), p. 134-160

[BSP90] S. Böcherer and R. Schulze-Pillot, *On a theorem of Waldspurger and on Eisenstein series of Klingen type*, Math. Ann. 288 (1990) 361–388.

[BSP92] S. Böcherer and R. Schulze-Pillot, *The Dirichlet series of Koecher and Maass and modular forms of weight* $\frac{3}{2}$, Math. Z. 209 (1992), no. 2, 273–287.

[BFH] D. Bump, S. Friedberg and J. Hoffstein *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Invent. Math. V. 102, 1990, p. 543-618.

[Gr] B. Gross, *Heights and Special Values of L-series*, Conference Proceedings of the CMS Vol. 7, 115-187 AMS 1986

[Ll] P. Llorente, *Correspondencia entre formas ternarias enteras y rdenes cuaterninicos*, Rev. R. Acad. Cienc. Exactas Fs. Nat. (Esp.) **94**, no. 3, 2000, 397–416.

[Pa-Vi] A. Pacetti and F. Rodriguez-Villegas, *Computing Weight* 2 *Modular forms of level* p^2, Math. Comp. **74** (2005), 1545–1557.

[Pa-To] A. Pacetti and G. Tornaría, *Examples of Shimura correspondence for level* p^2 *and real quadratic twists*, To appear in "Ranks of elliptic curves and random matrix theory."

[Pi] A. Pizer, *Theta Series and Modular Forms of Level* p^2M Compositio Mathematica, Vol. **40** Fasc. 2, 1980, p. 177–241.

[Sh] G. Shimura *On Modular Forms of Half Integral Weight* The Annals of Mathematics, Vol. 97, 1971, p. 440-481.

[To] Gonzalo Tornaría, *Data about the central values of the L-series of (imaginary and real) quadratic twists of elliptic curves*, http://www.ma.utexas.edu/users/tornaria/cnt/, 2004.

[Ue88] M. Ueda, *The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators*, J. Math. Kyoto Univ. 28-3 (1988) 505–555.

[Ue93] M. Ueda, *On twisting operators and newforms of half-integral weight*, Nagoya Mathematical Journal Vol. 131,1993, p. 135-205.

[Wa] J-L Waldspurger *Sur les coefficients de Fourier des formes modulaires de poids demi-entier* Journal de Mathematiques Pures et Appliquees, Tome 60 Fas. 4, 1981 p. 375-484

Departamento de Matemática, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria. C.P:1428, Buenos Aires, Argentina

E-mail address: apacetti@dm.uba.ar

Department of Mathematics, University of Texas at Austin, Austin, TX 78712

E-mail address: tornaria@math.utexas.edu