BINOMIAL RESIDUES

EDUARDO CATTANI, ALICIA DICKENSTEIN AND BERND STURMFELS

ABSTRACT. A binomial residue is a rational function defined by a hypergeometric integral whose kernel is singular along binomial divisors. Binomial residues provide an integral representation for rational solutions of A-hypergeometric systems of Lawrence type. The space of binomial residues of a given degree, modulo those which are polynomial in some variable, has dimension equal to the Euler characteristic of the matroid associated with A.

1. INTRODUCTION

By a *binomial residue* we mean a rational function in $2n$ variables $x_1, \ldots, x_n,\, y_1, \ldots, y_n$, which is defined by a residue integral of the form

$$(1.1)\quad R_\Gamma(x,y) \;:=\; \int_\Gamma \frac{t^\gamma}{(x_1+t^{a_1}y_1)^{\beta_1}\cdots(x_n+t^{a_n}y_n)^{\beta_n}}\,\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_d}{t_d}.$$

Here $a_1, a_2, \ldots, a_n$ are non-zero lattice vectors which span $\mathbb{Z}^d$, γ is any vector in $\mathbb{Z}^d$, $\beta_1, \ldots, \beta_n$ are positive integers, and Γ ranges over a certain collection, specified in (3.7) below, of compact d-cycles in the torus $(\mathbb{C}^*)^d$. In this paper we study analytic, combinatorial, and geometric properties of binomial residues. On the analytic side, we view binomial residues as hypergeometric integrals [18, page 223] and, consequently, as rational solutions of a certain A-hypergeometric system of differential equations, in the sense of Gel'fand, Kapranov and Zelevinsky [11, 12]. The A-hypergeometric system annihilating (1.1) is the left ideal in the $2n$-dimensional Weyl algebra generated by the operators

$$(1.2)\quad \begin{array}{c} \partial_x^u\,\partial_y^v \;-\; \partial_x^v\,\partial_y^u \;\text{ whenever }\; u,v\in\mathbb{N}^n \;\text{ and }\; \sum_{i=1}^n u_i a_i = \sum_{i=1}^n v_i a_i, \\ x_i\partial_{x_i} + y_i\partial_{y_i} + \beta_i \quad\text{for}\quad i=1,2,\ldots,n, \qquad\text{and} \\ a_{j1}y_1\partial_{y_1} + a_{j2}y_2\partial_{y_2} + \cdots + a_{jn}y_n\partial_{y_n} + \gamma_j \quad\text{for}\quad j=1,2,\ldots,d. \end{array}$$

Here $\partial_x^u = \partial_{x_1}^{u_1}\cdots\partial_{x_n}^{u_n}$ for $u\in\mathbb{N}^n$. In the notation of [11, 12, 18], this is the system $H_A(-\beta,-\gamma)$ associated with the $(n+d)\times 2n$-matrix

$$(1.3)\qquad A \quad:=\quad \begin{pmatrix} I_n & I_n \\ 0\,0\cdots 0 & a_1\,a_2\,\cdots\,a_n \end{pmatrix},$$

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*AMS Subject Classification: Primary 33C60, Secondary 05B35, 14M25, 32A27 where I_n denotes the $n \times n$ identity matrix. The matrix A is called the *Lawrence lifting* of $a_1, a_2, \ldots, a_n$. Such matrices play an important role in combinatorics [3, §9.3] and Gröbner bases [17, §7, page 55].

We next introduce a combinatorial invariant associated with a configuration of vectors. For the Lawrence lifting A, this invariant agrees with that of the submatrix $M := (a_1, \ldots, a_n)$; see Remark 2.9. The *matroid complex* of M is the simplicial complex $\Delta(M)$ consisting of all subsets $I \subseteq \{1, \ldots, n\}$ such that the corresponding vectors $a_i, i \in I$ are linearly independent. Let $\chi(M)$ denote the Euler characteristic of the matroid complex $\Delta(M)$, i.e. the sum of $(-1)^{|I|}$ for $I \in \Delta(M)$. The integer $|\chi(A)| = |\chi(M)|$ equals the Möbius invariant of the dual matroid [2, Proposition 7.4.7] and, via Zaslavsky's Theorem [21], it counts the regions of the hyperplane arrangement (2.5). Lemma 2.11 implies

$$|\chi(A)| \quad \leq \quad \binom{n-1}{d}, \tag{1.4}$$

with equality if all d-tuples $\{a_{i_1}, \ldots, a_{i_d}\}$ are linearly independent.

We note that $\chi(A) = 0$ if and only if A has a *coloop*, i.e., some linear functional on $\mathbb{R}^d$ vanishes on all but one of the points $a_1, \ldots, a_n$. If this is the case, then every A-hypergeometric function is a monomial times a solution of a smaller system (1.2) gotten by *contracting the coloops.* Thus, we will assume without loss of generality that $\chi(A) \neq 0$.

A rational function f in $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ is called *unstable* if it is annihilated by some iterated derivative $\partial_x^u \, \partial_y^v$. Otherwise we say that f is *stable.* Thus f is unstable if it is a linear combination of rational functions that depend polynomially on at least one of the variables. We denote by $R(\beta, \gamma)$ the vector space of rational solutions of $H_A(-\beta, -\gamma)$, by $U(\beta, \gamma)$ the subspace of unstable rational solutions, and we set

$$\mathcal{S}(\beta, \gamma) \; := \; R(\beta, \gamma)/U(\beta, \gamma)\,. \tag{1.5}$$

Our main result gives an integral representation for stable rational A-hypergeometric functions, when A is the Lawrence configuration (1.3).

Theorem 1.1. *Let $\beta \in \mathbb{Z}^n_{>0}$ and $\gamma \in \mathbb{Z}^d$. The space $\mathcal{S}(\beta, \gamma)$ of stable rational A-hypergeometric functions of degree $(-\beta, -\gamma)$ has dimension $|\chi(A)|$ and is spanned by binomial residues $R_\Gamma(x, y)$.*

We illustrate this theorem with three examples. First consider $d = 1, n = 3, a_1 = a_2 = a_3 = 1 \in \mathbb{Z}^1$, $\beta_1 = \beta_2 = \beta_3 = 1$, and $\gamma = 3$. The Euler characteristic is $\chi(A) = -2$. The binomial residues are the integrals

$$\int_\Gamma \frac{t^3}{(x_1 + ty_1)(x_2 + ty_2)(x_3 + ty_3)} \frac{dt}{t}. \tag{1.6}$$

By integrating around the three poles $t = -x_i/y_i$, we obtain

$$R_1 = \frac{x_1^2}{(x_1y_2 - x_2y_1)(x_1y_3 - x_3y_1)y_1}$$
$$R_2 = \frac{x_2^2}{(x_3y_2 - x_2y_3)(x_1y_2 - x_2y_1)y_2}$$
$$R_3 = \frac{x_3^2}{(x_2y_3 - x_3y_2)(x_1y_3 - x_3y_1)y_3}$$

These residues form a solution basis for the hypergeometric system

$$\begin{aligned} H_A(-\beta,-\gamma) \;=\; \{ & \partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1},\, \partial_{x_1}\partial_{y_3} - \partial_{x_3}\partial_{y_1},\, \partial_{x_2}\partial_{y_3} - \partial_{x_3}\partial_{y_2}, \\ & x_1\partial_{x_1} + y_1\partial_{y_1} + 1,\; x_2\partial_{x_2} + y_2\partial_{y_2} + 1,\; x_3\partial_{x_3} + y_3\partial_{y_3} + 1, \\ & y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3} + 3 \}. \end{aligned}$$

This is the *Aomoto-Gel'fand system* for a 2×3-matrix, which is holonomic of rank 3; see [18, §1.5]. The space $\mathcal{S}(\beta,\gamma)$ of rational solutions modulo unstable rational solutions has dimension $2 = |\chi(A)|$, since

$$R_1 + R_2 + R_3 \;=\; \frac{1}{y_1y_2y_3}$$

contains no x_i and is hence unstable. This identity expresses the fact that the sum of all local residues of a rational 1-form over $\mathbb{P}^1$ is zero.

