École d'Été CIMPA

Systèmes d'équations polynomiales : de la géométrie algébrique aux applications industrielles.
Près de Buenos Aires, Argentine.

Cours

Tableaux récapitulatifs

Première semaine : 3 cours de base, de 10 heures chacun

David A. Cox (Amherst College)	Eigenvalue and eigenvector methods for solving polynomial equations
Alicia Dickenstein (Univ. de Buenos Aires)	Introduction to residues and resultants
Lorenzo Robbiano (Univ. Genova)	Polynomial systems and some applications to statistics

Deuxième semaine : 6 séminaires avancés de 4 heures chacun

Ioannis Emiris (INRIA Sophia-Antipolis)	Toric resultants and applications to geometric modeling
André Galligo (Univ. de Nice)	Absolute multivariate polynomial factorization and algebraic variety decomposition
Bernard Mourrain (INRIA Sophia-Antipolis)	Symbolic Numeric tools for solving polynomial equations and applications
Juan Sabia (Univ. de Buenos Aires)	Efficient polynomial equation solving: algorithms and complexity
Michael Stillman (Cornell U.)	Computational algebraic geometry and the Macaulay2 system
Jan Verschelde (U. Illinois at Chicago)	Numerical algebraic geometry

Résumés disponibles

David A. Cox, *Eigenvalue and eigenvector methods for solving polynomial equations.*

Bibliography:

W. Auzinger and H.J. Stetter, An Elimination Algorithm for the Computation of all Zeros of a System of Multivariate Polynomial Equations, In *Proc. Intern. Conf. on Numerical Math.*, Intern. Series of Numerical Math., vol. 86, pp. 12–30, Birkhäuser, Basel, 1988,

H.M. Möller and H.J. Stetter, Multivariate Polynomial Equations with Multiple Zeros Solved by Matrix Eigenproblems, *Numer. Math.*, 70:311-329, 1995.

Paper of Moller-Tenberg.

Alicia Dickenstein, *Introduction to residues and resultants.*

Some basics of commutative algebra and Groebner bases: Zero dimensional ideals and their quotients, complete intersections. Review of residues in one variable. Multidimensional polynomial residues: properties and applications. Introduction to elimination theory. Resultants. The classical projective case: all known determinantal formulas of resultants. Relation between residues and resultants. Applications to polynomial system solving.

Bibliography:

E. Becker, J.P. Cardinal, M.-F. Roy and Z. Szafraniec, Multivariate Bezoutians, Kronecker symbol and Eisenbud-Levine formula. In *Algorithms in Algebraic Geometry and Applications*, L. González-Vega & T. Recio eds., Progress in Mathematics, vol. 143, p. 79–104, Ed. Birkhäuser, 1996.

J.P. Cardinal and B. Mourrain, Algebraic Approach of Residues and Applications, In *The Mathematics of Numerical Analysis*, Lectures in Applied Mathematics, vol. 32, pp. 189–210, 1996, AMS.

E. Cattani, A. Dickenstein and B. Sturmfels, Residues and Resultants, *J. Math. Sciences, Univ. Tokyo*, 5: 119–148, 1998.

E. Cattani, A. Dickenstein & B. Sturmfels: *Computing Multidimensional Residues*. Algorithms in Algebraic Geometry and Applications, L. González-Vega & T. Recio eds., Progress in Mathematics, vol. 143, p. 135–164, Ed. Birkhäuser, 1996.

D. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, Springer GTM, 1998.

C. D'Andrea & A. Dickenstein, Explicit Formulas for the Multivariate Resultant, *J. Pure & Applied Algebra*, 164/1-2:59–86, 2001.

M. Elkadi and B. Mourrain, *Géométrie Algébrique Effective en dimension 0 : de la théorie à la pratique*, Notes de cours, DEA de Mathématiques, Université de Nice. 2001.

I.M. Gelfand, M. Kapranov and A. Zelevinsky: Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.

