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Inventiones mathematicae

Canonical representatives in moderate cohomology

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To our very dear Professor and adviser Miguel Herrera. In memoriam

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1. Introduction

Let X be a n-dimensional complex manifold and Y a complex subspace of X of pure codimension p, $0<p\leqq n$, which is locally a complete intersection.

Denote $'\mathscr{D}^{r,p}$ the sheaf of currents on X of bidegree (r,p). We prove in this paper the following decomposition property for $\bar\partial$-closed currents $T\in\Gamma_Y(X,'\mathscr{D}^{r,p})$ with support on Y:

a) $T=R+\bar\partial S$, where $R\in\Gamma_Y(X,'\mathscr{D}^{r,p})$ is a locally residual current and $S\in\Gamma_Y(X,'\mathscr{D}^{r,p-1})$.

b) if $T=R'+\bar\partial S'$ is a similar decomposition, then $R=R'$ and $\bar\partial S=\bar\partial S'$.

The current R is called locally residual if it is equal locally to some current $R_{Y_1,\dots,Y_p}[\tilde\omega]$, where $R_{Y_1,\dots,Y_p}[\tilde\omega]$ is the residue operator (cf. [C-H]) associated to some family $\mathscr{Y}=\{Y_1,\dots,Y_p\}$ of complex hypersurfaces in an open set $U\subseteq X$, such that $Y_1\cap\dots\cap Y_p=Y\cap U$, and where $\tilde\omega\in\Gamma(U,\Omega^r(*\cup\mathscr{Y}))$ is a meromorphic form on U with poles on $\bigcup\mathscr{Y}=Y_1\cup\dots\cup Y_p$. A locally residual current is always $\bar\partial$-closed.

This result provides canonical residual representatives for the classes in the global moderate local cohomology groups $H^p_{[Y]}(X,\Omega^r)$, which are defined as the group of global sections of the moderate local cohomology sheaves $\mathscr{H}^p_{[Y]}(X,\Omega^r)$ (cf. Ramis [R]) and can be actually calculated as the p-cohomology of the complex $(\Gamma_Y(X,'\mathscr{D}^{r,\cdot}),\bar\partial)$. These groups cannot be identified with classical local cohomology $H^p_Y(X,\Omega^r)$, since the resolution $('\mathscr{D}^{r,\cdot},\bar\partial)$ of Ω^r has $\mathscr{O}_x$-injective fibers, $x\in X$, but it is not an injective resolution on X. In fact, it holds $\mathscr{H}^p_Y(X;\mathscr{O})\simeq\mathscr{D}^\infty\otimes_{\mathscr{D}}\mathscr{H}^p_{[Y]}(X;\mathscr{O})$ for the local cohomology sheaves, where $\mathscr{D}$ (resp. $\mathscr{D}^\infty$) denotes the sheaf of differential operators of finite (resp. infinite) order (cf. Mebkhout [M], Chap. II).

As an immediate consequence of our result we deduce that the support – in the sense of sheaves – of any class in $H^p_{[Y]}(X,\Omega^r)$ is always analytic.

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Conversely, we show in §7 that locally residual currents generalize analytic cycles; precisely: given T a cycle of codimension p and $x \in X$, for every family of hypersurfaces $\mathscr{Y} = \{Y_1, \ldots, Y_p\}$ such that $\operatorname{supp}(T) \subseteq \bigcap \mathscr{Y}$ near x, there exists a meromorphic p-form τ with poles on $\bigcup \mathscr{Y}$ such that $T = R_{Y_1, \ldots, Y_p}[\tau]$.

Our methods also give a duality theorem for residual currents, which is the natural extension of Grothendieck's duality theorem for scalar residues. We apply this duality theorem to obtain a result about fibration of analytic ideals.

Finally, we exhibit a straightforward construction for the cup-product

$$H^p_{[Y]}(X, \Omega^r) \otimes H^{p'}_{[Y']}(X, \Omega^{r'}) \to H^{p+p'}_{[Y \cap Y']}(X, \Omega^{r+r'})$$

for subspaces Y and Y' of X in the proper intersection position.

2. Background

Let X be a complex manifold of dimension n and $\mathscr{Y} = \{Y_1, \ldots, Y_{q+1}\}$ $(0 \leqq q < n)$ a family of hypersurfaces in X. As usual, Ω^r (resp. $\Omega^r(* \cup \mathscr{Y})$) will denote the sheaf of holomorphic $\left(\text{resp. meromorphic with poles on } \bigcup \mathscr{Y} = \bigcup_{i=1}^{q+1} Y_i\right)$ r-forms on X and, for any analytic subspace Z, $({}'\mathscr{D}_Z^{r,\cdot}, \bar\partial)$ will denote the complex of sheaves of currents on X of bidegree $(r, \cdot)$ supported on Z, with differential $\bar\partial$: ${}'\mathscr{D}_Z^{r,\cdot} \to {}'\mathscr{D}_Z^{r,\cdot+1}$

$$\bar\partial(T)(\alpha) = (-1)^{r+\cdot}\, T(\bar\partial \alpha)$$

for α a germ of $\mathscr{C}^\infty$-form of degree $2n - r - 1 - \cdot$.

Also, $\bigcap \mathscr{Y} = \bigcap_{i=1}^{q+1} Y_i$ and $\mathscr{Y}(j) = \mathscr{Y} - \{Y_j\}$ $(1 \leqq j \leqq q+1)$.

We have the following commutative diagram:

$$\begin{tikzcd} \Omega^r(* \cup \mathscr{Y}) \arrow[r, "RP_{\mathscr{Y}}"] \arrow[dr, "R_{\mathscr{Y}}"'] & {}'\mathscr{D}^{r,q}_{\cap \mathscr{Y}(q+1)} \arrow[d, "\bar\partial"] \\ & {}'\mathscr{D}^{r,q+1}_{\cap \mathscr{Y}} \end{tikzcd} \tag{2.1}$$

where $RP_{\mathscr{Y}}$ (resp. $R_{\mathscr{Y}}$) is the q-multiple residue-principal value operator (resp. $q+1$-multiple residue operator) in the sense of Coleff-Herrera (cf. [C-H]).

As an obvious consequence, $\bar\partial(R_{\mathscr{Y}}[\tilde\omega]) = 0$ for every meromorphic form $\tilde\omega$ (2.2).

When needed, we shall write

$$\begin{aligned} \operatorname{Res}_{Y_1, \ldots, Y_q} P_{Y_{q+1}} &= RP_{\mathscr{Y}} \\ \operatorname{Res}_{Y_1, \ldots, Y_{q+1}} &= R_{\mathscr{Y}} \end{aligned} \tag{2.3}$$

and, for equations $\{\phi_1, \ldots, \phi_{q+1}\}$ defining $\{Y_1, \ldots, Y_{q+1}\}$ respectively,

$$\operatorname{Res}_{\phi_1, \ldots, \phi_{q+1}} = \operatorname{Res}_\phi = R_{\mathscr{Y}} \tag{2.4}$$

(for equations we only mean $Z(\phi_i) = Y_i$).

We say that $T\in\Gamma(X,'\mathcal{D}^{r,q})$ is a *locally residual current* if there exist locally a family $\mathcal{Y}=\{Y_1,\dots,Y_q\}$ of hypersurfaces and a meromorphic r-form $\tilde{\omega}$ with poles on $\bigcup\mathcal{Y}$ such that

$$T=R_{\mathcal{Y}}[\tilde{\omega}].$$

If $\dim_{\mathbb{C}}(\bigcap\mathcal{Y})=n-(q+1)$ ($\mathcal{Y}$ having complete intersection), the following properties hold:

i) If $\tilde{\omega}\in\Omega^r(*\bigcup\mathcal{Y}(j))$ for some fixed j (i.e. $\tilde{\omega}$ is regular on Y_j), then

$$R_{\mathcal{Y}}[\tilde{\omega}]=0. \tag{2.5}$$

Moreover, if $j\leqq q$

$$RP_{\mathcal{Y}}[\tilde{\omega}]=0 \tag{2.6}$$

and if $j=q+1$,

$$RP_{\mathcal{Y}}[\tilde{\omega}]=R_{\mathcal{Y}(q+1)}[\tilde{\omega}]. \tag{2.7}$$

ii) For any $\mathcal{Y}'=\{Y_1',\dots,Y_{q+1}'\}$ family of hypersurfaces in X such that $Y_j\subseteq Y_j'$ for every j, $1\leqq j\leqq q+1$, and $\dim(\bigcap\mathcal{Y}')=n-(q+1)$

$$R_{\mathcal{Y}}[\tilde{\omega}]=R_{\mathcal{Y}'}[\tilde{\omega}], \tag{2.8}$$

$$RP_{\mathcal{Y}}[\tilde{\omega}]=RP_{\mathcal{Y}'}[\tilde{\omega}] \tag{2.9}$$

for $\tilde{\omega}\in\Omega^r(*\bigcup\mathcal{Y})$.