Our second example is the Lawrence lifting of the twisted cubic curve:

$$(1.7) \qquad d = 2,\, n = 4, \qquad A \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Fix $\beta = (1,1,1,1)$ and $\gamma = (1,1)$. The space of A-hypergeometric functions is 10-dimensional, and the subspace of rational solutions is 3-dimensional. A basis for $R(\beta,\gamma)$ consists of the three binomial residues

$$R_{23} = \frac{x_2y_3^2y_2}{(x_3^2y_2y_4 - x_2x_4y_3^2)(x_1x_3y_2^2 - x_2^2y_1y_3)}$$
$$R_{24} = \frac{y_4y_2(x_2^2y_3y_1 + x_1x_3y_2^2)}{(x_3^2y_2y_4 - x_2x_4y_3^2)(x_2^3y_1^2y_4 - x_1^2x_4y_2^3)}$$
$$R_{34} = \frac{x_3x_4y_3^3y_4}{(y_2y_4x_3^2 - y_3^2x_2x_4)(y_3^3x_4^2x_1 - x_3^3y_1y_4^2)}$$

Other residues can be computed by the *Orlik-Solomon relations* (cf. §5):

$$R_{14} = -R_{24} - R_{34}, \quad R_{13} = -R_{23} + R_{34}, \quad R_{12} = R_{23} + R_{24}.$$

For our third example take $\{a_1, \ldots, a_n\}$ to be the positive roots in the root system of type A_d. This means $n = \binom{d}{2}$ and (1.1) looks like

$$\int_\Gamma \frac{t_1^{\gamma_1} \cdots t_{d-1}^{\gamma_{d-1}}}{\prod_{1 \leq i < j \leq d} (x_{ij} + t_i t_j^{-1} y_{ij})^{\beta_{1j}}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{d-1}}{t_{d-1}},$$

where $t_d = 1$. This is the *Selberg type integral* studied by Kaneko [14] and many others; see [18, Example 5.4.7]. The holonomic rank of the associated A-hypergeometric system equals d^{d-2}, the number of labeled trees on d vertices. The following explicit formula for the number of stable rational hypergeometric functions of Selberg type is given in [15]:

$$|\chi(A_d)| \quad = \quad (d-2) \cdot \sum_{k=0}^{[(d-3)/2]} \binom{d-3}{2k} (d-1)^{d-3-2k} \cdot \prod_{i=1}^{k} (2i-1).$$

This paper is organized as follows. In §2 we examine hypergeometric Laurent series solutions, and we derive the upper bound in Theorem 1.1. In §3 we establish the connection to toric geometry, by expressing binomial residues as *toric residues* in the sense of Cox [10]; see also [5, 6]. Formulas and algorithms for computing binomial residues are presented in §4. In §5, we complete the proof of Theorem 1.1, and we prove Conjecture 5.7 from our previous paper [8] in the Lawrence case.

2. Laurent series expansions and Gale duality

In this section we establish the upper bound in Theorem 1.1 for arbitrary rational A-hypergeometric functions. The Lawrence hypothesis is not needed for this. The main idea is to look at series expansions, which leads to counting cells in a hyperplane arrangement. We fix an arbitrary integer $r \times s$-matrix A of rank r and an integer vector $\alpha \in \mathbb{Z}^r$.

Definition 2.1. [11, 12, 18]. The *A-hypergeometric system* is the left ideal $H_A(\alpha)$ in the Weyl algebra $\mathbb{C}\langle x_1, \ldots, x_s, \partial_1, \ldots, \partial_s \rangle$ generated by the *toric operators* $\partial^u - \partial^v$, for $u, v \in \mathbb{N}^s$ such that $A \cdot u = A \cdot v$, and the *Euler operators* $\sum_{j=1}^s a_{ij} x_j \partial_j - \alpha_i$ for $i = 1, \ldots, r$. A function $f(x_1, \ldots, x_s)$, holomorphic on an open set $U \subset \mathbb{C}^s$, is said to be *A-hypergeometric of degree α* if it is annihilated by the left ideal $H_A(\alpha)$.

A rational A-hypergeometric function admits Laurent series expansions convergent in a suitable open set. In the terminology of [18] these are *logarithm-free hypergeometric series* with integral exponents. We review their construction and refer to [18, §3.4] for proofs and details.

Given a vector $v \in \mathbb{C}^s$, we define its *negative support* by

$$\mathrm{nsupp}(v) \; := \; \left\{ \, i \in \{1, \ldots, s\} \; : \; v_i \in \mathbb{Z}_{<0} \, \right\}$$

A vector $v \in \mathbb{C}^s$ is said to have *minimal negative support* if there is no integer vector u in the kernel of A such that $\mathrm{nsupp}(u+v)$ is properly contained in $\mathrm{nsupp}(v)$. The following set of integer vectors,

$$N_v \;:=\; \{u \in \ker_{\mathbb{Z}}(A) : \mathrm{nsupp}(u+v) = \mathrm{nsupp}(v)\},$$

is used to define the formal Laurent series

$$\phi_v(x) \quad := \quad \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} \cdot x^{v+u} \tag{2.1}$$

where $[v]_{u_-} = \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (v_i - j + 1)$ and $[v+u]_{u_+} = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (v_i + j)$.

The Weyl algebra acts on formal Laurent series by multiplication and differentiation. The following is Proposition 3.4.13 in [18]:

Proposition 2.2. *Let $\alpha = A \cdot v$. The series $\phi_v(x)$ is annihilated by $H_A(\alpha)$ if and only if the vector $v \in \mathbb{C}^s$ has minimal negative support.*

In order to ensure that the A-hypergeometric series $\phi_v(x)$ have a common domain of convergence, we fix a generic weight vector $w \in \mathbb{R}^s$. A vector $v \in \mathbb{C}^s$ is called an *exponent for $H_A(\alpha)$ with respect to w* if v has minimal negative support and

$$A \cdot v = \alpha \qquad \text{and} \qquad \langle w, v\rangle \;=\; \min\{\, \langle w, u\rangle \;:\; u \in v + N_v \,\}. \tag{2.2}$$

The following is a restatement of [18, Theorem 3.4.14, Corollary 3.4.15]:

Theorem 2.3. *The set $\big\{ \phi_v \;:\; v \in \mathbb{Z}^s \text{ and } v \text{ is an exponent} \big\}$ is a basis for the space of hypergeometric functions of degree α admitting a Laurent expansion convergent in a certain open subset U_w of $\mathbb{C}^s$.*

For a more precise description of hypergeometric Laurent series, we next introduce the oriented hyperplane arrangement defined by the Gale dual (or matroid dual) to A. Set $m := s - r$ and let B be an integral $s \times m$ matrix whose columns are a $\mathbb{Z}$-basis of $\ker_{\mathbb{Z}}(A)$. The matrix B has rank m and $A \cdot B = 0$. Note that B is well-defined modulo right multiplication by elements of $GL(m, \mathbb{Z})$. We identify B with its set of row vectors, and we call this configuration the *Gale dual* of A:

$$B \;=\; \{b_1, \ldots, b_s\} \;\subset\; \mathbb{Z}^m .$$

Our assumption $\chi(A) \neq 0$ translates into the condition $b_j \neq 0$ for all $j = 1, \ldots, s$. As remarked in the Introduction, the study of A-hypergeometric functions, for arbitrary A, easily reduces to this case.