J.-P. Jouanolou: *Formes d'inertie et résultant: un Formulaire*. Advances in Mathematics 126(1997), 119–250.

E. Kunz, *Kähler differentials*, Appendices, F. Vieweg & Son, 1986.

A. Tsikh, *Multidimensional residues and Their Applications*, Trans. of Math. Monographs, vol. 103, AMS, 1992.

Lorenzo Robbiano, *Polynomial systems and some applications to statistics.*

In the first part of my lectures we study systems of polynomial equations from the point of view of Groebner bases. In the second part we introduce problems from Design of Experiments, a branch of Statistics, and show how to use computational commutative algebra to solve some of them.

Bibliography:

Kreuzer and L. Robbiano, *Computational Commutative Algebra, vol. 1.* Springer, 2000.

L. Robbiano, Groebner Bases and Statistics. In *Groebner Bases and Applications, Proc. Conf. 33 Years of Groebner Bases*, London Mathematical Society Lecture Notes Series, B. Buchberger and F. Winkler (eds.), Cambridge University Press, Vol. 251 (1998), pp. 179–204.

Ioannis Emiris, *Toric resultants and applications to geometric modeling.*

Toric (or sparse) elimination theory uses combinatorial and discrete geometry to model the structure of a given system of algebraic equations. The basic objects are the Newton polytope of a polynomial, the Minkowski sum of a set of convex polytopes, and a mixed polyhedral subdivision of such a Minkowski sum. It is thus possible to describe certain algebraic properties of the given system by combinatorial means. In particular, the generic number of isolated roots is given by the mixed volume of the corresponding Newton polytopes. This also gives the degree of the toric (or sparse) resultant, which generalizes the classical projective resultant.

This seminar will provide an introduction to the theory of toric elimination and toric resultants, paying special attention to the algorithmic and computational issues involved. Different matrices expressing the toric resultants shall be discussed, and effective methods for their construction will be defined based on discrete geometric operations, as well as linear algebra, including the subdivision-based methods and the incremental algorithm which is especially relevant for the systems studied by A. Zelevinsky and B. Sturmfels. Toric resultant matrices generalizing Macaulay's matrix exhibit a structure close to that of Toeplitz matrices, which may reduce complexity by almost one order of magnitude. These matrices reduce the numeric approximation of all common roots to a problem in numerical linear algebra, as described in the courses of this School. In addition to a survey of recent results, the seminar shall point to open questions regarding the theory and the practice of toric elimination methods for system solving.

Available software on Maple (from library `multires`) and in C (from library `ALP`) shall be described, with exercises designed to familiarize the user with is main aspects. The goal is to provide an arsenal of efficient tools for system solving by exploiting the fact that systems encountered in engineering applications are, more often than not, characterized by some structure. This claim shall be substantiated by examples drawn from several application domains discussed in other courses of this School including robotics, vision, molecular biology and, most importantly, geometric and solid modeling and design.

Bibliography:

J.F. Canny and I.Z. Emiris, A Subdivision-Based Algorithm for the Sparse Resultant, *J. ACM*, 47(3):417–451, 2000.

J. Canny and P. Pedersen, An Algorithm for the Newton Resultant, Tech. report 1394 Comp. Science Dept., Cornell University, 1993.

D. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, Springer GTM, 1998.

C. D'Andrea, Macaulay-style formulas for the sparse resultant, *Trans. of the AMS*, 2002. To appear.

C. D'Andrea and I.Z. Emiris, Solving Degenerate Polynomial Systems, In *Proc. AMS-IMS-SIAM Conf. on Symbolic Manipulation, Mt. Holyoke, Massachusetts*, pp. 121–139, AMS Contemporary Mathematics, 2001.

I. Emiris, *Notes creuses sur l'élimination creuse*, Notes au DEA de Maths, U. Nice, 2001, ftp://ftp-sop.inria.fr/galaad/emiris/publis/NOTcreuxDEA.ps.gz.

