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An effective residual criterion for the membership problem in $\mathbf{C}[z_1, \ldots, z_n]$

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§1. Introduction.

The purpose of this paper is to give an effective test for the membership problem in the case of a polynomial complete intersection.

Let $P_1, \ldots, P_r \in \mathbf{C}[z_1, \ldots, z_n]$ be polynomials defining a complete intersection in $\mathbf{C}^n$ (i.e. $\{P_1 = \ldots = P_r = 0\}$ is not empty and has pure dimension $n - r$) and denote $I(P_1, \ldots, P_r) = \left\{ \sum_{i=1}^{r} Q_i P_i\,, \;\; Q_i \in \mathbf{C}[z_1, \ldots, z_n] \right\}$.

For any natural number k, we associate to the ideal $I(P_1, \ldots, P_r)$ a homogeneous system of linear equations S_k, which characterizes membership to $I(P_1, \ldots, P_r)$ up to degree k in the following sense: Given a polynomial $Q \in \mathbf{C}[z_1, \ldots, z_n]$ of degree bounded by k, the vector of coefficients of Q is a solution of S_k iff $Q \in I(P_1, \ldots, P_r)$.

We first solve the case $r = n$ based on the local duality given by the punctual residual operators. Next, we reduce the general complete intersection case to the punctual case with the aid of appropriate fibrations.

The entries of the matrix of S_k can be expressed rationally in the coefficients of $P_1, \ldots, P_r$. As a consequence, when the coefficients of the given polynomials $P_1, \ldots, P_r$ lie in some subfield K of $\mathbf{C}$, the entries of the matrix also belong to K.

Moreover, the matrix of S_k can be effectively computed from the inputs $P_1, \ldots, P_r$ in sequential time $k^{O(n)} + d^{O(n^2)}$, where $d \geq \max\{\deg(P_i), \;\; i = 1, \ldots, r\}$ and $d \geq 3$.

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§2. The punctual case

We analyze first the case of n polynomials $P_1, \ldots, P_n$ having a finite zero set Z in $\mathbf{C}^n$. For $Q \in \mathbf{C}[z_1, \ldots, z_n]$ and $x \in Z$, $R_{P,x}[Q]$ will denote the operator which sends each germ φ of holomorphic function at x to the complex number

$$\mathrm{Res}_x\Big(\frac{Q(z)\varphi(z)dz_1 \wedge \ldots \wedge dz_n}{P_1(z)\ldots P_n(z)}\Big) \qquad \text{(cf. [7], [3])}$$

and $R_P[Q]$ will denote the operator on global holomorphic functions

$$R_P[Q](\varphi) = \sum_{x \in Z} R_{P,x}[Q](\varphi), \qquad \varphi \in \mathcal{O}(\mathbf{C}^n)\,.$$

The operator $R_{P,x}[Q]$ is, in fact, a polynomial in holomorphic derivatives of the delta distribution δ_x (cf. [3]); more precisely, there exist $n_x \in \mathbf{N}_0$ and complex constants $(c_{\alpha,x} : \alpha \in \mathbf{N}_0^n,\ 0 \le |\alpha| \le n_x)$ such that for every $\varphi \in \mathcal{O}_x$.

$$R_{P,x}[Q](\varphi) = \sum_{0 \le |\alpha| \le n_x} c_{\alpha,x}\, D_\alpha \varphi(x) \qquad (*)$$

(as usual, $|\alpha|$ denotes the number $\alpha_1 + \cdots + \alpha_n$ and $D_\alpha = \dfrac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$).

We note $\mathrm{ord}\, R_{P,x}[Q] = \max\{|\alpha| : c_{\alpha,x} \neq 0\} + 1$, and $\mathrm{ord}\, R_P[Q] = \sum_{x \in Z} \mathrm{ord}\, R_{P,x}[Q]$.

Proposition 2.1. $R_{P,x}[Q] \equiv 0 \quad \forall x \in Z \Leftrightarrow R_P[Q](S) = 0$ *for any polynomial* S *with* $\deg(S) \le$ *ord* $R_P[Q] - 1$.

<u>*Proof*</u>. By the local description (*) of the residual operator, the proof is reduced to finding polynomials of total degree at most $\mathrm{ord}\, R_P[Q] - 1$ with prescribed derivatives at each $x \in Z$ up to order $\mathrm{ord}\, R_{P,x}[Q] - 1$.