iii) *Transformation law:*

Let ϕ_i, $\psi_i\,(1\leqq i\leqq q)$ be holomorphic functions in $\mathbb{C}^n$ such that

$$\bigcap_{i=1}^{q} Z(\phi_i)=\bigcap_{i=1}^{q} Z(\psi_i)$$

is a complete intersection. If $\psi_i=\sum_{j=1}^{q} a_{ij}\phi_j$ with $M=\|a_{ij}\|\in\mathcal{O}^{q\times q}(\mathbb{C}^n)$, then for every $\omega\in\Omega^r(\mathbb{C}^n)$

$$\operatorname{Res}_{\phi}\left[\frac{\omega}{\phi_1\cdots\phi_q}\right]=\operatorname{Res}_{\psi}\left[\frac{\omega\cdot\det M}{\psi_1\cdots\psi_q}\right]. \tag{2.10}$$

Note. For i) and ii) cf. [C-H].

iii) can be deduced from the puntual case proved by Griffiths-Harris ([G-H], p. 657) by virtue of the locally semi-meromorphic fibered residue function (cf. [C-H]).

We shall refer to $\Lambda(j,q)$ as the set of increasing families of j elements running between $\{1,2,\dots,q\}$ and, for $J\in\Lambda(j,q)$, Y_J will denote the hypersurface $\bigcup_{i\in J} Y_i$. For the general theory of double complexes, cohomology and cup-product, see [G].

3. Characterization of the moderate cohomology sheaf

Let X be a n-dimensional complex manifold and $Y\subseteq X$ an analytic subspace of codimension p. Our main purpose in this section is the construction of an explicit isomorphism between two representations of the moderate cohomology sheaf with coefficients in Ω^r and supports on Y, given one by meromorphic r-forms and the other by currents on X, in order to exhibit a canonical representative in each class of the quotient sheaf $\mathscr{H}^p_{\bar\partial}('\mathscr{D}^{r,\cdot}_Y)$ (Theorem 3.6).

As a consequence, we get the following local characterization theorem for currents on X supported on Y:

Theorem 3.1. *Let X be a n-complex manifold and Y an analytic subspace of pure dimension $n-p$. Let $x \in X$ and $T \in {}'\mathscr{D}^{r,p}_{Y,x}$ be a germ of a $\bar\partial$-closed current supported on Y. If there is a family of p hypersurfaces $\mathscr{Y}=\{Y_1, \dots, Y_p\}$ in some neighborhood W of x such that $Y \cap W = \bigcap \mathscr{Y}$, then:*

a) *There exist a meromorphic r-form $\omega \in \Omega^r_x(*\bigcup \mathscr{Y})$ and $S \in {}'\mathscr{D}^{r,p-1}_{Y,x}$ such that*

$$T = R_{\mathscr{Y}}[\omega] + \bar\partial S.$$

b) *This splitting is unique in the following sense: If $\mathscr{Y}'=\{Y'_1, \dots, Y'_p\}$ is another family of hypersurfaces with $\bigcap \mathscr{Y}' = Y$ near x, $\omega' \in \Omega^r_x(*\bigcup \mathscr{Y}')$ and $S' \in {}'\mathscr{D}^{r,p-1}_{Y,x}$ such that*

$$T = R_{\mathscr{Y}'}[\omega'] + \bar\partial S'$$

then

$$R_{\mathscr{Y}}[\omega] = R_{\mathscr{Y}'}[\omega'].$$

3.2. For I a sheaf of ideals on X, we denote ${}'\mathscr{D}^{r,\cdot}_I$ the subsheaf of ${}'\mathscr{D}^{r,\cdot}$ whose stalk at $x \in X$ is

$${}'\mathscr{D}^{r,\cdot}_{I,x} = \{T \in {}'\mathscr{D}^{r,\cdot}_x : f \cdot T = 0, \forall f \in I_x\}.$$

Lemma. *If $Z(I) = Y$, then*

$$\mathscr{H}^\cdot_{\bar\partial}('\mathscr{D}^{r,\cdot}_Y) = \operatorname{inj\,lim}_k \mathscr{H}^\cdot_{\bar\partial}('\mathscr{D}^{r,\cdot}_{I^k})$$

(*the direct limit is defined by the inclusions* ${}'\mathscr{D}^{r,\cdot}_{I^k} \subseteq {}'\mathscr{D}^{r,\cdot}_{I^{k+1}}$).

Proof. The lemma follows from the identity ${}'\mathscr{D}^{r,\cdot}_Y = \operatorname{inj\,lim}_k {}'\mathscr{D}^{r,\cdot}_{I^k}$, which is easily deduced from a theorem of Schwartz ([S], Th. XXVII, Chap. III).

3.3. Let $U \subseteq X$ be open, $(f_i)_{i=1}^q \in \mathcal{O}^q(U)$. We denote $I = \langle f_1, \dots, f_q \rangle$ the generated sheaf of ideals and, for $k \in \mathbb{N}$, $I_k = \langle f_1^k, \dots, f_q^k \rangle$. As $I_{k \cdot q} \subseteq I^{k \cdot q} \subseteq I_k$ for every k, the following sheafs on U are isomorphic:

$$\operatorname{inj\,lim}_k \mathscr{H}^\cdot_{\bar\partial}('\mathscr{D}^{r,\cdot}_{I^k}) = \operatorname{inj\,lim}_k \mathscr{H}^\cdot_{\bar\partial}('\mathscr{D}^{r,\cdot}_{I_k}).$$

3.4. Let $\mathscr{Y} = \{Y_1, \dots, Y_q\}$ be a family of hypersurfaces such that $\bigcap \mathscr{Y} = Y$. Let $\mathscr{A}$ denote the following subsheaf of $\Omega^r(*\bigcup \mathscr{Y})$:

$$\mathscr{A} = \left\{ \sum_{i=1}^q \tilde\omega(i) \colon \tilde\omega(i) \text{ is regular on } Y_i \right\}$$

and

$$\mathscr{Q} = \Omega^r(*\bigcup \mathscr{Y})/\mathscr{A}.$$

Lemma. *If $(f_i)_{i=1}^{q} \in \mathcal{O}^q(U)$ satisfy $(f_i=0)=Y_i \cap U$ $(1 \leqq i \leqq q)$, there is an isomorphism of sheaves on U*

$$\operatorname{inj}\lim_k \Omega^r/I_k \otimes \Omega^r \xrightarrow{\beta_f} \mathcal{Q}$$

where the direct limit is constructed by means of the homomorphisms

$$\Omega^r/I_k \otimes \Omega^r \xrightarrow{\sigma_k} \Omega^r/I_{k+1} \otimes \Omega^r,$$

$$\omega_x \mapsto \overline{f_1 \dots f_p \omega_x}.$$

Proof. For $k \in \mathbb{N}$, the morphisms

$$\beta_k \colon \Omega^r/I_k \otimes \Omega^r \to \mathcal{Q},$$

$$\overline{\omega_x} \mapsto \overline{\frac{\omega_x}{f_1^k \dots f_q^k}}$$

define the required isomorphism

$$\beta_f \colon \operatorname{inj}\lim_k \Omega^r/I_k \otimes \Omega^r \to \mathcal{Q}.$$

Proposition 3.5. *If $\mathcal{Y}=\{Y_1, \dots, Y_p\}$ is a family of hypersurfaces having complete intersection, $(f_i)_{i=1}^{p} \in \mathcal{O}^p(U)$ are equations for each Y_i in an open set $U \subseteq X$ and $I = \langle f_1, \dots, f_p \rangle$, there is an isomorphism R_f*

$$\Omega^r/I \otimes \Omega^r \xrightarrow{R_f} \mathcal{H}^p_{\bar{\partial}}({}'\mathcal{D}_I^{r,\cdot})$$

induced by the mapping

$$\Omega^r \to {}'\mathcal{D}_I^{r,p},$$

$$\omega_x \mapsto R_{\mathcal{Y}}\left[\frac{\omega_x}{f_1 \dots f_p}\right].$$

Note. We are going to show the $1-1$ correspondence at each stalk; we will omit the point when no confusion may arise.