Fix an exponent $v \in \mathbb{Z}^s$. We identify the lattice $\mathbb{Z}^m$ with the sublattice $\text{image}_{\mathbb{Z}}(B) + v = \ker_{\mathbb{Z}}(A) + v$ of $\mathbb{Z}^s$ via the affine isomorphism $\lambda \mapsto B \cdot \lambda + v$. Under this identification, the affine hyperplane

$$\{\lambda \in \mathbb{R}^m \ : \ \langle b_j, \lambda \rangle = -v_j\} \tag{2.3}$$

corresponds to the coordinate hyperplane $x_j = 0$ in $\ker_{\mathbb{Z}}(A) + v \subset \mathbb{Z}^s$. Let $\mathcal{H}$ denote the arrangement in $\mathbb{R}^m$ consisting of the hyperplanes (2.3) for $j = 1, \ldots, s$. We define the *negative support* of a vector λ in $\mathbb{R}^m$ as the negative support of its image under the above isomorphism:

$$\text{nsupp}(\lambda) \quad := \quad \big\{ j \in \{1, 2, \ldots, s\} \ : \ \langle b_j, \lambda \rangle \ < \ -v_j \big\}.$$

The set of points with the same negative support will be called a *cell* of the hyperplane arrangement $\mathcal{H}$. Note that our definition of cell differs slightly from the familiar subdivision into relatively open polyhedra by the hyperplanes in $\mathcal{H}$. Our cells are unions of these: they are also polyhedra but they are usually not relatively open.

Consider the following attributes of a cell Σ in $\mathcal{H}$. We say that:

- Σ is *bounded* if Σ is a bounded subset of $\mathbb{R}^m$.
- Σ is *minimal* if the set $\Sigma \cap \mathbb{Z}^s$ is nonempty and the support of the elements in this set is minimal with respect to inclusion.
- Σ is *w-positive*, for a given vector w on $\mathbb{R}^n$, if there exists a real number ρ such that $\langle w, \lambda \rangle \geq \rho$ for all $\lambda \in \Sigma$.

We can now rewrite the hypergeometric series (2.1) as follows:

$$\phi_\Sigma \quad := \quad \phi_v \quad = \quad \sum_{\lambda \in \Sigma \cap \mathbb{Z}^m} \frac{[v]_{(B\lambda)_-}}{[v + B\lambda]_{(B\lambda)_+}} \cdot x^{B\lambda + v} \tag{2.4}$$

If Σ is bounded then ϕ_Σ is a Laurent polynomial, and if Σ is w-positive then ϕ_Σ lies in the Nilsson ring (cf. [18, §3.4]) associated with w, and hence defines an A-hypergeometric function on U_w when Σ is minimal.

The following is an immediate consequence of Theorem 2.3:

Proposition 2.4. *The series ϕ_Σ where Σ runs over all w-positive minimal cells in $\mathcal{H}$ form a basis for the space of A-hypergeometric functions of degree α admitting a Laurent expansion convergent in $U_w \subset \mathbb{C}^s$. Restricting to bounded cells Σ, we get a basis for the subspace of hypergeometric Laurent polynomials.*

Recall that we say that a rational function f is *unstable* if there exists $u \in \mathbb{N}^s$ such that the partial derivative $\partial^u(f)$ is identically zero. We extend this terminology to functions admitting a Laurent series expansion.

Lemma 2.5. *A-hypergeometric Laurent polynomials are unstable.*

Proof. By Proposition 2.4, it is enough to show that for any bounded minimal chamber Σ, the common negative support of all monomials in ϕ_Σ does not equal $\{1,\ldots,s\}$. In fact, suppose

$$\Sigma \quad = \quad \big\{\lambda \in \mathbb{R}^m \,:\, \langle b_j, \lambda\rangle \;<\; -v_j \text{ for all } j=1,2,\ldots,m\,\big\}.$$

The negative support of any lattice point in $\mathbb{Z}^m \backslash \Sigma$ is a proper subset of $\{1,2,\ldots,n\}$, and we conclude that Σ is not minimal. $\square$

If we differentiate an A-hypergeometric function of degree α with respect to x_i then we get an A-hypergeometric function of degree $\alpha - a_i$. If we iterate this process long enough, for all variables, then only the stable functions survive. The following definition is intended to make this more precise. The *Euler-Jacobi cone* is the open cone in $\mathbb{R}^d$:

$$-\mathrm{Int}(\mathrm{pos}(A)) \;=\; \big\{\nu_1 a_1 + \nu_2 a_2 + \cdots + \nu_s a_s \,:\, \nu_i \in \mathbb{R}_{<0} \;\text{ for all }\; i\,\big\}.$$

Note that $(-\beta,-\gamma)$ lies in the Euler-Jacobi cone in Example (1.7).

Proposition 2.6. *If $\alpha \in -\mathrm{Int}(\mathrm{pos}(A))$, then every A-hypergeometric series of degree α is stable.*

Proof. Suppose there is an unstable A-hypergeometric series. Then, by Theorem 2.3, it can be written as a linear combination of series ϕ_v as in (2.1) with minimal negative support. Since negative supports are preserved under derivation, it is then sufficient to show that a hypergeometric series ϕ_v cannot be unstable. Fix a strictly negative vector $\nu \in \mathbb{Q}^s_{<0}$ with $A\nu = Av = \alpha$. Let k be a positive integer such that $k\nu \in \mathbb{Z}^s$. For each integer $\ell \in \mathbb{N}$, the vector $v + \ell(v-\nu)$ has negative support contained in $\mathrm{nsupp}(v)$. Since v is minimal, we conclude that $\mathrm{nsupp}(v+\ell(v-\nu)) = \mathrm{nsupp}(v)$ for all $\ell \in N$. Let $I := \{i \in \{1,\ldots,s\} \,:\, v_i \geq 0\}$. For all $i \in I$, we have $v_i > \nu_i$, and so all the coordinates in I of the vectors $v + \ell(v-\nu)$ strictly increase with ℓ. This shows that ϕ_v cannot be decomposed as a finite sum of Laurent series that depend polynomially on one variable. $\square$

Theorem 2.7. *If $\alpha \in -\mathrm{Int}(\mathrm{pos}(A))$, then the dimension of the space of A-hypergeometric Laurent series of degree α with a common domain of convergence is bounded above by the Euler characteristic $|\chi(A)|$.*

Proof. Consider the central hyperplane arrangement gotten from $\mathcal{H}$ by translating all s hyperplanes so as to pass through the origin. This central arrangement consists of the s hyperplanes

$$(2.5) \qquad \big\{\,\lambda \in \mathbb{R}^m \;:\; \langle b_j, \lambda\rangle \;=\; 0\,\big\} \qquad \text{for } j=1,2,\ldots,s.$$

Since α is in the Euler-Jacobi cone, the minimal cells Σ of $\mathcal{H}$ are all unbounded and correspond to certain maximal cones of the central arrangement (2.5). Fix a generic linear functional w on $\mathbb{R}^m$. A basis for the relevant space of A-hypergeometric Laurent series is indexed by the w-bounded, minimal cells of $\mathcal{H}$. Their number is bounded above by the number of w-bounded maximal cones in the central arrangement.

A classical result in combinatorics due to Zaslavsky [21] states that the number of w-bounded maximal cones is the absolute value of the Möbius invariant $\mu(B)$ of the matroid associated with B. Our assertion now follows from the following identity from [2, Proposition 7.4.7 (i)]:

$$|\mu(B)| \quad = \quad |\chi(A)|. \tag{2.6}$$

In words, the Möbius invariant of a matroid equals (up to sign) the Euler characteristic of the dual matroid. □

The following Corollary implies the upper bound in Theorem 1.1:

Corollary 2.8. *For any $\alpha \in \mathbb{Z}^d$, the complex vector space of rational A-hypergeometric functions of degree α modulo the subspace of unstable functions has dimension at most $|\chi(A)|$.*

Proof. We represent the rational A-hypergeometric functions by Laurent series expansions which have a common domain of convergence. Hence it suffices to prove the asserted dimension bound for the space of convergent A-hypergeometric Laurent series modulo unstable ones.