In fact, if $Z = \{x_1, \ldots, x_k\}$ and $n_j := \mathrm{ord}\, R_{P,x_j}[Q] - 1$, given any i and $\varphi \in \mathcal{O}_{x_i}$, one can construct a polynomial S of degree at most $\mathrm{ord}\, R_P[Q] - 1 = \sum_{j=1}^{k} n_j + k - 1$ such that

$$D_\alpha S(x_i) = D_\alpha \varphi(x_i) \qquad \text{for all} \quad 0 \le |\alpha| \le n_i\,,$$

and $\forall\, j \neq i$

$$D_\alpha S(x_j) = 0 \qquad \text{for all} \quad 0 \le |\alpha| \le n_j\,.$$

Then, $R_{P,x_i}[Q](\varphi) = R_P[Q](S)$.

S can be obtained as follows:

- If $k=1$, $S=\sum_{0\le|\alpha|\le n_1} D_\alpha\varphi(x_1)\,\dfrac{(z-x_1)^\alpha}{\alpha!}$.

- • If $k>1$, choose for each $j\neq i$ a coordinate $e(j)$ such that $z_{e(j)}(x_j)\neq z_{e(j)}(x_i)$, and let h be the following polynomial of degree $\sum_{j\neq i} n_j+k-1$:

$$h=\prod_{j\neq i}(z_{e(j)}-z_{e(j)}(x_j))^{n_j+1}.$$

We seek g of degree n_i such that $S=g\cdot h$. Then, the polynomial g must satisfy the conditions:

$$\begin{cases} S(x_i)=g(x_i)\cdot h(x_i) \\ D_\alpha S(x_i)=D_\alpha g(x_i)\cdot h(x_i)+\displaystyle\sum_{\substack{\alpha-\beta\in\mathbf{N}_0^n\\ \beta\neq\alpha}} C_{\beta,\alpha}\, D_\beta g(x_i)\cdot D_{\alpha-\beta}h(x_i), \\ \qquad \forall\,\alpha: 0<|\alpha|\le n_i\,. \end{cases}$$

On setting $D_\alpha S(x_i)=D_\alpha\varphi(x_i)$ for all $0\le|\alpha|\le n_i$, we can determine $D_\alpha g(x_i)$ recursively because $h(x_i)\neq 0$, and thus find g by means of its Taylor expansion at x_i as in the case $k=1$. $\diamondsuit$

Proposition 2.2. $\operatorname{ord} R_P[Q]\le\prod_{i=1}^n \deg(P_i)$.

<u>*Proof.*</u> The above inequality is a consequence of the following facts:

i) $\operatorname{ord} R_P[Q]\le \operatorname{ord} R_P[1]$.

ii) Let $\mathcal{M}_x$ denote the maximal ideal in $\mathcal{O}_x$ and $m(x)=\min\{m:\mathcal{M}_x^m\subseteq \mathcal{I}_x(P_1,\dots,P_n)$. Then, $\operatorname{ord} R_{P,x}[1]=m(x)$ (cf. [4]).

iii) If $(P_1,\dots,P_n)_x$ denotes the local intersection number $\dim_{\mathbf{C}}\mathcal{O}_x/\mathcal{I}_x(P_1,\dots,P_n)$, $m(x)\le(P_1,\dots,P_n)_x$.

iv) $\sum_{x\in Z}(P_1,\dots,P_n)_x\le\prod_{i=1}^n\deg(P_i)$. $\diamondsuit$

Theorem 2.3. *Given* $Q\in\mathbf{C}[z_1,\dots,z_n]$, $Q=\sum_{0\le|\beta|\le M} C_\beta z^\beta$, *the following two conditions are equivalent:*

a) $Q\in I(P_1,\dots,P_n)$.

b) *The vector of coefficients* $(C_\beta: 0\le|\beta|\le M)$ *verifies the following linear system:*

$$\sum_\beta C_\beta\cdot R_P[1](z^{\alpha+\beta})=0$$

$$\forall\, \alpha \in N_0^n \textit{ such that } 0 \le |\alpha| \le \prod_{i=1}^{n} \deg(P_i) - 1\,.$$

Proof. Since $\sum_{\beta} C_\beta R_P[1](z^{\alpha+\beta}) = R_P[Q](z^\alpha)$, by the two previous propositions condition b) is equivalent to

c) $R_{P,x}[Q] \equiv 0\,, \ \ \forall\, x \in Z\,.$

On the other hand, as a consequence of the local duality given by the residual operator (cf. [7]), condition c) is equivalent to $Q_x \in \mathcal{I}_x(P_1,\ldots,P_n)\,, \ \forall\, x \in Z$. Now, this last condition is equivalent to local algebraic membership for each $x \in Z$ (cf. [11]), which is in turn equivalent to the global algebraic condition a). ◊