Proof. R_f is well defined by virtue of (2.1), (2.2) and (2.5). We consider for each $x \in U$:

a) Koszul's projective resolution of $(\mathcal{O}/I)_x$

$$0 \to \Lambda^p \mathcal{O}^p \xrightarrow{\alpha_p} \Lambda^{p-1} \mathcal{O}^p \xrightarrow{\alpha_{p-1}} \dots \to \Lambda^1 \mathcal{O}^p \xrightarrow{\alpha_1} \mathcal{O} \xrightarrow{\pi} \mathcal{O}/I \to 0$$

$$\alpha_k(e_J) = \sum_{\nu=1}^{k} (-1)^{\nu-1} f_{j_\nu} e_{J-\{J_\nu\}}$$

where $J \in \Lambda(k,p)$; $(e_i)_{i=1}^{p}$ is the canonical basis in $\mathcal{O}^p$ and $e_J = e_{J_1} \wedge \dots \wedge e_{J_k}$, if $J_1 < \dots < J_k$ are the elements of J and b) the injective resolution of Ω_x^r: $({}'\mathcal{D}_x^{r,\cdot}, \bar{\partial})$.

Functor $\operatorname{Hom}_{\mathcal{O}_{X,x}}(\cdot, \Omega^r)$ applied to a) and functor $\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}/I, \cdot)$ applied to b) give two complexes with canonically isomorphic cohomologies representing the group $\operatorname{Ext}_{\mathcal{O}_{X,x}}(\mathcal{O}/I, \Omega^r)$. This isomorphism is constructed following the arrows in the double complex

$$(\mathrm{Hom}_{\mathcal{O}_{X,x}}(\Lambda^{\cdot}\mathcal{O}^p, '\mathcal{D}^{r,\cdot}), \bar\partial^*, \alpha^*).$$

Let

$$\phi = \left[\bar 1 \mapsto R_{\mathcal{Y}}\left[\frac{\omega}{f_1 \dots f_p}\right]\right] \in \mathrm{Hom}_x(\mathcal{O}/I, '\mathcal{D}^{r,p}) \cong '\mathcal{D}^{r,p}_{I,x}.$$

We define for each i, $1 \leqq i \leqq p$

$$\phi_i(e_J) = \begin{cases} 0 & \text{if } J \neq \{p-(i-2), \dots, p-1, p\} \\ R_{Y_1,\dots,Y_{p-i}} P_{Y_{p-i+1}}\left[\dfrac{\omega}{f_1 \dots f_{p-i+1}}\right] & \\ \qquad \text{if } J = \{p-(i-2), \dots, p\}, & \end{cases}$$

$$\phi_i \in \mathrm{Hom}_{\mathcal{O}}(\Lambda^{i-1}\mathcal{O}^p, '\mathcal{D}^{r,p-i}).$$

Properties (2.6) and (2.7) assure that

$$(\alpha_i^* \phi_i)(e_J) = \phi_i(\alpha_i(e_J)) = \begin{cases} 0 & \text{if } J \neq \{p-(i-1), \dots, p\} \\ R_{Y_1,\dots,Y_{p-i}}\left[\dfrac{\omega}{f_1 \dots f_{p-i}}\right] & \\ \qquad \text{if } J = \{p-(i-1), \dots, p\}. & \end{cases}$$

Therefore

$$\alpha_i^* \phi_i = \bar\partial \phi_{i+1}, \qquad i = 1, \dots, p-1,$$

$$\bar\partial \phi_1 = \phi$$

and

$$\alpha_p^* \phi_p(e_1 \wedge \dots \wedge e_p) = P_{Y_1}[\omega] = \int \omega \wedge \cdot = i^* \eta_\omega(e_1 \wedge \dots \wedge e_p)$$

where $\eta_\omega \in \mathrm{Hom}(\Lambda^p \mathcal{O}^p, \Omega^r)$ is the homomorphism

$$\eta_\omega(e_1 \wedge \dots \wedge e_p) = \omega.$$

To complete the proof we should show that

$$\sigma(\alpha_p^*(\mathrm{Hom}_{\mathcal{O}}(\Lambda^{p-1}\mathcal{O}^p, \Omega^r))) = I \otimes \Omega^r$$

where σ denotes the isomorphism

$$\sigma\colon\ \mathrm{Hom}_{\mathcal{O}}(\Lambda^p \mathcal{O}^p, \Omega^r) \to \Omega^r$$

$$\eta \longmapsto \eta(e_1 \wedge \dots \wedge e_p).$$

In fact, for $\psi \in \mathrm{Hom}_{\mathcal{O}}(\Lambda^{p-1}\mathcal{O}^p, \Omega^r)$

$$\sigma(\alpha_p^*(\psi)) = \sum_{i=1}^{p} (-1)^{i-1} f_i \cdot \psi(e_1 \wedge \dots \wedge \hat e_i \wedge \dots \wedge e_p)$$

as we wanted.

Now, we can state:

Theorem 3.6. *Let $\mathscr{Y}=\{Y_1,\ldots,Y_p\}$ be a family of complex hypersurfaces in X such that $Y=\bigcap\mathscr{Y}$ has pure dimension $n-p$.*

The following morphism of sheaves on X

$$\Omega^r(*\bigcup\mathscr{Y})\to{}'\mathscr{D}_Y^{r,p},$$
$$\tilde{\omega}\mapsto R_{\mathscr{Y}}[\tilde{\omega}]$$

induces an isomorphism R

$$\mathscr{Q}\xrightarrow{R}\mathscr{H}_{\bar\partial}^p({}'\mathscr{D}_Y^{r,\cdot}).$$

Proof. Locally, there are $(f_i)_{i=1}^p\in\mathcal{O}^p(U)$ equations for each Y_i. If $I_k=\langle f_1^k,\ldots,f_p^k\rangle$ for every $k\in\mathbb{N}$, the mappings

$$\Omega^r/I_k\otimes\Omega^r\xrightarrow{R_{f^k}}\mathscr{H}_{\bar\partial}^p({}'\mathscr{D}_{I_k}^{r,\cdot})$$

are isomorphisms (by Proposition 3.5), which commute with the respective direct limit morphisms (3.2, 3.3 and 3.4) to give the quoted isomorphism R.

Corollary 3.7. *Let $\tilde{\omega}\in\Omega_x^r(*\bigcup\mathscr{Y})$ and $S\in{}'\mathscr{D}_{Y,x}^{r,p-1}$ such that*

$$R_{\mathscr{Y}}[\tilde{\omega}]=\bar\partial S$$

then

$$R_{\mathscr{Y}}[\tilde{\omega}]=0.$$

Proof. It is an immediate consequence of Theorem 3.6 and property (2.5).

We finish this section with the proof of Theorem 3.1:

a) is a consequence of the surjectivity of the mapping R in Theorem 3.6.

b) is clear in the case $\mathscr{Y}'=\mathscr{Y}$. Then, it is sufficient to show the following:

Let $\mathscr{Y}'=\{Y_1',\ldots,Y_p'\}$ be another family of hypersurfaces with $Y=\bigcap\mathscr{Y}'$ near x, and $\omega'\in\Omega_x^r(\bigcup\mathscr{Y}')$. There exists $\omega\in\Omega_x^r(\bigcup\mathscr{Y})$ such that

$$R_{\mathscr{Y}'}[\omega']=R_{\mathscr{Y}}[\omega]. \tag{3.9}$$

In fact, given $\sigma\in\Omega_x^r$, $(g_i)_{i=1}^p\in\mathcal{O}_x^p$ local equations for $(Y_i')_{i=1}^p$ such that $\omega'=\dfrac{\sigma}{g_1\cdots g_p}$, and $(f_i)_{i=1}^p\in\mathcal{O}_x^p$ defining $(Y_i)_{i=1}^p$ near x, by the Nullstellensatz, there exist $q\in\mathbb{N}$ and $A=(a_{ij})\in\mathcal{O}_x^{p\times p}$ such that

$$f_j^q=\sum_{i=1}^p a_{ij}g_i \qquad j=1,\ldots,p.$$

Then, $\omega=\dfrac{\det A\cdot\sigma}{\prod f_i^q}$ verifies (3.9), thanks to the Transformation Law (2.10).

4. Duality Law and fibration of regular ideals

4.1. As an immediate consequence of (2.5) and (3.5) we have the

Duality Law: Let $x \in X$ *and* $(f_i)_{i=1}^p \in \mathcal{O}_x^p$ *such that*

$$\dim_x \bigcap_{i=1}^p (f_i = 0) = n - p.$$

For $\omega \in \Omega_x^r$, *the following statements are equivalent:*

i) $\operatorname{Res}_f \left[\dfrac{\omega}{f_1 \dots f_p}\right] = 0$,

ii) $\omega \in I_x \otimes \Omega_x^r$, *where* $I_x = \langle f_1, \dots, f_p \rangle$.

4.2. *Remark.* The Duality Law is not valid without the hypothesis of complete intersection, as the following examples show:

Let $X = \mathbb{C}^2$, $f_1 = z_1$, $f_2 = z_1 \cdot z_2$, $\mathcal{Y} = \{Z(f_1), Z(f_2)\}$.