Choose $u \in \mathbb{N}^s$ so that $\alpha - Au$ lies in the Euler-Jacobi cone. The operator ∂^u induces a monomorphism from $\mathcal{S}(\alpha)$ into $\mathcal{S}(\alpha - Au)$. By Proposition 2.6, $\mathcal{S}(\alpha - Au) \cong R(\alpha - Au)$, hence the dimension bound follows from Theorem 2.7 applied to $\alpha - Au$. □

Remark 2.9. Passing from $\{a_1, \ldots, a_n\}$ to its Lawrence lifting (1.3) corresponds under Gale duality to the operation of replacing $\{b_1, \ldots, b_n\}$ by its symmetrization $\{b_1, \ldots, b_n, -b_1, \ldots, -b_n\}$; see [3, Proposition 9.3.2]. This process does not change the geometry of the hyperplane arrangement (2.5) and hence it does not change the Möbius invariant $\mu(B)$. In view of (2.6), we conclude that the Euler characteristic of $\{a_1, \ldots, a_n\}$ equals the Euler characteristic of its Lawrence lifting as stated in the Introduction.

Corollary 2.10. *The space $\mathcal{S}(\beta, \gamma)$ has dimension at most $|\chi(A)|$*

We conclude this section with one more result from matroid theory which we need to complete the proof of Theorem 1.1. A maximally independent subset of B is a *basis* of B. Note that $\{b_j : j \in J\}$ is a basis of B if and only if $\{a_j : j \notin J\}$ is a basis of A. A minimally-dependent subset of B is a *circuit* of B. If $C = \{b_{i_1}, \ldots, b_{i_t}\}$ is a circuit and $i_1 < \cdots < i_t$ then the set $C \backslash \{b_{i_t}\}$ is a *broken circuit.* A basis of B is called an *nbc-basis* if it contains no broken circuits.

Lemma 2.11. *The number of nbc-bases of B equals $|\chi(A)|$.*

Proof. This result follows from (2.6) and Proposition 7.4.5 in [2]. □

3. Binomial residues and toric geometry

This section is concerned with global residues of meromorphic forms whose polar divisor is a union of hypersurfaces defined by binomials. The analogous case when the polar divisor is defined by linear forms has been extensively studied, for instance, by Varchenko [20] and Brion-Vergne [4]. Our situation can be regarded as a multiplicative analogue to that theory. The binomial hypersurfaces are embedded in a suitable projective toric variety, which places binomial residues into the framework of toric residues [5, 6, 10]. This will allow us in §5 to find bases of A-hypergeometric stable rational functions for Lawrence liftings in terms of binomial residues, and to give a geometric meaning to the linear dependencies among binomial residues. We refer to [13, 19] for the definition and basic properties of Grothendieck residues.

Let X be a complete d-dimensional toric variety and S its homogeneous coordinate ring in the sense of Cox [9]. Let $T \simeq (\mathbb{C}^*)^d$ denote the dense torus in X. The zero-locus of a homogeneous polynomial in S is a Weil divisor in X. Suppose $G_0, G_1, \ldots, G_d$ are homogeneous polynomials in S whose divisors D_i satisfy

$$D_0 \cap D_1 \cap \cdots \cap D_d = \emptyset. \tag{3.1}$$

Any homogeneous polynomial H of critical degree determines a meromorphic d-form on X with polar locus contained in $D_0 \cup \cdots \cup D_d$,

$$\Phi(H) \quad = \quad \frac{H\,\Omega_X}{G_0 G_1 \cdots G_d},$$

where Ω_X is a choice of an Euler form on X [1]. The d-form $\Phi(H)$ defines a Čech cohomology class $[\Phi(H)] \in H^d(X, \widehat{\Omega}^d_X)$ relative to the open cover $\{X \backslash D_i\}_{i=0,\ldots,d}$ of X. Here $\widehat{\Omega}^d_X$ denotes the sheaf of Zariski d-forms on X [10]. The class $[\Phi(H)]$ is alternating with respect to permutations of $G_0, \ldots, G_d$. If H lies in the ideal $\langle G_0, \ldots, G_d\rangle$ of S then $\Phi(H)$ is a Čech coboundary. Thus, $[\Phi(H)]$ depends only on the image of the polynomial H in the quotient ring $S/\langle G_0, \ldots, G_d\rangle$.

The *toric residue* $\mathrm{Res}^X_G(\Phi(H)) \in \mathbb{C}$ is given by the formula

$$\mathrm{Res}^X_G(\Phi(H)) \;=\; \mathrm{Tr}_X([\Phi(H)]),$$

where $\mathrm{Tr}_X : H^d(X, \widehat{\Omega}^d_X) \to \mathbb{C}$ is the trace map.

The following proposition can be deduced from Stokes Theorem (cf. [13], [19, §7.2]). It follows directly from the definition of toric residue.

Proposition 3.1. *If the polar locus of the d-form $\Phi(H)$ is contained in the union of only d divisors, say $D_1 \cup \cdots \cup D_d$, then* $\mathrm{Res}_G^X(\Phi(H)) = 0$.

The relationship between toric residues and the usual notion of multidimensional residues is given by the following result.

Theorem 3.2. *Let $G_0, \ldots, G_d \in S$ satisfy (3.1) and suppose*

$$V^0 := D_1 \cap \cdots \cap D_d \;\subset\; T \tag{3.2}$$

Then

$$\mathrm{Res}_G^X(\Phi) \;=\; \sum_{\xi \in V^0} \mathrm{Res}_\xi(\Phi|_T) \tag{3.3}$$

where $\mathrm{Res}_\xi(\Phi|_T)$ denotes the (local) Grothendieck residue at ξ of the meromorphic form Φ restricted to the torus and relative to the divisors $D_1 \cap T, \ldots, D_d \cap T$.

Proof. We note, first of all, that (3.1) implies that V^0 is a finite set and hence the sum in (3.3) makes sense. Moreover, as shown in [19, §II.7.2], the local residues in the right-hand side of (3.3) depend only on the divisors $D_i \cap T$ and not on the choice of local defining equations.

If X is simplicial, then (3.3) is the content of Theorem 0.4 in [5]. For general X we argue as in the proof of Theorem 4 in [6]. □

We consider now the binomial case which is relevant in this paper. Let $a_1, \ldots, a_n \in \mathbb{Z}^d$ as in the Introduction. Let Δ_i denote the segment $[0, a_i] \subset \mathbb{R}^d$ and $\Delta = \Delta_1 + \cdots + \Delta_n$ their Minkowski sum. This is a *zonotope*, that is, a polytope all of whose faces are centrally symmetric [3, §2.2]. Let $\eta_1, \ldots, \eta_{2p}$ denote the inner normals of the facets of the zonotope Δ, where $\eta_j = -\eta_{p+j}$. We can write

$$\Delta \;=\; \Big\{\, m \in \mathbb{R}^d \;:\; \langle m, \eta_j \rangle \geq \sum_{i : \langle \eta_j, a_i \rangle < 0} \langle \eta_j, a_i \rangle \,;\; j = 1, \ldots, 2p \,\Big\}$$

We consider the associated projective toric variety X_Δ. The homogeneous coordinate ring of X_Δ is the polynomial ring $S = \mathbb{C}[z_1, \ldots, z_{2p}]$. The monomials $t_j := \prod_{i=1}^{2p} \big(z_i^{\eta_{ij}}\big)$, for $j = 1, 2, \ldots, d$, have degree zero and define coordinates in the torus $T \subset X_\Delta$.