§3. Algebraic behaviour of the residual operator associated to a polynomial complete intersection

3.1. Given a regular sequence $P_1(z),\ldots,P_r(z)$, it is possible to find a linear change of coordinates $z = w \cdot M$, given by $M \in GL(n,\mathbf{Q})$, and polynomials $F_1(w),\ldots,F_r(w)$, $A_{ij}(w)$ $(1 \le i,j \le r)$, such that:

i) $F_j(w) = \sum_{j=1}^{r} A_{ji}(w) \cdot P_i(w)\,, \ j = 1,\ldots,r\,.$

ii) Each F_j depends only on $w_j, w_{j+1},\ldots,w_n$ and $\deg_{w_j}(F_j) = \deg(F_j)$. (A polynomial with this last property will be called semimonic in w_j).

iii) The coefficients of $F_j(w)$ and $A_{ij}(w)$ are rational polynomial functions in the coefficients of $P_1(w),\ldots,P_r(w)$, and so they can be expressed as rational polynomials in the coefficients of $P_1(z),\ldots,P_r(z)$ (rational polynomial means a polynomial with coefficients in $\mathbf{Q}$).

These polynomials can be classically obtained by iterated elimination of variables by means of resultants (cf. [9]), and they are the key for the polynomial behaviour of the residual operators described in theorem 3.3 below. However, this effective method is not satisfactory from the complexity point of view, since the bounds for the degrees of F_i and A_{ij} become doubly exponential in r. The algorithms recently given in [6] solve the problem of finding M, F_i and A_{ij} verifying i) and ii) with simply exponential complexity bounds (see §5).

Remarks 3.2.

1) The procedures of [9] and [6] involve effective algorithmic choices of changes of variables to get semimonic polynomials. However, these choices are not algebraic functions of the coefficients of the data $P_1,\ldots,P_r$.

2) If the hypothesis of complete intersection is not verified, the procedures of both [9] and [6] break down at some step.
3) The coordinates w verify the following property:
The linear projection $\pi : \mathbf{C}^n \to \mathbf{C}^{n-r}$, $\pi(w) = (w_{r+1}, \ldots, w_n)$, is a proper map from the common zero locus of the polynomials $P_1, \ldots, P_r$ onto $\mathbf{C}^{n-r}$.

Theorem 3.3. *Given $P_1, \ldots, P_r$ and coordinates w as in 3.1, denote $w' = (w_1, \ldots, w_r)$ and $x = (w_{r+1}, \ldots, w_n)$. Then, for any $Q \in \mathbf{C}[w] = \mathbf{C}[w', x]$, the mapping*

$$R : \mathbf{C}^{n-r} \to \mathbf{C}$$
$$x \mapsto R_{P(w',x)}[Q(w',x)](1)$$

is a polynomial function of x. Its coefficients are effectively computable; they are rational functions in the coefficients of $P_1(w',x), \ldots, P_r(w',x)$, and they depend linearly on the coefficients of Q.

Proof. By the Transformation Law (cf. [7]), we have

$$R_{P(w',x)}[Q(w',x)](1) = R_{F(w',x)}[\det(A_{ij}(w',x)) \cdot Q(w',x)](1)$$

(F_j and A_{ij} as in 3.1).

Taking into account 3.1 iii) and the linearity of the residual operator, the theorem will be proved if we show that, for each $\beta \in \mathbf{N}_0^r$, the mapping

$$x \mapsto R_{F(w',x)}[w'^{\beta}](1)$$

is an effectively computable polynomial function whose coefficients are rational functions in the coefficients of $F_1, \ldots, F_r$.

Now, taking into account 3.1 ii), for each $x \in \mathbf{C}^{n-r}$, $R_{F(w',x)}[w'^{\beta}](1)$ can be computed by iterating residues in a single variable. Namely,

$$R_{F(w',x)}[w'^{\beta}](1) = R_{F_r(w_r,x)}[w_r^{\beta_r}](R_{F_{r-1}(w_{r-1},w_r,x)}[w_{r-1}^{\beta_{r-1}}](\ldots(R_{F_1(w',x)}[w_1^{\beta_1}](1))\ldots),$$