1) $\operatorname{Res}_{f_1, f_2} \left[\dfrac{1}{f_1 \cdot f_2}\right] = 0$ because $V_e(\mathcal{Y}) = \emptyset$ (for the notion of $V_e(\mathcal{Y})$ cf. [C-H]) and obviously $1 \notin \langle z_1, z_1 \cdot z_2 \rangle$.

2) If $h = z_1$

$$\operatorname{Res}_{f_2, f_1} \left[\frac{h}{f_1 \cdot f_2}\right] = \operatorname{Res}_{f_2, f_1} \left[\frac{1}{z_1 \cdot z_2}\right] = -\frac{1}{2\pi i} \delta_{\{0\}} \neq 0$$

and $h \in \langle f_1, f_2 \rangle$.

4.3. Let X be a complex manifold of dimension n and I a regular sheaf of ideals, i.e. I is locally generated by a regular sequence of holomorphic functions $f_1, \dots, f_p$ (which implies $\dim_{\mathbb{C}} Z(I) = n - p$).

Let $x_0 \in Z(I)$ and (U, φ) be a coordinate neighborhood of x_0. Let us denote $P_x = \{z \in U / \varphi_i(z) = \varphi_i(x),\ i = n-p, \dots, n\}$ for $x \in U$. We have:

Theorem 4.3. *Under the hypothesis* $P_{x_0} \cap Z(I) = \{x_0\}$ *the following statements are equivalent for* $h \in \mathcal{O}(U)$:

i) $h|_{P_x} \in I|_{P_x; x}$ *for every* $x \in \operatorname{reg} Z(I) \cap U$,

ii) $h \in I_{x_0}$.

Proof. As this is a local property, the theorem follows from the Duality Law and Proposition 4.4 below.

Proposition 4.4. *Let* $\Delta \subseteq \mathbb{C}^n$ *an open set,* $I = \langle f_1, \dots, f_p \rangle$ *a regular ideal in* $\mathcal{O}(\Delta)$ *and* $\pi\colon \Delta \to \mathbb{C}^{n-p}$, $\pi(x_1, \dots, x_n) = (x_{p+1}, \dots, x_n)$, *verifying* $\dim_{\mathbb{C}} \pi^{-1}(\pi(x)) \cap Z(I) = 0$ *for every* $x \in Z(I)$.

For $h \in \mathcal{O}(\Delta)$, *we denote* $\tilde{h} \in \mathcal{M}(\Delta)$, $\tilde{h} = \dfrac{h}{f_1 \dots f_p}$.

The following statements are equivalent:

i) $\operatorname{Res}_f [\tilde{h}] = 0$

ii) *The restricted currents* $\mathrm{Res}_{P_x;f|_{P_x}}[\tilde{h}|_{P_x}]$ *are null on the p-plane* $P_x=\{z\in\Delta:\ \pi(z)=\pi(x)\}$ *for all* $x\in\operatorname{reg}Z(I)$.

Proof. Only ii) $\Rightarrow$ i) has to be proved.

Let $x_0\in\operatorname{reg}Z(I)$ and $(V,(z_1,\dots,z_n))$ be a coordinate system such that $z(x_0)=0$ and

$$U:=V\cap Z(I)=\{z_1=\dots=z_p=0\}.$$

For every $a\in U$,

$$\left\langle \frac{\partial}{\partial z_{p+1}}\bigg|_a,\dots,\frac{\partial}{\partial z_n}\bigg|_a\right\rangle$$

and

$$\left\langle \frac{\partial}{\partial x_1}\bigg|_a,\dots,\frac{\partial}{\partial x_p}\bigg|_a\right\rangle$$

are respective basis for $T_a(U)$ and $T_a(P_a)$. Let $g\in\mathcal{O}(U)$

$$g=\det\begin{vmatrix}\dfrac{\partial x_{p+1}}{\partial z_{p+1}} & \cdots & \dfrac{\partial x_{p+1}}{\partial z_n}\\ \vdots & & \vdots\\ & & \dfrac{\partial x_n}{\partial z_n}\end{vmatrix}$$

and $S=\{b\in U/g(b)=0\}$. As g is the determinant of the jacobian matrix of the mapping

$$\pi|_U:\ U\to\pi(U)\subseteq\mathbb{C}^{n-p}\quad\text{open}$$

and $\dim_{\mathbb{C}}(U)=\dim_{\mathbb{C}}\pi(U)=n-p$, we get $g\not\equiv 0$ on U and $\operatorname{codim}_{U,\mathbb{C}}S\geqq 1$.

Now, let $a\in U-S$ and W be a neighborhood of a in Δ verifying

i) $W\cap Z(I)\subseteq U-S$,

ii) $\exists r\in\mathbb{N}$, and $A=(a_{ij})\in\mathcal{O}^{p\times p}(W)$ such that

$$z_i^r=\sum_{j=1}^{p}a_{ij}f_j\qquad\forall i=1,\dots,p$$

(this is possible by virtue of the Nullstellensatz because

$$(z_1=\dots=z_p=0)\cap V=(f_1=\dots=f_p=0)\cap V,$$

iii) $(z_1,\dots,z_p,x_{p+1},\dots,x_n)$ is a coordinate system in W (as $g(a)\neq 0$, $T_a(P_a)$ and $T_a(U)$ meet transversely at a).

For $\alpha\in\mathcal{D}^{2n-p}(W)$, we have by (2.10)

$$\mathrm{Res}_f[\tilde{h}](\alpha)=\mathrm{Res}_{z_1,\dots,z_p}\left[\frac{\det A\cdot h}{z_1^r\dots z_p^r}\right](\alpha).$$

We must prove the above term is zero only for

$$\alpha = g(z,x)\,dz_1 \wedge \ldots \wedge dz_p \wedge dx_{p+1} \wedge \overline{dx_{p+1}} \wedge \ldots \wedge dx_n \wedge \overline{dx_n}, \qquad g \in \mathscr{C}_0^\infty(W)$$

(cf. [C-H], Prop. 2.13).

In this case,

$$\operatorname{Res}_f[\tilde{h}](\alpha) = \lim_{\delta \to 0} \int_{Z(I) \cap W \cap (|\rho| > \delta)} \mathit{res}(y)\, \pi^*(dx_{p+1} \wedge \overline{dx_{p+1}} \wedge \ldots \wedge \overline{dx_n})$$

where the semimeromorphic fibered residue function

$$\begin{aligned} \mathit{res}(y) &= \operatorname{Res}_{P_y; z_1, \ldots, z_p; y} \left[\left. \frac{\det A \cdot h \cdot g\, dz_1 \wedge \ldots \wedge dz_p}{z_1^r \ldots z_p^r} \right|_{P_y} \right] \\ &= \operatorname{Res}_{P_y; f; y} [\tilde{h}|_{P_y}] (g \cdot dz_1 \wedge \ldots \wedge dz_p|_{P_y}) = 0 \end{aligned}$$

for all $y \in W \cap Z(I)$ (cf. [C-H], Th. 1.8.3).

Hence, $\operatorname{supp}(\operatorname{res}_f[\tilde{h}]) \cap U \subseteq S$, which implies $\operatorname{supp}(\operatorname{Res}_f[\tilde{h}]) \cap U = \emptyset$, because the support of the residual current is empty, or equal to the union of some irreducible components of $Z(I)$ (cf. [C-H], Theorem 1.7.6).

Finally, we have proved that $x_0 \notin \operatorname{supp}(\operatorname{Res}_f[\tilde{h}])$, for all $x_0 \in \operatorname{reg} Z(I)$, and so, by the same property of purity of the support of residual currents we deduce that $\operatorname{Res}_f[\tilde{h}] = 0$ as wanted.

