TEST YOUR SKILLS IN TROPICAL AND REAL GEOMETRY!

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Exercises marked by $\circ$ are supposed to be very easy. Exercises marked by $\ast$ might require (a bit) more work than the others.

(1) $\circ$ Let $(M,+)$ be an idempotent monoid with neutral element $e$ (i.e. $x + x = x$ for all $x$ in $M$). Show that $e$ is the only invertible element in $M$.

(2) $\circ$ Draw the graph of the tropical polynomials $P(x) = "x^3 + 2x^2 + 3x + (-1)"$ and $Q(x) = "x^3 + (-2)x^2 + 2x + (-1)"$, and locate their tropical roots.

(3) Determine the tropical roots of a degree 3 tropical polynomial $"a + bx + cx^2 + dx^3"$ in terms of $a, b, c,$ and $d$.

(4) Prove that $x_0$ is a tropical root of order $k$ of the tropical polynomial $P(x)$ if and only if $P(x) = "(x + x_0)^kQ(x)"$ where $Q(x)$ is a tropical polynomial which does not admit $x_0$ as a root.

(5) $\circ$ Let $val : \mathbb{K} \rightarrow \mathbb{T}$ be a non-archimedean valuation on a field $\mathbb{K}$. Show that if $val(a) \neq val(b)$ then $val(a + b) = \max(val(a), val(b))$.

(6) $\circ$ Find an equation for each of the tropical curves depicted in Figure 1

(7) $\circ$ Draw the tropical curves defined by the tropical polynomials $P(x,y) = "5 + 5x + 5y + 4xy + 1y^2 + x^2"$ and $Q(x,y) = "7 + 4x + y + 4xy + 3y^2 + (-3)x^2"$, as well as their dual subdivision.

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(8) Prove that a rational (i.e. with integer slopes) weighted balanced polyhedral complex of pure dimension \( n - 1 \) in \( \mathbb{R}^n \) is a tropical hypersurface.

(9) Draw the dual subdivision and find an equation for each of the tropical cubics depicted in Figure 2 (for some position of the vertices). Note that the primitive direction of edges which are neither horizontal, vertical, nor of slope 1 can be deduced from the balancing condition.

![Figure 2](image)

(10) Let \( d \geq 1 \) be a fixed integer. Prove that there exist finitely many possible primitive directions for an edge of a tropical curve of degree \( d \).

(11) Let \( \Delta \subset \mathbb{R}^n \) be an integer polytope of lattice area 1, and suppose for simplicity that the origin is a vertex of \( \Delta \). Prove that there exists a map in \( GL_n(\mathbb{Z}) \) which maps the standard simplex \( \text{Conv}\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} \) to \( \Delta \).

**Hint**: if \( \Delta \) is a simplex (i.e. has \( n + 1 \) vertices \( v_0, \ldots, v_n \)), then the volume of \( \Delta \) is equal to \( |\det(v_0v_1, \ldots, v_0v_n)| \).

(12) ° By perturbing the union of two ellipses meeting in 4 real points, construct all possible isotopy types of a non-empty non-singular quartic in \( \mathbb{R}P^2 \).

(13) ° Let us consider an arrangement of \( k \) ovals and \( l \) pseudolines in \( \mathbb{R}P^2 \), all of them being disjoint (so in particular \( l = 0 \) or 1). Prove that this arrangement is realizable by a non-singular real algebraic curve of degree \( 2k + l \).

(14) Classify non-singular real algebraic curves of degree 5 in \( \mathbb{R}P^2 \).

(15) ° Using patchworking, construct a non-singular real algebraic quartic in \( \mathbb{R}P^2 \) made of 2 ovals, one containing the other.

(16) ° Using patchworking, construct the two maximal real algebraic curves of degree 6 in \( \mathbb{R}P^2 \) originally constructed by Harnack and Hilbert.
(17) Construct as much as possible of maximal real algebraic curves of degree 7 in $\mathbb{RP}^2$. You can start with the subdivision of the Newton polygon depicted in Figure 3.

(18) Using patchworking, prove that there exists a maximal real algebraic curve of any degree $d \geq 1$.

(19) Draw the amoeba of the complex polynomials $\pm 2 + z + w + zw$ and $1 + zw^3 + z^2 + w^2$.

(20) We consider $P(z, w) = z + w - 1$ as a polynomial with coefficients in the field of transfinite Puiseux series. Compute $W(V(P)) \cap \text{Log}^{-1}(0, 0)$.

(21) Prove that the genus of a plane non-singular complex real algebraic curve $C$ is equal to the number of interior integer points of $\Delta(C)$.

(22) * Prove Bernstein Theorem for plane curves:

If $C_1$ and $C_2$ are two generic complex algebraic curves in $(\mathbb{C}^*)^2$, then the number of intersection points of $C_1$ and $C_2$ in $(\mathbb{C}^*)^2$ is exactly

$$\frac{A(\Delta(C_1) + \Delta(C_2)) - A(\Delta(C_1)) - A(\Delta(C_2))}{2}$$

You can assume as known that this number of intersection points in $(\mathbb{C}^*)^2$ is constant for two generic curves $C_1$ and $C_2$ as soon as their Newton polygons are fixed.