To each binomial $f_i := x_i + y_i t^{a_i}$ in the denominator of the kernel of (1.1) we associate the homogeneous polynomial

$$F_i(z) \;:=\; x_i \prod_{\langle \eta_j, a_i \rangle < 0} z_j^{-\langle \eta_j, a_i \rangle} + y_i \prod_{\langle \eta_j, a_i \rangle > 0} z_j^{\langle \eta_j, a_i \rangle}.$$

The divisor $Y_i := \{F_i(z) = 0\} \subset X_\Delta$ is the closure of the divisor $\{f_i(t) = 0\} \subset T$. Moreover, for $\beta \in \mathbb{Z}^n_{>0}$ and $\gamma \in \mathbb{Z}^d$, the d-form on T,

$$\phi(\beta,\gamma) \quad = \quad \frac{t^\gamma}{f_1^{\beta_1}\cdots f_n^{\beta_n}}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_d}{t_d}, \tag{3.4}$$

extends to the following meromorphic d-form on the toric variety X_Δ:

$$\Phi(\beta,\gamma) \quad = \quad \frac{z^{h(\beta,\gamma)}}{F_1^{\beta_1}\cdots F_n^{\beta_n}}\,\Omega_\Delta, \tag{3.5}$$

$$\text{where}\quad h_j(\beta,\gamma) \quad = \quad \langle \eta_j,\gamma\rangle - \sum_{\langle\eta_j,a_i\rangle<0}\langle\eta_j,\beta_i a_i\rangle - 1,\quad j=1,\dots,2p.$$

The polar divisor of $\Phi(\beta,\gamma)$ is the union of the divisors $Y_1,\dots,Y_n$ and coordinate divisors $\{z_\ell = 0\}$ for indices ℓ with $h_\ell(\beta,\gamma) < 0$. For degrees in the Euler-Jacobi cone such indices ℓ do not exist. Indeed,

$$-\mathrm{Int}(\mathrm{pos}(A)) = \big\{(-\beta,-\gamma)\in\mathbb{R}^{n+d}\ :\ \beta_i > 0;\ h_j(\beta,\gamma)+1>0\big\} \tag{3.6}$$

Thus, if $(-\beta,-\gamma)$ lies in the Euler-Jacobi cone, the polar divisor of $\Phi(\beta,\gamma)$ equals $Y_1\cup\cdots\cup Y_n$.

We are now prepared to give a precise definition of binomial residues. Fix an index set $I = \{1 \le i_1 < \cdots < i_d \le n\}$ such that the corresponding vectors a_i, $i \in I$, are linearly independent. For $k = 1,\dots,d$, set $G^I_k = F_{i_k}$ and $D_k = \{G^I_k = 0\}$. For generic values of the coefficients x_i, y_i, $i \in I$, the divisors $D_1,\dots,D_d$ satisfy (3.2).

Definition 3.3. For $\beta \in \mathbb{Z}^n_{>0}$ and $\gamma \in \mathbb{Z}^d$, let

$$G^I_0 \ = \ \Big(\prod_{\ell:h_\ell(\beta,\gamma)<0} z_\ell\Big)\ \cdot\ \Big(\prod_{j\notin I} F_j\Big).$$

Define the following quantity which depends on $x_1,\dots,x_n,y_1,\dots,y_n$:

$$R_I(\beta,\gamma) \quad := \quad \mathrm{Res}^X_{G^I}(\Phi(\beta,\gamma)).$$

Each local residue in the right-hand side of (3.3) may be written as an integral over a d-cycle "around" the point $\xi \in V^0$. Since for generic values of the coefficients, the map $f_I = (f_{i_1},\dots,f_{i_d})\colon T \to \mathbb{C}^d$ is proper, it follows from [19, §II.8] that the total sum of residues (3.3) may be written as a single integral,

$$R_I(\beta,\gamma) \quad = \quad \left(\frac{1}{2\pi i}\right)^d \int_{\Gamma(I,x,y)} \frac{t^\gamma}{f_1^{\beta_1}\cdots f_n^{\beta_n}}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_d}{t_d}, \tag{3.7}$$

where $\Gamma(I,x,y)$ is the compact real d-cycle $\Gamma(I,x,y) \subset T$ defined by $\{|f_{i_1}| = \varepsilon_1,\dots,|f_{i_d}| = \varepsilon_d\}$ for small positive $\varepsilon_1,\dots,\varepsilon_d$. Moreover, the cycle $\Gamma(I,x,y)$ can be locally replaced by a cohomologous cycle $\Gamma(I)$ independent of $(x_1,\dots,x_n,y_1,\dots,y_n)$. See [18, §5.4] for further details.

We close this section with the observation that the "basic binomial residue" $R_I(\beta,\gamma)$ is indeed a rational A-hypergeometric function.

Lemma 3.4. *The toric residue $R_I(\beta,\gamma)$ is a rational function of (x,y) and is annihilated by the hypergeometric system (1.2).*

Proof. For any choice of polynomials $G_0,\dots,G_d$, the trace map Tr_X in the definition of the toric residue has its image in the subfield of $\mathbb{C}$ generated by the coefficients of the G_i. This implies that $R_I(\beta,\gamma)$ is an element in the rational function field $\mathbb{Q}(x_1,\dots,x_n,y_1,\dots,y_n)$.

The kernel of the integral (3.4) is annihilated by the toric operators $\partial_x^u\,\partial_y^v - \partial_x^v\,\partial_y^u$ in (1.2). Hence so is the integral itself, by differentiating under the integral sign. Specifically, it follows from [6, Lemma 6] that

$$\partial_{x_i} R_I(\beta,\gamma) \;=\; -\beta_i\, R_I(\beta+e_i,\gamma), \quad \text{and} \tag{3.8}$$

$$\partial_{y_i} R_I(\beta,\gamma) \;=\; -\beta_i\, R_I(\beta+e_i,\gamma+a_i), \tag{3.9}$$

where $e_1,\dots,e_d$ is the standard basis of $\mathbb{R}^d$. The verification of the homogeneity equations is immediate from the expression (3.4) for the form $\phi(\beta,\gamma)$. Hence $R_I(\beta,\gamma)$ is a rational solution of $H_A(-\beta,-\gamma)$. □

4. Computing binomial residues

In this section we present methods for computing the binomial residue $R_I(\beta,\gamma)$. Here $I=\{i_1,\dots,i_d\}$ is a fixed column basis of the matrix $M=(a_1,\dots,a_n)$. Let M_I denote the non-singular $d\times d$ matrix with columns a_i, $i\in I$. Write $M_I^{-1}=(\mu_{ij})\in GL(d,\mathbb{Q})$. We set $V_I=\{\xi\in T : f_i(\xi)=0 \text{ for all } i\in I\}$. The points in V_I are in bijection with the characters $\theta\in\mathrm{Hom}(\mathbb{Z}^d,\mathbb{C}^*)$ satisfying $\theta(a_i)=-1$, for all $i\in I$. The point $\xi^\theta=(\xi_1^\theta,\dots,\xi_d^\theta)\in V_I$ indexed by θ has coordinates

$$\xi_j^\theta \;=\; \theta(e_j)\cdot\prod_{i\in I}\left(\frac{x_i}{y_i}\right)^{\mu_{ij}}$$

There are $\det(M_I)$-many simple roots ξ^θ provided all x_i,y_i are nonzero.

Let g be a function meromorphic on the torus $T=(\mathbb{C}^*)^d$ and regular at a simple root $\xi\in V_I$. Then the local Grothendieck residue of the meromorphic d-form $\frac{g}{f_{i_1}\cdots f_{i_d}}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_d}{t_d}$ at the point ξ equals

$$R_{I,\xi}[g] \;=\; \frac{g(\xi)}{J_I(\xi)} \tag{4.1}$$

where J_I denotes the *toric Jacobian* of the binomials $f_i = x_i + y_i t^{a_i}$:

$$J_I(t) \;=\; \det\left(t_j \frac{\partial f_i}{\partial t_j}\right)_{i\in I}^{j=1,\dots,d} \;=\; \det M_I \cdot (\prod_{i\in I} y_i) \cdot t^{a_I}.$$

Here $a_I = a_{i_1} + \cdots + a_{i_d}$. We deduce the following identity

$$J_I(\xi) \;=\; (-1)^d \cdot \det M_I \cdot (\prod_{i\in I} x_i) \qquad \text{for all } \xi \in V_I. \tag{4.2}$$

We obtain the following procedure for summing (4.1) over all $\xi \in V_I$.