5. The global moderate cohomology group

Let $Y \subseteq X$ be an analytic subspace of codimension p. We define the p^{th} moderate cohomology group with coefficients in Ω^r and supports on Y to be the group of global sections:

$$H^p_{[Y]}(X, \Omega^r) := \Gamma(X, \mathscr{H}^p_{[Y]}(\Omega^r)).$$

Proposition 5.1. *If Y is locally a complete intersection, we have*

$$H^p_{[Y]}(X, \Omega^r) \simeq H^p_{\bar{\partial}}(\Gamma(X, {}'\mathscr{D}_Y^{r,\cdot})).$$

Proof. As we have already seen, the moderate cohomology sheaves $\mathscr{H}^{\cdot}_{[Y]}(\Omega^r)$ can be obtained from the presheaf

$$U \mapsto \operatorname{Ker} \bar{\partial} \subseteq \Gamma_Y(U, {}'\mathscr{D}^{r,\cdot}) / \bar{\partial}(\Gamma_Y(U, {}'\mathscr{D}^{r,\cdot-1})).$$

Now, one is able to check that the natural homomorphism

$$H^p_{\bar{\partial}}(\Gamma(X, {}'\mathscr{D}_Y^{r,\cdot})) \to \Gamma(X, \mathscr{H}^p_{\bar{\partial}}({}'\mathscr{D}_Y^{r,\cdot}))$$

is in fact an isomorphism, observing that the proof due to Siu-Trautmann ([S-T], Lemma 0.6) can be adapted to the moderate case by the following facts:

i) $H^q(X, {}'\mathscr{D}_Y^{r,\cdot}) = 0$ for every $q \geqq 1$, because ${}'\mathscr{D}_Y^{r,\cdot}$ are fine sheaves.

ii) $\mathscr{H}^j_{[Y]}(\Omega^r) = 0\ \forall j \neq p$, as the stalk of a point $x \in X$ is given by

$$\operatorname{inj}\lim_k \operatorname{Ext}^j_{\mathscr{O}_{X;x}}((\mathscr{O}/I_k)_x, \Omega^r_x)$$

where I_k are regular ideals. Functor $\mathscr{H}om$ applied to Koszul's resolution of $\mathcal{O}/I_k$ gives a complex with zero j-cohomology for $j \neq p$. (cf. [G-H], p. 690).

Remark. Proposition 5.1 is true for a general subspace Y of codimension p, because $\mathscr{H}^j_{[Y]}(\Omega^r)=0$ holds for $j<p$.

We give now characterization theorems for the global moderate cohomology group:

Theorem 5.2. *Let X be a holomorphic manifold of dimension n, Y and analytic subspace of pure codimension p and $\mathscr{Y}=\{Y_1, \dots, Y_p\}$ a family of hypersurfaces such that $\bigcap \mathscr{Y}=Y$.*

Let $T \in \Gamma_Y(X, '\mathscr{D}^{r,p})$ a $\bar{\partial}$-closed current

a) *There exist*

i) *$\mathscr{U}=(U_\alpha)_{\alpha\in A}$ an open covering of X,*

ii) *a family of meromorphic forms $(\omega_\alpha)_{\alpha\in A}$,*
$\omega_\alpha \in \Gamma(U_\alpha,\ \Omega^r(*\cup\mathscr{Y}))$, defining a global section of $\mathscr{Q}$ (3.4), i.e. $\omega_\alpha-\omega_\beta=\sum_{i=1}^{p}\omega(i)$, *where*
$\omega(i) \in \Gamma(U_\alpha \cap U_\beta,\ \Omega^r(*\cup\mathscr{Y}(i)))$ is regular on Y_i.

iii) *$S \in \Gamma_Y(X, '\mathscr{D}^{r,p-1})$*
such that

$$T=R_{\mathscr{Y}}[(\omega_\alpha)]+\bar{\partial}S.$$

(*The collection $(R_{\mathscr{Y}}[\omega_\alpha])_{\alpha\in A}$ defines a global current $R_{\mathscr{Y}}[(\omega_\alpha)]$ by virtue of compatibility in* ii).)

b) *This splitting is unique in the following sense:*

Let $\mathscr{Y}'=\{Y_1', \dots, Y_p'\}$ be another family such that $\bigcap \mathscr{Y}'=Y$, $\mathscr{V}=(V_\beta)_{\beta\in B}$ an open covering, $(\omega_\beta')_{\beta\in B} \in \Gamma(X, \mathscr{Q}')$ (where $\mathscr{Q}'$ is the quotient sheaf associated to $\mathscr{Y}'$), and $S' \in \Gamma_Y(X, '\mathscr{D}^{r,p-1})$ such that:

$$T=R_{\mathscr{Y}'}[(\omega_\beta')]+\bar{\partial}S'$$

then

$$R_{\mathscr{Y}}[(\omega_\alpha)]=R_{\mathscr{Y}'}[(\omega_\beta')]$$

and

$$\bar{\partial}S=\bar{\partial}S'.$$

Proof. By 5.1, the sheaf isomorphism

$$\mathscr{Q} \xrightarrow{R} \mathscr{H}^p_{\bar{\partial}}('\mathscr{D}^{r,\cdot}_Y)$$

provides a group isomorphism

$$\Gamma(X, \mathscr{Q}) \xrightarrow{R} H^p_{[Y]}(X, \Omega^r),$$

$$(\omega_\alpha)_{\alpha\in A} \mapsto \overline{R_{\mathscr{Y}}[(\omega_\alpha)]},$$

proving a).

b) is in fact a local result and has been proved in Theorem 3.1.

Note. Under the assumptions

$$H^q(X, \Omega^r(*\bigcup_{i\in J} Y_i))=0 \quad \text{for every } q\geq 1$$

and every $J\in A(j,p)(1\leq j\leq p)$, the collection $(\omega_\alpha)_{\alpha\in A}$ of a) in Theorem 5.2 can be replaced by a global form $\omega\in\Gamma(X,\Omega^r(*\cup\mathcal{Y}))$ if and only if T has zero class in $H^p(X,\Omega^r)$ (cf. [D-H-S], Th. 2.9).

As a consequence, we obtain the following result:

Proposition 5.3. *Under the above hypotheses, for every $T\in\Gamma_Y(X,'\mathcal{D}^{r,p})$ the following conditions are equivalent:*

i) *There exists $S\in\Gamma(X,'\mathcal{D}^{r,p-1})$ such that $\bar\partial S=T$.*

ii) *There exist an analytic subspace Y' containing Y of pure codimension $p-1$ and $S'\in\Gamma_{Y'}(X,'\mathcal{D}^{r,p-1})$ such that $\bar\partial S'=T$.*

Proof. If i) is verified, let ω be a global meromorphic form with poles on $\bigcup\mathcal{Y}$ such that $T=R_{\mathcal{Y}}[\omega]+\bar\partial T_1$, $\operatorname{supp}(T_1)\subseteq Y$. Then, $Y'=\bigcap\mathcal{Y}(p)$ and $S'=RP_{\mathcal{Y}}[\omega]+T_1$ satisfy ii).

Gathering all the above information, we conclude the following:

Theorem 5.4. *Let X be a complex manifold and $Y\subseteq X$ an analytic subspace which is locally a complete intersection. For every $T\in\Gamma_Y(X,'\mathcal{D}^{r,p})$ $\bar\partial$-closed there exists one and only one locally residual current R such that*

$$T=R+\bar\partial S$$

with $S\in\Gamma_Y(X,'\mathcal{D}^{r,p-1})$.

Remark. As a consequence, each class in $H^p_{[Y]}(X,\Omega^r)$ has a representative with purely analytic support.

6. Cup-product

Given an analytic subspace Y of codimension p in X, which is the intersection $\bigcap\mathcal{Y}$ of a family $\mathcal{Y}=\{Y_1,\dots,Y_p\}$ of hypersurfaces, Theorem 5.2 provides a canonical representative for each class $T\in H^p_{[Y]}(X,\Omega^r)$; namely, $T=\overline{R_{\mathcal{Y}}[(\omega_\alpha)]}$ for a suitable collection $(\omega_\alpha)_{\alpha\in A}$ of meromorphic r-forms having its poles on $\bigcup\mathcal{Y}$.

Let $Y'=\bigcap\mathcal{Y}'$, $\mathcal{Y}'=\{Y'_1,\dots,Y'_q\}$, be an analytic subspace of codimension q such that $\operatorname{codim}_{\mathbb{C}}(Y\cap Y')=p+q$. In this situation, for any $T\in H^p_{[Y]}(X,\Omega^r)$ and $T'\in H^q_{[Y']}(X,\Omega^s)$ we can assign a residual current, whose class in $H^{p+q}_{[Y\cap Y']}(X,\Omega^{r+s})$ represents their "cup-product" in the following sense:

Theorem 6.1. *The linear mapping*

$$H^p_{[Y]}(X,\Omega^r)\otimes H^q_{[Y']}(X,\Omega^s)\to H^{p+q}_{[Y\cap Y']}(X,\Omega^{r+s}),$$

$$T\otimes T'\mapsto \overline{R_{\mathcal{Y}\cup\mathcal{Y}'}[(\phi_\alpha)]}$$

where $T=\overline{R_{\mathcal{Y}}[(\omega_\alpha)]}$, $T'=\overline{R_{\mathcal{Y}'}[(\omega'_\alpha)]}$ and $\phi_\alpha=\omega_\alpha\wedge\omega'_\alpha$, makes the following diagram commutative:

$$\begin{array}{ccc} H^p_{[Y]}(X,\Omega^r)\otimes H^q_{[Y']}(X,\Omega^s) & \longrightarrow & H^{p+q}_{[Y\cap Y']}(X,\Omega^{r+s}) \\ \downarrow & & \downarrow \\ H^p(X,\Omega^r)\otimes H^q(X,\Omega^s) & \xrightarrow{\ \cup\ } & H^{p+q}(X,\Omega^{r+s}) \end{array}$$

for $\cup$ *the standard cup-product in cohomology with coefficients in* $\Omega^{\bullet}$:

Remark. $\mathscr{Y}\cup\mathscr{Y}'$ denotes the ordered family $\{Y_1,\dots,Y_p,Y_1',\dots,Y_q'\}$. The compatibility conditions required for $(\omega_\alpha)_{\alpha\in A}$ and $(\omega'_\alpha)_{\alpha\in A}$ imply that the collection $(\phi_\alpha)_{\alpha\in A}$ actually defines a global residual current associated to the family $\mathscr{Y}\cup\mathscr{Y}'$.