where $R_{F_1(w',x)}[w_1^{\beta_1}](1)$ is the function which assigns to any fixed $(w_2, \ldots, w_r, x)$ the complex number $\sum_{\lambda \in \mathbf{C}} \mathrm{Res}_{\lambda}\left(\frac{w_1^{\beta_1} dw_1}{F_1(w_1, \ldots, x)}\right)$. This is just the property of transitivity of the algebraic residues ([8]), which in the complex case can also be proved as follows: Let $B(x)$ be an open ball in $\mathbf{C}^r$ containing the zero set $Z(x) = \{w' \in \mathbf{C}^r / F_1(w',x) = F_2(w',x) = \ldots = F_r(w',x) = 0\}$. Then, for any sufficiently small $\varepsilon > 0$,

$$R_{F(w',x)}[w'^{\beta}](1) = \frac{1}{(2\pi i)^r} \int_{\{w' \in B(x) : |F_i(w',x)| = \varepsilon,\ 1 \le i \le r\}} \frac{w'^{\beta} dw'}{F_1(w',x) \ldots F_r(w',x)}$$

The stated iterated formula is now a consequence of Fubini's theorem.

We will show that $R_{F_1(w',x)}[w_1^{\beta_1}](1)$ is a polynomial function in $(w_2,\dots,w_r,x)$; it will follow that $R_{F(w',x)}[w'^{\beta}](1)$ is also a polynomial function in x by iterating the same argument.

In fact, let $F_1(w',x) = c(w_1^k + \sum_{i=1}^{k} G_i(w_2,\dots,x)w_1^{k-i})$, with $c \in \mathbf{C}-\{0\}$, $\deg(G_i) \le i$ and denote $Q = 1+\sum_{i=1}^{k} G_i w_1^i \in \mathbf{C}[w',x]$. Then, $R_{F_1(w',x)}[w_1^{\beta_1}](1) =$

$$= -\mathrm{Res}\Big(\frac{w_1^{\beta_1}dw_1}{F_1}\Big) = \frac{1}{c}\,\mathrm{Res}_0\Big(\frac{dw_1}{w_1^{\beta_1+2-k}\cdot Q}\Big) =$$

$$= \begin{cases} 0 & \text{if } \beta_1 \le k-2. \\ \dfrac{1}{c}\dfrac{1}{(\beta_1+1-k)!}\dfrac{\partial^{\beta_1+1-k}}{\partial w_1^{\beta_1+1-k}}\Big(\dfrac{1}{Q}\Big)\Big|_{w_1=0} & \text{if } \beta_1 \ge k-1. \end{cases}$$

These derivatives $B_j(w_2,\dots,x) := \frac{1}{j!}\frac{\partial^j}{\partial w_1^j}(\frac{1}{Q})\Big|_{w_1=0}$ are polynomial functions of degree at most j which can be recursively determined from the equality

$$\Big(\sum_{j=0}^{\infty} B_j w_1^j\Big)\cdot\Big(1+\sum_{i=1}^{k} G_i w_1^i\Big) = 1\,. \qquad \diamondsuit$$

We want to remark that similar ideas for the computation of punctual multiple residues are already used for instance in [12].

§4. The linear equation for the membership problem in the general complete intersection case

Proposition 4.1. *Given* $P_1,\dots,P_r$, $Q \in \mathbf{C}[z_1,\dots,z_n]$ *and* (w',x) *a coordinate system verifying:* $\forall\, x_0 \in \mathbf{C}^{n-r}$, $\dim_{\mathbf{C}} \bigcap_{i=1}^{r}\{P_i(w',x_0)=0\} = 0$, *the following conditions are equivalent:*

a) $Q \in I(P_1,\dots,P_r)$.

b) $\forall\, x \in \mathbf{C}^{n-r}$, $R_{P(w',x)}[Q(w',x)](w'^{\beta}) = 0$, $\forall\, \beta \in \mathbf{N}_0^r$ *s.t.* $|\beta| \le \prod_{i=1}^{r}\deg(P_i) - 1$.

Proof. As in the proof of theorem 2.3, a) is equivalent to:

c) $Q_z \in \mathcal{I}_z(P_1,\dots,P_r)$, $\forall\, z \in \mathbf{C}^n$.