Algorithm 4.1. *(Computing global residues using Gröbner bases)*
Input: A $d\times d$-integer matrix M_I of rank d, a Laurent polynomial $g(t)$.
Output: The *global residue*

$$R_I[g] := \sum_{\xi\in V_I} R_{I,\xi}[g]$$

(1) Fix the field $K = \mathbb{Q}(x_1,\dots,x_n,y_1,\dots,y_n)$ and write the Laurent polynomial ring over K as a quotient of a polynomial ring:
$$K[t_1,\dots,t_d,t_1^{-1},\dots,t_d^{-1}] \;=\; K[t_0,t_1,\dots,t_d]/\langle t_0t_1\cdots t_d - 1\rangle.$$
(2) Compute any Gröbner basis G for the ideal $\langle f_{i_1},\dots,f_{i_d}\rangle$.
(3) Let B be the set of standard monomials for G in $K[t_0,\dots,t_d]$
(4) Compute the *trace* of g modulo B as follows:
$$\sum_{\xi\in V_I} g(\xi) \;=\; \sum_{t^b\in B} \mathrm{coeff}_{t^b}\big(\mathrm{normalform}_G(t^b\cdot g(t))\big)$$
(5) Output the result of step (4) divided by the monomial in (4.2).

The output produced by the above algorithm is a rational function in x_i, y_i and the coefficients of g. In the case when g is a Laurent monomial, one can give a completely explicit formula for that output.

Lemma 4.2. *Let $\gamma \in \mathbb{Z}^d$. If $\nu = M_I^{-1}\cdot\gamma$ lies in the lattice $\mathbb{Z}^d$ then*

$$R_I[t^\gamma](x,y) \;=\; (-1)^{|\nu|+d}\cdot \prod_{i\in I} x_i^{\nu_i-1}\, y_i^{-\nu_i}. \tag{4.3}$$

Otherwise the global residue $R_I[t^\gamma]$ is zero.

Proof. It follows from (4.1) and (4.2) that

$$R_{I,\xi^\theta}[t^\gamma](x,y) \;=\; \theta(\gamma)\cdot\frac{(-1)^d}{\det(M_I)}\cdot\prod_{i\in I} x_i^{\nu_i-1}\, y_i^{-\nu_i}$$

where $\nu_i := \sum_{j=1}^{d} \mu_{ij}\gamma_j$, $i \in I$. Thus, the global residue is given by

$$R_I[t^\gamma](x,y) \;=\; \left(\sum_\theta \theta(\gamma)\right) \cdot \frac{(-1)^d}{\det(M_I)} \cdot \prod_{i\in I} x_i^{\nu_i - 1}\, y_i^{-\nu_i}$$

and consequently it vanishes unless $\gamma \in M_I \cdot \mathbb{Z}^d$. In this case we have (4.3) for $\gamma = \sum_{i\in I} \nu_i a_i$, and $|\nu| := \sum_{i\in I} \nu_i$. □

We now compute the binomial residue $R_I(\beta,\gamma)$ for $I = \{i_1,\dots,i_d\}$ as above. In view of (3.8) and (3.9), it suffices to consider the case $\beta = \mathbf{1} := (1,\dots,1)$. Set $J := \{1,\dots,n\}\backslash I$ and let M_J denote the matrix whose columns are the vectors $a_j, j \in J$. Since the coefficients are generic, none of the polynomials f_j, $j \in J$ vanishes on any point of V_I and hence

$$R_I(\mathbf{1},\gamma) \;=\; R_I[\, t^\gamma / f_J(t)\,](x,y) \qquad \text{where} \quad f_J(t) = \prod_{j\in J} f_j(t). \tag{4.4}$$

This gives rise to the following symbolic algorithm for binomial residues.

Algorithm 4.3. *(Computing binomial residues)*
Input: Vectors $a_1,\dots,a_n$ and γ as above, and a basis $I = \{i_1,\dots,i_d\}$.
Output: The rational function $R_I(\mathbf{1},\gamma)$ of $x_1,\dots,x_n,y_1,\dots,y_n$.

(1) Run steps (1), (2) and (3) of Algorithm 4.1.
(2) Using linear algebra over the field K, compute the unique polynomial $g(t) = \sum_{t^b\in B} c_b \cdot t^b$ such that all c_b lie in K and $g(t)\cdot f_J(t) - t^\gamma$ reduces to zero modulo the Gröbner basis G.
(3) Run steps (4) and (5) of Algorithm 4.1.

The output of this algorithm is an element of the field K. It is nonzero and has the following expansion as a Laurent series in x_i, y_i.

Proposition 4.4. *Suppose $\gamma \in M \cdot \mathbb{Z}^n$. Then $R_I(\mathbf{1},\gamma) \neq 0$ and*

$$R_I(\mathbf{1},\gamma) \;=\; \sum (-1)^{d+|\nu|+|\mu|} \cdot \prod_{i\in I} \frac{x_i^{\nu_i-1}}{y_i^{\nu_i}} \cdot \prod_{j\in J} \frac{y_j^{\mu_j}}{x_j^{\mu_j+1}}, \tag{4.5}$$

where the sum is over $\nu \in \mathbb{Z}^I$ and $\mu \in \mathbb{N}^J$ such that $M_I\cdot\nu - M_J\cdot\mu = \gamma$. Moreover, for every $\beta \in \mathbb{Z}^n_{>0}$, the residue $R_I(\beta,\gamma)$ is a stable rational hypergeometric function.

Proof. We expand

$$t^\gamma \cdot f_J(t)^{-1} \;=\; \sum_{\mu\in\mathbb{N}^J} \prod_{j\in J} \left(y_j^{\mu_j} \cdot x_j^{-\mu_j-1}\right)\; t^{\gamma + M_J\cdot\mu}. \tag{4.6}$$

Applying (4.3) to each term of (4.6) yields the Laurent expansion (4.5).

Suppose now that $\gamma = M_I \cdot \nu_0 - M_J \cdot \mu_0$, $\nu_0 \in \mathbb{Z}^I$, $\mu_0 \in \mathbb{Z}^J$. There exists a vector $m \in \mathbb{Z}^J_{>0}$ such that $m_j a_j \in M_I \cdot \mathbb{Z}^d$. Hence for $k \in \mathbb{N}$,

$$\gamma + M_J \cdot (\mu_0 + km) \in M_I \cdot \mathbb{Z}^d$$

and $\mu_0 + km$ is non-negative for $k \gg 0$. Hence, the series (4.5) contains infinitely many non-zero terms. This shows that $R_I(\mathbf{1}, \gamma) \neq 0$.

Suppose now that $\beta \in \mathbb{Z}^n_{>0}$ is arbitrary. In view of (3.8), it suffices to show that the derivative $\partial_x^{\beta - \mathbf{1}}$ of the series (4.5) contains infinitely many powers of each of the variables x_ℓ, $\ell = 1, \ldots, n$. The previous argument shows that this is indeed the case for x_j, $j \in J$ and also for a variable x_{i_0}, $i_0 \in I$, unless every vector a_j, $j \in J$, is in the $\mathbb{Q}$-span of $\{\, a_i \,:\, i \in I, i \neq i_0\}$. But this would mean that the points a_k, $k \neq i_0$ define a coloop in A, which is impossible by assumption. $\square$

Our final task in this section is to identify the irreducible factors in the denominators of these binomial residues. Let $C \subseteq \{1, \ldots, n\}$ be a *circuit*, i.e., the set $\{a_i, i \in C\}$ obeys a unique (up to sign) linear relation $\sum_{i \in C} m_i a_i = 0$ over $\mathbb{Z}$ such that $\gcd(m_i,\, i \in C) = 1$. Then

$$\mathrm{Res}(C; x, y) \quad = \quad \prod_{m_i > 0} x_i^{m_i} \prod_{m_j < 0} y_j^{m_j} - (-1)^{|C|} \prod_{m_i > 0} y_i^{m_i} \prod_{m_j < 0} x_j^{m_j}$$

is the *resultant* of the binomials $f_i, i \in C$. In fact, the singular locus of $H_A(-\beta, -\gamma)$ is described by the product of all the variables and all the resultants $\mathrm{Res}(C; x, y)$ as C ranges over the circuits (cf. [8],[11]). Let I be a basis as above. Note that for each $j \notin I$, there exists a unique subset $I'(j) \subseteq I$, such that $I(j) := I'(j) \cup \{j\}$ is a circuit.

