TEST YOUR SKILLS IN TROPICAL AND REAL GEOMETRY !

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Exercices marked by $^\circ$ are supposed to be very easy. Exercices marked by * might require (a bit) more work than the others.

(1) $^\circ$ Let $(M, +)$ be an idempotent monoid with neutral element e (i.e. $x + x = x$ for all x in M). Show that e is the only invertible element in M.

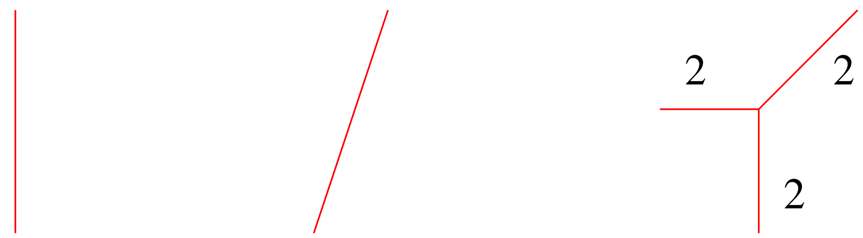
(2) $^\circ$ Draw the graph of the tropical polynomials $P(x) =$ "$x^3 + 2x^2 + 3x + (-1)$" and $Q(x) =$ "$x^3 + (-2)x^2 + 2x + (-1)$", and locate their tropical roots.

(3) Determine the tropical roots of a degree 3 tropical polynomial "$a + bx + cx^2 + dx^3$" in term of a, b, c, and d.

(4) Prove that x_0 is a tropical root of order k of the tropical polynomial $P(x)$ if and only if $P(x) =$ "$(x + x_0)^k Q(x)$" where $Q(x)$ is a tropical polynomial which does not admit x_0 as a root.

(5) $^\circ$ Let $val : \mathbb{K} \to \mathbb{T}$ be a non-archimedean valuation on a field $\mathbb{K}$. Show that if $val(a) \neq val(b)$ then $val(a + b) = \max(val(a), val(b))$.

(6) $^\circ$ Find an equation for each of the tropical curves depicted in Figure 1



a) the curve contains the point $(0, 1)$ and has primitive direction $(0, 1)$

b) the curve contains the point $(1, 0)$ and has primitive direction $(1, 3)$

c) the vertex is the origin $(0, 0)$ and the three directions are $(-1, 0)$, $(0, -1)$, and $(1, 1)$

FIGURE 1

(7) $^\circ$ Draw the tropical curves defined by the tropical polynomials $P(x, y) =$ "$5 + 5x + 5y + 4xy + 1y^2 + x^2$" and $Q(x, y) =$ "$7 + 4x + y + 4xy + 3y^2 + (-3)x^2$", as well as their dual subdivision.

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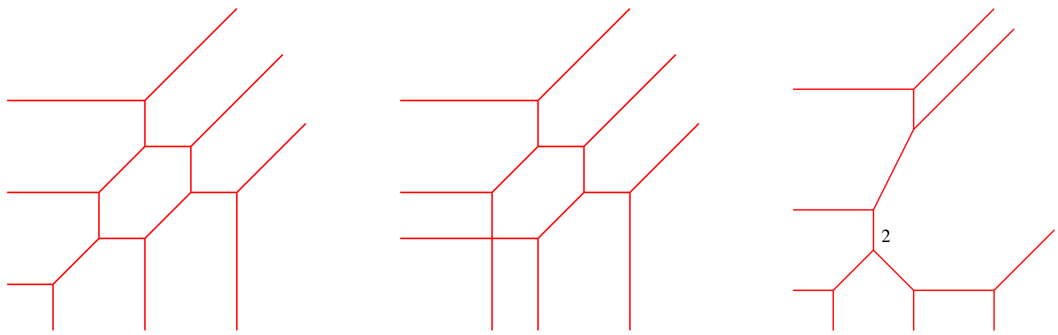
(8) Prove that a rational (i.e. with integer slopes) weighted balanced polyhedral complex of pure dimension $n-1$ in $\mathbb{R}^n$ is a tropical hypersurface.

(9) Draw the dual subdivision and find an equation for each of the tropical cubics depicted in Figure 2 (for some position of the vertices). Note that the primitive direction of edges which are neither horizontal, vertical, nor of slope 1 can be deduced from the balancing condition.



a) b) c)

FIGURE 2

(10) Let $d \geq 1$ be a fixed integer. Prove that there exist finitely many possible primitive directions for an edge of a tropical curve of degree d.

(11) Let $\Delta \subset \mathbb{R}^n$ be an integer polytope of lattice area 1, and suppose for simplicity that the origin is a vertex of Δ. Prove that there exists a map in $GL_n(\mathbb{Z})$ which maps the standard simplex $Conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$ to Δ.

Hint : if Δ is a simplex (i.e. has $n+1$ vertices $v_0,\ldots,v_n$), then the volume of Δ is equal to $|det(\vec{v_0v_1},\ldots,\vec{v_0v_n})|$.

(12) $^\circ$ By perturbing the union of two ellipses meeting in 4 real points, construct all possible isotopy types of a non-empty non-singular quartic in $\mathbb{R}P^2$.

(13) $^\circ$ Let us consider an arrangement of k ovals and l pseudolines in $\mathbb{R}P^2$, all of them beeing disjoint (so in particular $l = 0$ or 1). Prove that this arrangement is realizable by a non-singular real algebraic curve of degree $2k+l$.

(14) Classify non-singular real algebraic curves of degree 5 in $\mathbb{R}P^2$.

(15) $^\circ$ Using patchworking, construct a non-singular real algebraic quartic in $\mathbb{R}P^2$ made of 2 ovals, one containing the other.

(16) $^\circ$ Using patchworking, construct the two maximal real algebraic curves of degree 6 in $\mathbb{R}P^2$ originally constructed by Harnack and Hilbert.

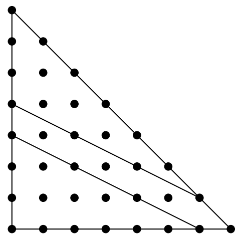


FIGURE 3

(17) Construct as much as possible of maximal real algebraic curves of degree 7 in $\mathbb{R}P^2$. You can start with the subdivision of the Newton polygon depicted in Figure 3.

(18) Using patchworking, prove that there exists a maximal real algebraic curve of any degree $d \geq 1$.

(19) Draw the amoeba of the complex polynomials $\pm 2 + z + w + zw$ and $1 + zw^3 + z^2 + w^2$.

(20) We consider $P(z, w) = z + w - 1$ as a polynomial with coefficients in the field of transfinite Puiseux series. Compute $W(V(P)) \cap \mathrm{Log}^{-1}(0, 0)$.

(21) Prove that the genus of a plane non-singular complex real algebraic curve C is equal to the number of interior integer points of $\Delta(C)$.

(22) * Prove Bernstein Theorem for plane curves :

If C_1 and C_2 are two generic complex algebraic curves in $(\mathbb{C}^*)^2$, then the number of intersection points of C_1 and C_2 in $(\mathbb{C}^*)^2$ is exactly

$$\frac{\mathcal{A}(\Delta(C_1) + \Delta(C_2)) - \mathcal{A}(\Delta(C_1)) - \mathcal{A}(\Delta(C_2))}{2}$$

You can assume as known that this number of intersection points in $(\mathbb{C}^*)^2$ is constant for two generic curves C_1 and C_2 as soon as their Newton polygons are fixed.

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