Preliminary result

Given $\mathscr{Y}=\{Y_1,\dots,Y_t\}$ a family of hypersurfaces in complete intersection position, let us consider the resolution $(\mathscr{A}^{\bullet},\delta)$ of Ω^r by meromorphic r-forms with poles on $\mathscr{Y}$ and Čech type sheaf homomorphisms (cf. [D-H-S], Prop. 2.2):

$$0\to\Omega^r\to\bigoplus_{i=1}^{t}\Omega^r(*Y_i)\to\bigoplus_{i<j}\Omega^r(*Y_i\cup Y_j)\to\dots\to\Omega^r(*\cup\mathscr{Y})\to\mathscr{Q}\to 0.$$

The exactness of $(\mathscr{A}^{\bullet},\delta)$ tells us in particular that for any family $(\alpha_i)_{i=1,\dots,t}$ of meromorphic forms, $\alpha_i\in\Omega^r(*\cup\mathscr{Y}(i))$, such that $\sum_{i=1}^{t}\alpha_i=0$, there exists, for all $i=1,\dots,t$, a family $(\beta_{ij})_{j\neq i}$ of $t-1$ meromorphic r-forms $\beta_{ij}\in\Omega^r(*\cup\mathscr{Y}(i)(j))$ such that $\alpha_i=\sum_{j\neq i}\beta_{ij}$. When this is the case, we will say that α_i *splits* and β_{ij} is a *j-regular summand* of α_i. Moreover, if $\alpha_i\in\Gamma(W,\ \Omega^r(*\cup\mathscr{Y}(i))$ for some Stein open set W where there are global equations for each hypersurface of $\mathscr{Y}$, then, α_i splits globally in W, i.e. there exist j-regular summands $\beta_{ij}\in\Gamma(W,\Omega^r(*\cup\mathscr{Y}(i)(j)))$ for all $j\neq i$. (cf. [D-H-S], Lemma 3.4).

Now, we give the proof of Theorem 6.1:

First step. Let $\omega=(\omega_\alpha)_{\alpha\in A}$ be a compatible collection of meromorphic r-forms associated to an open Leray covering $\mathscr{U}=(U_\alpha)_{\alpha\in A}$ of X for the sheaf Ω^r, given by Stein open sets U_α. We will construct the image cocycle of the class of the residual current $T=\operatorname{Res}_{Y_1,\dots,Y_t}[\omega]$ through the canonical isomorphism

$$H^t_{\bar\partial}(\Gamma(X,{}'\mathscr{D}^{r,\cdot}))\to\check{H}^t(\mathscr{U},\Omega^r)=H^t(X,\Omega^r).$$

By the compatibility conditions, $(\check{\delta}\omega)_{\langle\alpha_0,\alpha_1\rangle}=\omega_{\alpha_1}-\omega_{\alpha_0}\in\Gamma(U_{\alpha_0}\cap U_{\alpha_1},\Omega^r(*\cup\mathscr{Y}))$ splits. We define $\sigma^1\in\check{\mathscr{C}}^1\left(\mathscr{U},\Omega^r\left(*\bigcup_{i=1}^{t-1}Y_i\right)\right)$ as follows:

$$\sigma^1_{\langle\alpha_0,\alpha_1\rangle}\text{ is a }t\text{-regular summand of }(\check{\delta}\omega)_{\langle\alpha_0,\alpha_1\rangle}.$$

Suppose that for $j<t-1$, we have already defined $\sigma^j\in\check{\mathscr{C}}^j\left(\mathscr{U},\Omega^r\left(*\bigcup_{i=1}^{t-j}Y_i\right)\right)$ such that $\sigma^j_{\langle\alpha_0,\dots,\alpha_j\rangle}$ is a $(t-(j-1))$-regular summand of $(\check{\delta}\sigma^{j-1})_{\langle\alpha_0,\dots,\alpha_j\rangle}$. As $\check{\delta}\check{\delta}\sigma^{j-1}=0$, by the preliminary result it holds that $(\check{\delta}\sigma^j)_{\langle\alpha_0,\dots,\alpha_{j+1}\rangle}$ splits

$\forall \alpha_0, \ldots, \alpha_{j+1} \in A$. We define then $\sigma^{j+1} \in \check{\mathscr{C}}^{j+1}\left(\mathscr{U}, \Omega^r\left(*\bigcup_{i=1}^{t-j-1} Y_i\right)\right)$ as follows:

$$\sigma^{j+1} \text{ is a } (t-j)\text{-regular summand of } (\check{\delta}\sigma^j)_{\langle \alpha_0, \ldots, \alpha_{j+1}\rangle}.$$

Finally, we have $\check{\delta}\sigma^{t-1} \in \check{\mathscr{C}}^t(\mathscr{U}, \Omega^r)$, and we note $\sigma^t = \check{\delta}\sigma^{t-1}$, $\sigma^0 = \omega$.

Now, for all $i = 0, \ldots, t-1$ and $\alpha_0, \ldots, \alpha_i \in A$, we consider the currents

$$T_{i\langle \alpha_0, \ldots, \alpha_i\rangle} = \operatorname{Res}_{Y_1, \ldots, Y_{t-1-i}} P_{Y_{t-i}}[\sigma^i_{\langle \alpha_0, \ldots, \alpha_i\rangle}]$$

which define a Čech cochain $T_i \in \check{\mathscr{C}}^i(\mathscr{U}, '\mathscr{D}^{r, t-i-1})$. It is easy to check that:

a) $\bar{\partial} T_0 = \operatorname{Res}_{Y_1, \ldots, Y_t}[\omega]$

b) $\forall i = 1, \ldots, t-1$, $\bar{\partial} T_i = \check{\delta} T_{i-1}$,

c) $\forall \alpha_0, \ldots, \alpha_t \in A$, $(\check{\delta} T_{t-1})_{\langle \alpha_0, \ldots, \alpha_t\rangle} = \int \sigma^t_{\langle \alpha_0, \ldots, \alpha_t\rangle} \wedge$. This proves that the canonical isomorphism

$$H^t_{\bar{\partial}}(\Gamma(X, '\mathscr{D}^{r,\cdot})) \to \check{H}^t(\mathscr{U}, \Omega^r)$$

sends

$$\overline{\operatorname{Res}_{Y_1, \ldots, Y_t}[\omega]} \mapsto \overline{\sigma^t}.$$

Second Step. Given T and T' as in the statement of Theorem 6.1, we can respectively construct by the first step two collections of cochains of meromorphic forms:

$$(\sigma^j)_{j=0, \ldots, p}, \quad \sigma^j \in \check{\mathscr{C}}^j\left(\mathscr{U}, \Omega^r\left(*\bigcup_{i=1}^{p-j} Y_i\right)\right)$$

and

$$(\sigma'^k)_{k=0, \ldots, q}, \quad \sigma'^k \in \check{\mathscr{C}}^k\left(\mathscr{U}, \Omega^s\left(*\bigcup_{i=1}^{q-k} Y'_i\right)\right)$$

satisfying

$$\sigma^0 = (\omega_\alpha)_{\alpha \in A}; \quad \sigma'^0 = (\omega'_\alpha)_{\alpha \in A};$$

$\sigma^{j+1}_{\langle \alpha_0, \ldots, \alpha_{j+1}\rangle}$ is a $(p-j)$-regular summand of $\check{\delta}\sigma^j_{\langle \alpha_0, \ldots, \alpha_{j+1}\rangle}$ for every $j = 0, \ldots, p-1$, $\sigma'^{j+1}_{\langle \alpha_0, \ldots, \alpha_{j+1}\rangle}$ is a $(q-j)$-regular summand of $\check{\delta}\sigma'^j_{\langle \alpha_0, \ldots, \alpha_{j+1}\rangle}$ for every $j = 0, \ldots, q-1$; $\sigma^p = \check{\delta}\sigma^{p-1}$; $\sigma'^q = \check{\delta}\sigma'^{q-1}$.