Theorem 4.5. *The binomial residue, defined by I, β, γ as above, equals*

$$R_I(\beta, \gamma) \quad = \quad \frac{P(x, y)}{x^a\, y^b \prod_{j \notin I} \mathrm{Res}(I(j); x, y)^{c_j}} \qquad \text{with all } c_j > 0 \tag{4.7}$$

where $P(x, y)$ is a polynomial relatively prime from the denominator.

Proof. We may assume that $\beta = \mathbf{1}$. It follows from a variant of Theorem 1.4 in [7] that $R_I(\mathbf{1}, \gamma)$ is a rational function whose denominator divides a monomial times

$$\prod_{j \notin I} \mathrm{Res}(f_{i_1}, \ldots, f_{i_d}, f_j)\,.$$

Since $\{a_k \,|\, k \in I(j)\}$ is the unique essential subset of $\{a_i \,|\, i \in I \cup \{j\}\}$, with "essential" as defined in [8], we have that

$$\mathrm{Res}(f_{i_1}, \ldots, f_{i_d}, f_j) \quad = \quad \mathrm{Res}(I(j); x, y).$$

We know by Proposition 4.4 that P is non zero. Moreover, if any of the factors $\mathrm{Res}(I(j);x,y)$ were missing from the denominator of $R_I(\mathbf{1},\gamma)$, then the Laurent series (4.5) would contain only finitely many powers of x_j. The formula in Proposition 4.4 implies that is impossible. $\square$

For unimodular bases, Theorem 4.5 can be refined as follows:

Proposition 4.6. *Suppose that $\{a_i \,|\, i \in I\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Then*

$$R_I(\mathbf{1},\gamma) \;=\; \frac{x^a y^b}{\prod_{j\notin I} \mathrm{Res}(I(j);x,y)} \tag{4.8}$$

where x^a and y^b are monomials specified in the proof.

Proof. Choose $\nu, n_j \in \mathbb{Z}^I$, $j \in J$, so that $\gamma = M_I \cdot \nu$, $a_j = M_I \cdot n_j$. Then

$$\gamma + \sum_{j\in J} \mu_j \cdot a_j \;=\; M_I \cdot (\nu + \sum_{j\in J} \mu_j \cdot m_j) \qquad \text{for all } \mu \in \mathbb{N}^J,$$

and consequently, the Laurent series (4.5) reduces, up to sign, to

$$\begin{aligned} R_I(\mathbf{1},\gamma) \;&=\; \frac{x_I^{\nu-\mathbf{1}}}{y_I^{\nu}\, x_J} \sum_{\mu\in\mathbb{N}^J} \prod_{j\in J}\prod_{i\in I} x_i^{n_{ij}\mu_j} y_i^{-n_{ij}\mu_j} y_j^{\mu_j} x_j^{-\mu_j} \\ &=\; \frac{x_I^{\nu-\mathbf{1}}}{y_I^{\nu}} \prod_{n_{ij}>0} y_i^{n_{ij}} \prod_{n_{ij}<0} x_i^{-n_{ij}} \prod_{j\in J} \mathrm{Res}(I(j);x,y)^{-1} \qquad \square \end{aligned}$$

5. The lower bound and the linear relations

In this section we establish the lower bound in Theorem 1.1 by exhibiting $|\chi(A)|$ many linearly independent binomial residues $R_I(\beta,\gamma)$ for fixed β,γ and fixed Lawrence matrix

$$A \quad := \quad \begin{pmatrix} I_n & I_n \\ 0\,0\cdots 0 & a_1\, a_2 \,\cdots\, a_n \end{pmatrix}.$$

We will show that all linear relations among the $R_I(\beta,\gamma)$ arise from Proposition 3.1 and correspond to Orlik-Solomon relations [16, §3.1].

The Gale dual to the Lawrence matrix A has the form

$$B \;=\; \{b_1,\ldots,b_n,-b_1,\ldots,-b_n\}, \tag{5.1}$$

where $B_0 = \{b_1,\ldots,b_n\} \subset \mathbb{Z}^{n-d}$ is a Gale dual of $\{a_1,\ldots,a_n\}$. According to Corollary 2.8 and Lemma 2.11, the dimension of the space of stable rational A-hypergeometric functions of degree $(-\beta,-\gamma)$ is at most the number of nbc-bases in B, which agrees with the number of nbc-bases in B_0. The following converse will imply Theorem 1.1.

Theorem 5.1. *Let $\beta \in \mathbb{Z}^n_{>0}$ and $\gamma \in \mathbb{Z}^d$. Then the set of binomial residues $R_I(\beta, \gamma)$, where $\{1, \ldots, n\}\backslash I$ runs over all nbc-bases of B_0, is linearly independent modulo the space of unstable rational functions.*

It is convenient to use the following characterization for being an nbc-basis of the dual matroid. The proof of Lemma 5.2 is straightforward.

Lemma 5.2. *The set $\{1, \ldots, n\}\backslash I$ is an nbc-basis of $B_0 = \{b_1, \ldots, b_n\}$ if and only if, for each $i_0 \in I$, there exists $j_0 \in \{1, \ldots, n\}\backslash I$ such that $j_0 > i_0$ and $I\backslash\{i_0\} \cup \{j_0\}$ is a basis of $\{a_1, \ldots, a_n\} \subset \mathbb{R}^d$.*

Proof of Theorem 5.1. Consider the space $\mathcal{S}(\beta, \gamma)$ of stable rational hypergeometric functions defined in the Introduction. The derivative ∂_{x_i} induces a monomorphism from $\mathcal{S}(\beta, \gamma)$ into $\mathcal{S}(\beta + e_i, \gamma)$, while ∂_{y_i} induces an monomorphism into $\mathcal{S}(\beta + e_i, \gamma + a_i)$. Binomial residues are mapped to binomial residues, with the set of irreducible factors in their denominators preserved. We may thus assume $\beta = \mathbf{1}$. All linear spaces in this proof are understood modulo unstable rational functions.

By Theorem 4.5, for any basis I of $\{a_1, \ldots, a_n\}$, the denominator of $R_I(\mathbf{1}, \gamma)$ equals a monomial multiplied by

$$\prod_{j \notin I} \mathrm{Res}(I(j); x, y) \tag{5.2}$$

Let $\mathcal{I}_0$ denote the set of indices I complementary to nbc-bases of B_0. Let $R_{\mathcal{I}_0}$ denote the linear span of binomial residues $R_I(\mathbf{1}, \gamma)$, $I \in \mathcal{I}_0$. Clearly, $n \notin I$ for any $I \in \mathcal{I}_0$. Our goal is to show $\dim_{\mathbb{C}}(R_{\mathcal{I}_0}) = \#\mathcal{I}_0$.

Let K be a circuit of $\{a_1, \ldots, a_n\}$ which contains the index n. Define $R_{\mathcal{I}_0}(K)$ to be the span of all binomial residues $R_I(\mathbf{1}, \gamma)$ with $I \in \mathcal{I}_0$ and $I(n) = K$, i.e., K is the unique circuit in $I \cup \{n\}$. We may decompose

$$R_{\mathcal{I}_0} = \bigoplus_K R_{\mathcal{I}_0}(K)\,. \tag{5.3}$$

The sum in (5.3) is direct because no element in $\sum_{K' \neq K} R_{\mathcal{I}_0}(K')$ contains $\mathrm{Res}(K; x, y)$ in its denominator, while all elements in $R_{\mathcal{I}_0}(K)$ do.