I.Z. Emiris, A General Solver Based on Sparse Resultants: Numerical Issues and Kinematic Applications, INRIA Tech. report 3110, 1997.

I.Z. Emiris and J.F. Canny. Efficient incremental algorithms for the sparse resultant and the mixed volume. *J. Symbolic Computation*, 20(2):117–149, August 1995. I.Z. Emiris and B. Mourrain, *Matrices in elimination theory*, J. Symb. Comput., 28:3–44, 1999.

I.M. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants,* Birkhäuser, Boston, 1994.

B. Mourrain and V.Y. Pan, Multivariate Polynomials, Duality and Structured Matrices, *J. Complexity*, 16(1):110–180, 2000.

B. Sturmfels, On the Newton Polytope of the Resultant, *J. of Algebr. Combinatorics*, 3:207–236, 1994.

B. Sturmfels and A. Zelevinsky, Multigraded Resultants of Sylvester Type, *J. of Algebra*, 163(1):115–127, 1994.

Bernard Mourrain, *Symbolic Numeric tools for solving polynomial equations and applications.*

This course will be divided into a tutorial part and a problem solving part. In the first part, we will gives an introductive presentation of symbolic and numeric methods for solving equations. We will briefly recall well-known analytic methods and less known subdivision methods and will move to algebraic methods. Such methods are based on the study of the quotient algebra A of the polynomial ring modulo the ideal $I = (f_1, \ldots, f_m)$. We show how to deduce the geometry of the solutions, from the structure of A and in particular, how solving polynomial equations reduces to eigencomputations on these multiplication operators. We will mention a new method for computing the normal of elements in A, used to obtain a representation of the multiplication operators. based on these formulations. We will describe iterative methods exploiting the properties of A, and which can be applied to select a root (among the other roots), which maximize or minimize some criterion, or to count or isolate the roots in a given domain. A major operation in effective algebraic geometry is the projection, which is closely related to the theory of resultants. We present different notions and constructions of resultants and different geometric methods for solving systems of polynomial equations

In a second part, we will consider problems from different areas such CAD, robotics, computer vision, computational biology, . . . and show how to apply the methods that we have presented before. Practical experimentations in maple with the package `multires` and with the library `ALP` (environment for symbolic and numeric computations) will illustrate these developments.

Bibliography:

David Cox, John Little and Donal O'Shea: *Ideals, Varieties, and Algorithms, second edition*, Undergraduate Texts in Mathematics, Springer, 1997.

D. Eisenbud, Commutative Algebra with a view toward Algebraic Geometry, Berlin, Springer-Verlag, Graduate Texts in Math. 150, 1994.

M. Elkadi and B. Mourrain. *Géométrie Algébrique Effective en dimension 0 : de la théorie à la pratique*, Notes de cours, DEA de Mathématiques, Université de Nice. 2001.

B. Mourrain, An introduction to algebraic methods for solving polynomial equations, Tutorial in *Workshop on Constructive Algebra and Systems Theory, Acad. Art and Science, Amsterdam, 2000*, 2001, submitted. `http://www-sop.inria.fr/galaad/mourrain/Cours/2001tutorial.ps.gz.`

B. Mourrain and H. Prieto, A framework for Symbolic and Numeric Computations, Rapport de Recherche 4013, INRIA, 2000.

L. Busé, M. Elkadi and B. Mourrain, Residual Resultant of Complete Intersection, *J. Pure & Applied Algebra*, 2001.

I.Z. Emiris and B. Mourrain, *Matrices in elimination theory*, J. Symb. Comput., 28:3–44, 1999.

Juan Sabia, *Efficient polynomial equation solving: algorithms and complexity.*

This course intends to familiarize the assistants with the notion of *algebraic complexity* when solving polynomial equation systems. First, it will deal with the notion of dense representation of multivariate polynomials. Some results about the algebraic complexities of the effective Nullstellensatz, of quantifier elimination processes and of decomposition of varieties when using this model will be exposed. Then it will be shown how these complexities are essentially optimal in the dense representation model. This leads to a change of encoding of polynomials to get lower bounds for the complexity: the sparse representation and the straight-line program *representation* will be discussed. Finally, some complexity results in the straight-line program representation model will be shown (effective Nullstellensatz, quantifier elimination procedures, deformation techniques, for example).