Now, by theorem 4.3 of [5], c) is equivalent to:

d) $\forall x \in \mathbf{C}^{n-r}$ the respective restrictions $\overline{Q}$, $\overline{P}_1, \ldots, \overline{P}_r$ to the fiber $\mathbf{C}^r \times \{x\}$ verify $\overline{Q}_w \in \mathcal{I}_w(\overline{P}_1, \ldots, \overline{P}_r)$, $\forall w' \in \mathbf{C}^r$.
Again as in §2, d) is equivalent to condition b). $\diamondsuit$

Gathering all the above information, we have the following result:

Theorem 4.2. *Given a complete intersection $P_1, \ldots, P_r \in \mathbf{C}[z_1, \ldots, z_n]$ and $k \in \mathbf{N}$, there is an effective choice of $M \in GL(n, \mathbf{Q})$ such that, in coordinates $(w', x) = = z \cdot M$, one can construct a "residual" homogeneous system of linear equations S_k satisfying:*

i) *$\forall Q \in \mathbf{C}[z_1, \ldots, z_n]$ with $\deg(Q) \leq k$, $Q \in I(P_1, \ldots, P_r)$ iff the vector of coefficients of $Q(w', x)$ is a solution of S_k.*

ii) *The entries of the matrix of S_k can be expressed rationally in the coefficients of $P_1(z), \ldots, P_r(z)$.*

Proof. Let (w', x) be as in §3. Then, by theorem 3.3 we know that $P_\beta(x) := R_{P(w',x)}[Q(w', x)](w'^\beta)$ is a polynomial function of x. Moreover, we can give a bound for $\deg_x(P_\beta(x))$ in terms of the data (see §5). Then, condition b) in proposition 4.1 is reduced to a finite number of linear equations in the coefficients of Q with the desired properties. $\diamondsuit$

Note. The hypothesis $\dim_{\mathbf{C}} \bigcap_{i=1}^{r} \{P_i(w', x_0) = 0\} = 0$ for all $x_0 \in \mathbf{C}^{n-r}$ in proposition 4.1, without some additional requirement such as semi-monicity, is not sufficient to get the polynomial behaviour of $P_\beta(x)$, as the following example shows:

Let $n = 2$, $r = 1$, $P_1 = P = w + w^2 x$, $Q = 1$, $\beta = 0$. Then,

$$R_{P(w,x)}[Q(w, x)](1) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

§5. The complexity of computing the matrices of the linear systems S_k

Given $P_1, \ldots, P_r \in \mathbf{C}[z_1, \ldots, z_n]$, denote $d_i := \deg(P_i)$, $D := \prod_{i=1}^{r} d_i$ and let $d \in \mathbf{N}_{\geq 3}$, $d \geq d_i$ for all $i = 1, \ldots, r$.

The algorithm described in [6], §1 solves the problem of finding a system of coordinates (w', x) in "Noether position" for $I(P_1, \ldots, P_r)$, with complexity bounds which are simply exponential in n. For any choice of such (w', x), it is possible to find polynomials F_i and A_{ij} $(i, j = 1, \ldots, r)$ verifying 3.1 i) and ii) with degrees bounded by $d^n(d^r+1)$ (cf. [6]) (These algorithms with simply exponential bounds are based on the affine versions of the effective Nullstellensatz after [1], see also [10]).

As a consequence, $P_\beta(x) = R_{P(w',x)}[Q(w',x)](w'^\beta)$ is a polynomial function of x for any polynomial Q (same proof as in theorem 3.3). We will next show a bound for its degree.

Lemma 5.1. *For each* $\beta \in \mathbf{N}_0^r$, $P_\beta(x) \equiv 0$ *or* $\deg_x(P_\beta(x)) \leq \deg(Q) + |\beta| + + rd^n(d^r+1)$.

Proof. Denote $A = (A_{ij})$. Then, $\deg(\det A) \leq rd^n(d^r+1)$ and $\det A \cdot Q(w',x) = \sum_{|\alpha+\gamma|\leq k} c_{\alpha\gamma} w'^\alpha x^\gamma$, where $k := rd^n(d^r+1) + \deg Q$.

As in theorem 3.3,

$$P_\beta(x) = R_{P(w',x)}[Q(w',x)](w'^\beta) = R_{F(w',x)}[\det A \cdot Q](w'^\beta) = = \sum_{|\alpha+\gamma|\leq k} c_{\alpha\gamma} x^\gamma R_F[w'^{\alpha+\beta}](1)$$

Now, for any $\mu = (\mu_1, \ldots, \mu_r) \in \mathbf{N}_0^r$, $R_F[w'^\mu](1)$ is computed by the iterated procedure:

$$R_{F_r(w_r,x)}[w_r^{\mu_r}](R_{F_{r-1}(w_{r-1},w_r,x)}[w_{r-1}^{\mu_{r-1}}](\ldots(R_{F_1(w',x)}[w_1^{\mu_1}](1))\ldots)$$

where $R_{F_1}[w_1^{\mu_1}](1) = 0$ or

$$\deg_{(w_2,\ldots,w_r,x)}(R_{F_1}[w_1^{\mu_1}](1)) \leq \mu_1 - \deg F_1 + 1 \quad \text{(see theorem 3.3)}.$$