The classes $\overline{\sigma^p} \in \check{H}^p(\mathscr{U}, \Omega^r)$ and $\overline{\sigma'^q} \in \check{H}^q(\mathscr{U}, \Omega^s)$ represent T and T' respectively.

Let us call $Z_i = Y_i$, $i = 1, \ldots, p$, and $Z_i = Y'_{i-p}$, $i = p+1, \ldots, p+q$.

We define a new collection of meromorphic cochains

$$(\gamma^l)_{l=0, \ldots, p+q}, \quad \gamma^l \in \check{\mathscr{C}}^l\left(\mathscr{U}, \Omega^{r+s}\left(*\bigcup_{i=1}^{p+q-l} Z_i\right)\right)$$

as follows:

a) for $j = 0, \ldots, q$

$$\gamma^j_{\langle \alpha_0, \ldots, \alpha_j\rangle} = \omega_{\alpha_0} \wedge \sigma'^j_{\langle \alpha_0, \ldots, \alpha_j\rangle}.$$

b) for $j = 0, \ldots, p$

$$\gamma^{q+j}_{\langle \alpha_0, \ldots, \alpha_{q+j}\rangle} = \sigma^j_{\langle \alpha_0, \ldots, \alpha_j\rangle} \wedge \sigma'^q_{\langle \alpha_j, \ldots, \alpha_{q+j}\rangle}.$$

It is straightforward to verify that:

i) $\gamma^0_{\langle\alpha\rangle}=\omega_\alpha\wedge\omega'_\alpha$, $\forall\,\alpha\in A$,

ii) $\gamma^{j+1}_{\langle\alpha_0,\ldots,\alpha_{j+1}\rangle}$ is a $(p+q-j)$-regular summand of

$$\check{\delta}\gamma^j_{\langle\alpha_0,\ldots,\alpha_{j+1}\rangle},\qquad \forall j=0,\ldots,p+q-1,$$

iii) $\gamma^{p+q}=\check{\delta}\gamma^{p+q-1}$, and

iv) $\gamma^{p+q}_{\langle\alpha_0,\ldots,\alpha_{p+q}\rangle}=\sigma^p_{\langle\alpha_0,\ldots,\alpha_p\rangle}\wedge\sigma'^q_{\langle\alpha_p,\ldots,\alpha_{p+q}\rangle}$.

By the first step and conditions i), ii) and iii), we know that γ^{p+q} is a representing cocycle for the image of $\overline{\mathrm{Res}_{\mathscr{Y}\cup\mathscr{Y}'}[(\omega_\alpha\wedge\omega'_\alpha)]}$ in $\check{H}^{p+q}(\mathscr{U},\Omega^{r+s})$. The last condition shows that γ^{p+q} is also a representing cocycle for the cup-product of σ^p and σ'^q in Čech cohomology of $\Omega^\cdot$, which completes the proof of the theorem.

We show in the next lemma that the definition is independent of the choice of the families $\mathscr{Y}$ and $\mathscr{Y}'$.

Lemma 6.2. *Let* $\mathscr{Y}=\{Y_1,\ldots,Y_p\}$, $\mathscr{Z}=\{Z_1,\ldots,Z_p\}$, $\mathscr{Y}'=\{Y'_1,\ldots,Y'_q\}$, $\mathscr{Z}'=\{Z'_1,\ldots,Z'_q\}$ *be ordered families of hypersurfaces in* X *such that* $\bigcap\mathscr{Y}=\bigcap\mathscr{Z}=Y$, $\bigcap\mathscr{Y}'=\bigcap\mathscr{Z}'=Y'$, $\operatorname{codim}_{\mathbb{C}}(Y\cap Y')=p+q$.

Let U *be an open set,* $\tilde{\omega}\in\Gamma(U,\Omega^r(*\cup\mathscr{Y}))$, $\tilde{\gamma}\in\Gamma(U,\Omega^r(*\cup\mathscr{Z}))$ *such that* $R_{\mathscr{Y}}[\tilde{\omega}]=R_{\mathscr{Z}}[\tilde{\gamma}]$ and $\tilde{\omega}'\in\Gamma(U,\Omega^s(*\cup\mathscr{Y}'))$, $\tilde{\gamma}'\in\Gamma(U,\Omega^s(*\cup\mathscr{Z}'))$ *such that* $R_{\mathscr{Y}'}[\tilde{\omega}']=R_{\mathscr{Z}'}[\tilde{\gamma}']$. *Then,*

$$R_{\mathscr{Y}\cup\mathscr{Y}'}[\omega\wedge\omega']=R_{\mathscr{Z}\cup\mathscr{Z}'}[\gamma\wedge\gamma'].$$

Proof. Let $x\in X$. There exist a neighborhood U_x of x and $(f_i)_{i=1}^p$, $(f'_i)_{i=1}^q$, $(g_i)_{i=1}^p$, $(g'_i)_{i=1}^q$ in $\mathcal{O}(U_x)$ such that:

i)
$$Y_i=(f_i=0),\qquad Z_i=(g_i=0)\quad 1\leqq i\leqq p,$$
$$Y'_i=(f'_i=0),\qquad Z'_i=(g'_i=0)\quad 1\leqq i\leqq q.$$

ii) There exist $A\in\mathcal{O}^{p\times p}(U_x)$ and $B\in\mathcal{O}^{q\times q}(U_x)$ such that

$$g=A\cdot f\quad\text{and}\quad g'=B\cdot f.$$

iii)
$$\tilde{\omega}=\frac{\omega}{f_1\cdots f_p},\qquad \tilde{\gamma}=\frac{\gamma}{g_1\cdots g_p},$$
$$\tilde{\omega}'=\frac{\omega'}{f'_1\cdots f'_q},\qquad \tilde{\gamma}'=\frac{\gamma'}{g'_1\cdots g'_q}$$

where ω, γ, ω' and γ' are holomorphic forms. Transformation Law (2.10) gives

$$R_{\mathscr{Z}}\left[\frac{\gamma}{g_1\cdots g_p}\right]=R_{\mathscr{Y}}[\tilde{\omega}]=R_{\mathscr{Z}}\left[\frac{\det A\cdot\omega}{g_1\cdots g_p}\right].$$

Duality Law (4.1) yields that $\det A\cdot\omega-\gamma\in I_x(g_1,\ldots,g_p)$. On the other hand, $\det B\cdot\omega'-\gamma'\in I_x(g'_1,\ldots,g'_q)$; therefore

$$\det A\cdot\det B\cdot\omega\wedge\omega'-\gamma\wedge\gamma'\in I_x(g_1,\ldots,g_p,g'_1,\ldots,g'_q).$$

As

$$\begin{vmatrix} g \\ g' \end{vmatrix} = \left|\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right| \begin{vmatrix} f \\ f' \end{vmatrix},$$

we finally get:

$$\begin{aligned} R_{\mathscr{Y}\cup\mathscr{Y}'}[\tilde{\omega}\wedge\tilde{\omega}'] &= R_{\mathscr{X}\cup\mathscr{X}'}\left[\frac{\det A\cdot\det B\cdot\omega\wedge\omega'}{g_1\dots g_p\cdot g_1'\dots g_q'}\right] \\ &= R_{\mathscr{X}\cup\mathscr{X}'}[\tilde{\gamma}\wedge\tilde{\gamma}'] \quad \text{near } x. \end{aligned}$$

7. Application to analytic cycles

As it is well known, each $(n-p)$-dimensional analytic cycle T of the complex manifold X defines a $\bar{\partial}$-closed section $[T]\in\Gamma(X,'\mathscr{D}^{p,p}_{\operatorname{supp} T})$. In case that $T=[f^{-1}(0)]$ is the inverse image cycle associated to a holomorphic mapping $f=(f_1,\dots,f_p)$, $[T]$ can be represented as a residual current, namely

$$[T]=\left(\frac{1}{2\pi i}\right)^p \operatorname{Res}_{\mathscr{Y}}\left[\frac{df_1}{f_1}\wedge\dots\wedge\frac{df_p}{f_p}\right],$$

for $\mathscr{Y}=\{Z(f_1),\dots,Z(f_p)\}$ (cf. [C-H], p. 52).

We are now ready to show the following general result:

Theorem 7.1. *Every analytic cycle $[T]$ is a locally residual current.*

Proof. Let $x\in X$ and $\mathscr{Y}=\{Y_1,\dots,Y_p\}$ a family in complete intersection position such that $Y=\bigcap\mathscr{Y}\supseteq\operatorname{supp} T:=|T|$ near x. We get by Theorem 5.2 the splitting

$$[T]=\operatorname{Res}_{\mathscr{Y}}[\tilde{\mu}]+\bar{\partial}S$$

for some $\tilde{\mu}\in\Gamma(U,\Omega^p(*\cup\mathscr{Y}))$ and $S\in\Gamma(U,'\mathscr{D}^{p,p-1}_Y)$ in an open neighbourhood U of x. Our aim is to show that $\bar{\partial}S=0$.