Bibliography:

D. Brownawell, Bounds for the degrees in the Nullstellensatz, Ann. Math. 2nd Series, 126 (3) (1987) 577-591.

L. Caniglia, A. Galligo and J. Heintz, Some new effective bounds in computational geometry, Lecture Notes in Computer Science 357, Springer, Berlin (1989), 131-151.

A. L. Chistov and D. Y. Grigor'ev, Subexponential time solving systems of algebraic equations, LOMI preprint E-9-83, Steklov Institute, Leningrad (1983).

A. L. Chistov and D. Y. Grigor'ev, Complexity of quantifier elimination in the theory of algebraically closed fields, Lecture Notes in Computer Science 176, Springer, Berlin (1984), 17-31.

M. Elkadi and B. Mourrain, A new algorithm for the geometric decomposition of a variety, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (1999).

N. Fitchas, A. Galligo and J. Morgenstern, Precise sequential and parallel complexity bounds for quantifier elimination over algebraically closed fields, J. Pure Appl. Algebra 67 (1990) 1-14.

M. Giusti and J. Heintz, Algorithmes -disons rapides- pour la décomposition d'une variété algébrique en composantes irréductibles et équidimensionnelles, Progress in Mathematics 94, Birkhauser (1991) 169-193.

M. Giusti, J. Heintz and J. Sabia, On the efficiency of effective Nullstellensatz, Comput. Complexity 3, (1993) 56-95.

J. Heintz and C. P. Schorr, Testing polynomials which are easy to compute, Monographie 30 de l'Enseignement Mathématique (1982) 237-254.

G. Jeronimo and J. Sabia, Effective equidimensional decomposition of affine varieties, to appear in J. Pure Appl. Algebra (2001). G. Lecerf, Computing an equidimensional decomposition of an algebraic variety by means of geometric resolutions, Proceedings of the ISSAC 2000 Conference (ACM) (2000).

J. Kollar, Sharp effective Nullstellensatz, J. AMS 1 (1988), 963-975.

S. Puddu and J. Sabia, An effective algorithm for quantifier elimination over algebraically closed fields using straight-line programs, J. Pure Appl. Algebra 129 (1998), 173-200.

Jan Verschelde, *Numerical algebraic geometry.*

In a 1996 paper, Andrew Sommese and Charles Wampler began developing new area, "Numerical Algebraic Geometry", which would bear the same relation to "Algebraic Geometry" that "Numerical Linear Algebra" bears to "Linear Algebra".

To approximate all isolated solutions of polynomial systems, numerical path following techniques have been proven reliable and efficient during the past two decades. In the nineties, homotopy methods were developed to exploit special structures of the polynomial system, in particular its sparsity. For sparse systems, the roots are counted by the mixed volume of the Newton polytopes and computed by means of polyhedral homotopies.

In Numerical Algebraic Geometry we apply and integrate homotopy continuation methods to describe solution components of polynomial systems. One special, but important problem in Symbolic Computation concerns the approximate factorization of multivariate polynomials with approximate complex coefficients. Our algorithms to decompose positive dimensional solution sets of polynomial systems into irreducible components can be considered as symbolic-numeric, or perhaps rather as numeric-symbolic, since numerical interpolation methods are applied to produce symbolic results in the form of equations describing the irreducible components.

Applications from mechanical engineering motivated the development of Numerical Algebraic Geometry. The performance of our software on several test problems illustrate the effectiveness of the new methods.

Bibliography:

J. Verschelde, Polynomial Homotopies for Dense, Sparse and Determinantal Systems, MSRI Preprint Number 1999-041.

Sophia-Antipolis, 18th of March, 2002.