Suppose w.l.o.g. $r = 2$ and $R_{F_1}[w_1^{\mu_1}](1) = \sum_{|\delta|+\ell\leq\mu_1-\deg F_1+1} d_{\delta\ell} x^\delta w_2^\ell$. Then,

$$R_{F_2}[w_2^{\mu_2}](R_{F_1}[w_1^{\mu_1}](1)) = \sum_{|\delta|+\ell\leq\mu_1-\deg F_1+1} d_{\delta\ell} x^\delta R_{F_2}[w_2^{\ell+\mu_2}](1)$$

Again, $R_{F_2}[w_2^{\ell+\mu_2}](1) \equiv 0$ or $\deg_x(R_{F_2(w_2,x)}[w_2^{\ell+\mu_2}](1)) \leq \ell + \mu_2 - \deg F_2 + 1$. Therefore, $\deg_x(R_{F_2}[w_2^{\mu_2}](R_{F_1}[w_1^{\mu_1}](1))) \leq \mu_1 - \deg F_1 + 1 + \mu_2 - \deg F_2 + 1$.

We get, for any $r \leq n$: $R_F[w^{\alpha+\beta}](1) \equiv 0$ or $\deg_x(x^\gamma R_F[w^{\alpha+\beta}](1)) \leq |\gamma| + |\alpha+\beta| - \sum_{i=1}^r \deg F_i + r \leq |\gamma| + |\alpha| + |\beta|$. Finally, $P_\beta(x) \equiv 0$ or $\deg_x(P_\beta(x)) \leq rd^n(d^r+1) + \deg(Q) + |\beta|$. $\diamond$

Remark 5.2. In fact, by the algorithms described in [6], §1, it is possible to find polynomials $F_i = F_i(w_i, x)$, $i = 1, \ldots, r$, depending only on $n-r+1$ variables with the same complexity bounds. If this is the case, $R_F[w'^\mu](1)$ becomes a product of r residues in a single variable.

Corollary 5.3. *Let $N = k+D-1+rd^n(d^r+1)$. The matrix of the linear system S_k associated to the polynomials $P_1, \ldots, P_r$ in 4.2 has at most $\binom{N+n-r}{n-r}\binom{D-1+r}{r}$ rows (and $\binom{n+k}{k}$ columns).*

In the particular case $r = n$, the number of equations of S_k is $\binom{D-1+n}{n}$, independently of k.

Proof. By lemma 5.1, $\deg_x(P_\beta(x)) \leq N$ for all $\beta \in \mathbf{N}_0^r$ verifying $|\beta| \leq D-1$. There are $\binom{D-1+r}{r}$ different such β, and each $P_\beta(x)$ has at most $\binom{N+n-r}{n-r}$ coefficients.

Remark 5.4. After corollary 5.3, the number of equations of the system S_k is bounded by $(k+D+rd^n(d^r+1))^{n-r} \cdot D^r$. Moreover, taking into account the complexity bounds in [6], §1, to get an appropiate coordinate system (w', x) together with polynomials F_i, A_{ij} $i, j = 1, \ldots, r$, and the proof of theorem 3.3 (which shows how to compute global residues), the matrix of S_k can be computed from the inputs $P_1, \ldots, P_r$ in sequential time $k^{O(n)} + d^{O(n^2)}$.

§6. Changing the field

When the coefficients of the polynomials $P_1, \ldots, P_r$ defining a complete intersection in $\mathbf{C}^n$ lie in some subfield K of $\mathbf{C}$ (e.g. $K = \mathbf{Q}$), all the entries of the matrices of the linear systems S_k also lie in K, as a consequence of the proof of theorem 3.3. The linear system S_k describes membership to the ideal generated by $P_1, \ldots, P_r$ in $K[x]$ up to degree k, for any $k \in \mathbf{N}$.

Given Grothendieck's notion of residues over an arbitrary algebraically closed field ([8]), it is natural to ask if our results are still valid in any field K with char $K = 0$. We thank the referee for bringing up this point. In fact, this seems to be the case although we have not checked the details. We leave it for the interested reader. Pertinent to this question are [8] and Angeniol, B.: Résidus et Effectivité. Preprint (1983).

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