For $y\in Y-|T|$, $\operatorname{Res}_{\mathscr{Y}}[\tilde{\mu}]=-\bar{\partial}S\in{}'\mathscr{D}^{p,p}_Y$, and so $\bar{\partial}S_y=0$ by 3.7. Then, $\operatorname{supp}(\bar{\partial}S)\subseteq|T|$.

For $y\in|T|\cap U$, take (V,z) a coordinate system near y such that all the coordinate projections $\pi\colon Y\cap V\to\mathbb{C}^{n-p}$ are branched covering maps. Let $\alpha\in\mathscr{D}^{n-p,n-p}(V)$ be a monomial, i.e. $\alpha=a_0\,dz_A\wedge d\bar{z}_B$ with $|A|=|B|=n-p$ and $a_0\in\mathscr{C}^\infty_0(V)$; so, $\tilde{\mu}\wedge\alpha=\mu\cdot a_0\,dz\wedge d\bar{z}_B$ with $\mu\in\mathscr{O}(*\cup\mathscr{Y})$ and $dz=dz_1\wedge dz_2\wedge\dots\wedge dz_n$. We assume $B=\{p+1,\dots,n\}$ and let $\pi\colon V\to\mathbb{C}^{n-p}$, $\pi(z)=(z_{p+1},\dots,z_n)$.

Recalling the fibered residue formula (cf. [C-H], p. 50), there exist $m\in\mathbb{N}$, $\rho\in\mathscr{O}(V)$ and for every $r\in\mathbb{N}_0^p$, $|r|\leq m$, a meromorphic function $k[r]\in\Gamma(Y\cap V,\mathscr{O}(*(\rho=0)))$, such that

$$\operatorname{Res}_{\mathscr{Y}}[\tilde{\mu}](\alpha)=P_{Y,\rho}\left(\sum_{|r|\leq m}\left.\frac{\partial^r a_0}{\partial z_1^{r_1}\dots\partial z_p^{r_p}}\right|_Y k[r]\,dz_B\wedge d\bar{z}_B\right)$$

where $P_{Y,\rho}$ denotes the principal value current on Y associated to $(\rho=0)$.

For each connected component of $(Y-|T|)\cap V$, we choose an open set $V'\subseteq V-|T|$ and p holomorphic functions $x_i\colon V'\to\mathbb{C}$ $1\leq i\leq p$, such that

$(V',(x_1,\dots,x_p,\ z_{p+1},\dots,z_n)$ is a coordinate system and $V'\cap Y=\{x_1=\dots=x_p=0\}$.

Thus, for every $a\in\mathscr{C}_0^\infty(V')$

$$0=P_{Y,\rho}\left(\sum_{|r|\leq m}\frac{\partial^r a}{\partial z_1^{r_1}\dots\partial z_p^{r_p}}\bigg|_Y\cdot k[r]\,dz_B\wedge d\bar z_B\right)$$
$$=P_{Y,\rho}\left(\sum_{|r|\leq m}\frac{\partial^r a}{\partial x_1^{r_1}\dots\partial x_p^{r_p}}\bigg|_Y\cdot k'[r]\,dz_B\wedge d\bar z_B\right)$$

where $k[r]=\sum_{|s|\leq m}\varphi_s^r\cdot k'[s]$ with $\varphi_s^r\in\mathcal{O}(V')$. Consequently, $k'[r]\equiv 0$ on $V'\cap Y$ for every r, $|r|\leq m$, which leads to $k[r]\equiv 0$ on $V'\cap Y$ for every r, $|r|\leq m$. Then, $k[r]\equiv 0$ on $Y-|T|$.

We may also choose $\rho'\in\mathcal{O}(V)$ with $(\rho'=0)$ containing $\operatorname{sing}|T|$ and the branch locus of π, and $k\in\mathcal{O}(V-(\rho'=0))$ such that

$$[T](\alpha)=P_{|T|,\rho'}(a_0|_{|T|}\cdot k\cdot dz_B\wedge d\bar z_B).$$

Then

$$\bar\partial S(\alpha)=([T]-\operatorname{Res}_{\mathscr{Y}}[\tilde\mu])(\alpha)$$
$$=P_{|T|,\rho\cdot\rho'}\left(\left\{a_0|_{|T|}\cdot(k-k[0])-\sum_{0<|r|\leq m}\frac{\partial^r a_0}{\partial z_1^{r_1}\dots\partial z_p^{r_p}}\bigg|_{|T|}k[r]\right\}\ dz_B\wedge d\bar z_B\right).$$

As above, for each connected component of $|T|\cap V$, we choose an open set V' and $(x_1,\dots,x_p)\in\mathcal{O}^p(V')$ such that $V'\cap Y=V'\cap|T|=\{x_1=\dots=x_p=0\}$ and $(V',(x_1,\dots,x_p,\ z_{p+1},\dots,z_n))$ is a coordinate system.

For $a\in\mathscr{C}_0^\infty(V')$,

$$\bar\partial S(a\,dz_A\wedge d\bar z_B)=P_{|T|,\rho\cdot\rho'}\left(\sum_{|r|\leq m}\frac{\partial^r a}{\partial x_1^{r_1}\dots\partial x_p^{r_p}}\bigg|_{|T|}h[r]\,dz_B\wedge d\bar z_B\right)$$

where $h[r]$ are holomorphic linear combinations of $k-k[0]$ and $k[r]$, $0<|r|\leq m$, in V'.

Let $a_1\in\mathscr{C}^\infty(V')$, $\alpha_1\in\mathscr{E}^{n-p,n-p}(V')$

$$a_1(x_1,\dots,x_p,\ z_{p+1},\dots,z_n):=\sum_{|t|\leq m}\frac{1}{t!}x_1^{t_1}\dots x_p^{t_p}\frac{\partial^t a}{\partial x_1^{t_1}\dots\partial x_p^{t_p}}\bigg|_{x_1=\dots=x_p=0}$$

$\alpha_1=a_1\,dz_A\wedge d\bar z_B$.

It holds:

i) $K=\operatorname{supp}\alpha_1\cap\operatorname{supp}S\subseteq\operatorname{supp}a_1\cap|T|\subseteq\operatorname{supp}a$ is compact.

ii) $\bar\partial\alpha_1=0$,

iii) $\dfrac{\partial^r a_1}{\partial x_1^{r_1}\dots\partial x_p^{r_p}}\bigg|_{|T|}=\dfrac{\partial^r a}{\partial x_1^{r_1}\dots\partial x_p^{r_p}}\bigg|_{|T|}\ \forall r,\quad |r|\leq m.$

Let S^* be the usual extension of S to $\mathscr{C}^\infty$-forms such that $\operatorname{supp}\beta\cap\operatorname{supp}S:=K_0$ is a compact set, i.e. $S^*(\beta)=S(\varphi\cdot\beta)$ for any $\varphi\in\mathscr{C}_0^\infty$ with $\varphi\equiv 1$ *in a* neighbourhood of K_0. It is easy to check that $(\bar\partial S)^*(\beta)=-S^*(\bar\partial\beta)$, provided that $\operatorname{supp}\beta\cap\operatorname{supp}S$ is compact.

If $\alpha := a\, dz_A \wedge d\bar{z}_B$

$$\begin{aligned}\bar{\partial} S(\alpha) &= P_{Y,\rho\cdot\rho'}\left(\sum_{|r|\leqq m} \left.\frac{\partial^r a}{\partial x_1^{r_1}\ldots\partial x_p^{r_p}}\right|_Y \cdot h[r]\, dz_B \wedge d\bar{z}_B\right)\\ &= P_{Y,\rho\rho'}\left(\sum_{|r|\leqq m} \left.\frac{\partial^r a_1}{\partial x_1^{r_1}\ldots\partial x_p^{r_p}}\right|_Y \cdot h[r]\, dz_B \wedge d\bar{z}_B\right)\\ &= (\bar{\partial} S)^*(\alpha_1) = -S^*(\bar{\partial}\alpha_1) = 0.\end{aligned}$$

Then, $h[r] \equiv 0$ on $|T| \cap V'$ for every r, $|r| \leqq m$, which leads to $k[r] \equiv 0$ on $|T| \cap V'$ for every r, $0 < |r| \leqq m$, and $k[0] \equiv k$ on $|T| \cap V'$. As a consequence, these last identities hold on $|T| \cap V$.

So, $\bar{\partial} S(\alpha) = 0$ for every $\alpha \in \mathscr{D}^{n-p,n-p}(V)$. q.e.d.

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