Thus, it suffices to fix $K = K_0$ and show that the binomial residues $R_I(\mathbf{1}, \gamma)$ with $I \in \mathcal{I}_0$ and $I(n) = K_0$ are linearly independent. Let

$$\mathcal{I}_1 \;=\; \{I \in \mathcal{I}_0 : I(n) = K_0\}\,.$$

Let n_1 denote the largest index which does not belong to K_0, then note that $n_1 \notin I$ for any $I \in \mathcal{I}_1$. Indeed, if $n_1 \in I$, $I \in \mathcal{I}_1$, then we would not be able to replace a_{n_1} by a_j with $j > n_1$ and still have a basis; this would contradict Lemma 5.2. This means that we can repeat the previous argument with $\mathcal{I}_1$ in place of $\mathcal{I}_0$ and n_1 in place of n and obtain a decomposition of $R_{\mathcal{I}_1}$ as a direct sum of subspaces $R_{\mathcal{I}_1}(K)$ spanned by binomial residues $R_I(\mathbf{1},\gamma)$ with $I \in \mathcal{I}_1$ and $I(n_1) = K$. Continuing in this manner, all subspaces $R_{\mathcal{I}_p}(K)$ will eventually be one-dimensional. Then, the desired result follows from Proposition 4.4. □

We next describe all linear relations among the binomial residues $R_I(\beta,\gamma)$ as I varies. In the identity below, it is essential to keep track of signs. Namely, if I' is taken to be ordered then we must multiply $R_{I'\cup\ell}(\beta,\gamma)$ by the sign of the permutation which orders $I' \cup \{\ell\}$.

Theorem 5.3. *Let I' be a $(d-1)$-subset of $\{1,\dots,n\}$ and* $\operatorname{ind} I'$ *the set of indices ℓ such that $\{a_\ell\} \cup \{a_i : i \in I'\}$ is a basis of $\mathbb{R}^d$. Then*

$$\sum_{\ell\in \operatorname{ind} I'} R_{I'\cup\ell}(\beta,\gamma) \;\equiv\; 0 \qquad \textit{modulo unstable rational functions},$$

and these span all the $\mathbb{C}$-linear relations relations among the $R_I(\beta,\gamma)$.

Proof. By Proposition 4.4, all $R_I(\beta,\gamma)$ residues are stable. We have established that the spaces $\mathcal{S}(\beta,\gamma)$ have the same dimension $|\chi(A)|$ for all β and γ. It follows that the maps $\partial_{x_i} : \mathcal{S}(\beta,\gamma) \to \mathcal{S}(\beta+e_i,\gamma)$ and $\partial_{y_i} : \mathcal{S}(\beta,\gamma) \to \mathcal{S}(\beta+e_i,\gamma+a_i)$ are isomorphisms. Iterating, we can assume that $(-\beta,-\gamma)$ lies in the Euler-Jacobi cone $-\mathrm{Int}(\mathrm{pos}(A))$. By Proposition 2.6, there are no unstable rational A-hypergeometric functions, so we are claiming that $\sum_{\ell\in\operatorname{ind} I'} R_{I'\cup\ell}(\beta,\gamma)$ is zero.

We may assume that $\{a_i : i \in I'\}$ is linearly independent. On the B-side, the complement of I' has $n-d+1$ elements and therefore defines a dependent set $\{b_i, i \notin I'\}$. We can consider as in §2, the central hyperplane arrangement $\mathcal{A}$ defined by $\mathcal{H}$. Consider the socle of the *Orlik-Solomon algebra* of that hyperplane arrangement [16, §3.1]. The linear relation in Theorem 5.3 is the translation to the A-side of the relation in the socle degree of the Orlik-Solomon algebra defined by $\{b_i, i \notin I'\}$. In view of [16, Theorem 3.4] and Theorem 1.1, it suffices to show that the asserted relations are valid. It will then follow by dimension reasons that they span all $\mathbb{C}$-linear relations.

We now prove the identity $\sum_{\ell\in\operatorname{ind} I'} R_{I'\cup\ell}(\beta,\gamma) = 0$ using the formulation in terms of toric residues given in §2. By (3.6), all $h_j(\beta,\gamma)$ are non negative, and so the polar divisor of the form $\Phi(\beta,\gamma)$ in (3.5) is contained in the union of the divisors $Y_i = \{F_i = 0\}, i = 1,\dots,n$.

For $k = 1,\dots,d-1$, set $G_k^{I'} = F_{i_k}$. Set also $G_d^{I'} = \prod_{j\notin I'} F_j$ and let $G_0^{I'} = z_1 \dots z_{2p}$. Then, $G_0^{I'},\dots,G_d^{I'}$ define divisors with empty intersection in $X = X_\Delta$ for generic values of the coefficients and moreover

$$\Phi(\beta,\gamma) \quad = \quad \frac{z^{h(\beta,\gamma)}\ \ \Omega_\Delta}{G_1\dots G_d}.$$

Proposition 3.1 implies that the corresponding toric residue vanishes:

$$\mathrm{Res}^X_{G^{I'}}(\Phi(\beta,\gamma)) \quad = \quad 0.$$

On the other hand, consider also the following $n-d+1$ families of divisors: for any $\ell \notin I'$, set $G_k^{I',\ell} = G_k^{I'}$ for any $k = 1,\dots,d-1$, $G_d^{I',\ell} = F_\ell$ and $G_0^{I',\ell} = \prod_{j\notin I'\cup\{\ell\}} F_j$. Again, these divisors have empty intersection on X for generic values of the coefficients and the poles of $\Phi(\beta,\gamma)$ are contained in their union, and so we can consider the toric residues $\mathrm{Res}^X_{G^{I',\ell}}(\Phi(\beta,\gamma))$. These toric residues are non-zero precisely when $\ell \in \mathrm{ind}\, I'$. We conclude that the following relations hold:

$$\sum_{\ell\in \mathrm{ind} I'} \mathrm{Res}^X_{G^{I',\ell}}(\Phi(\beta,\gamma)) \;=\; \sum_{\ell\notin j} \mathrm{Res}^X_{G^{I',\ell}}(\Phi(\beta,\gamma)) \;=\; \mathrm{Res}^X_{G^{I'}}(\Phi(\beta,\gamma)) \;=\; 0.$$

The second equality follows from a variation on [19, §II.7]. Translating back to binomial residues completes the proof of Theorem 5.3. □

In [8], we studied the problem of classifying vector configurations A for which there exist a rational A-hypergeometric function which is not a Laurent polynomial. We conjectured [8, Conjecture 1.3] that such a configuration has to have a facial subset which is an essential Cayley configuration. It is easy to see that Lawrence liftings are Cayley configurations of segments; they are essential if and only if $n = d+1$. We also conjectured [8, Conjecture 5.7] that a rational A-hypergeometric function has an iterated derivative which is a linear combination of toric residues associated with facial subsets of A.

Theorem 5.4. *Conjecture 5.7 in* [8] *holds for Lawrence configurations.*

Proof. Let A be a Lawrence configuration. The assertion of [8, Conjecture 5.7] is obvious for unstable rational hypergeometric functions. On the other hand, given a stable rational hypergeometric function, a suitable derivative will have degree in the Euler-Jacobi cone and hence, by Theorem 1.1, will be a linear combination of toric residues. □

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Eduardo Cattani: Department of Mathematics and Statistics. University of Massachusetts. Amherst, MA 01003, USA
E-mail address: cattani@math.umass.edu

ALICIA DICKENSTEIN: DEPARTAMENTO DE MATEMATICA, FCEYN. UNIVERSIDAD DE BUENOS AIRES. (1428) BUENOS AIRES, ARGENTINA
E-mail address: alidick@dm.uba.ar

BERND STURMFELS: DEPARTMENT OF MATHEMATICS. UNIVERSITY OF CALIFORNIA. BERKELEY, CA 94720, USA
E-mail address: bernd@math.berkeley.edu