Signs of vectors in a linear subspace (a gentle introduction)

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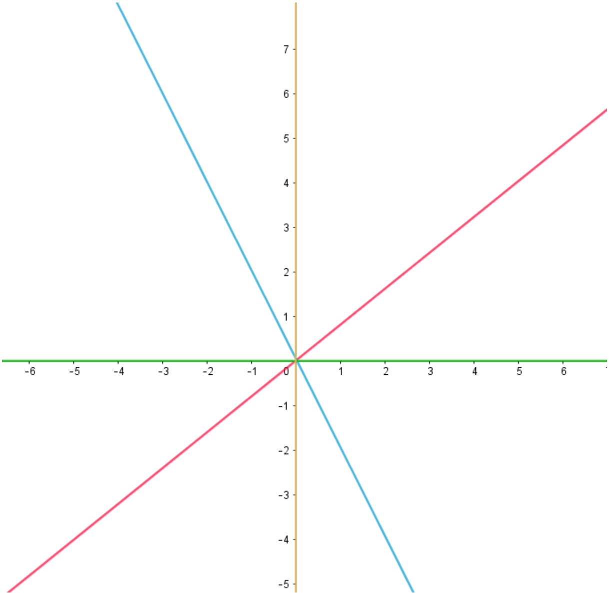
1 Which orthants does a linear subspace intersect?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Given a real matrix $C \in \mathbb{R}^{d\times n}$, when does there existe a positive vector $v \in \mathbb{R}^n_{>0}$ in the kernel of C?

Answer: When any nonzero vector in the row span of C has (at least) one positive coefficient and (at least) one negative coefficient.

Where does this condition come from and how can we verify it?

Quick answer: Computing signs of maximal minors of C.

Given a $V \subseteq \mathbb{R}^n$, we want to know in which orthants $\mathcal{O} \in \{-1, 0, 1\}^n$ in $\mathbb{R}^n$ there are vectors belonging to V. The *support* *supp*(v) is the subset of indices with nonzero coordinates (i.e. $\sigma_i = \text{sign}(v_i) \neq 0$). We will define the *circuits* of V and then we will see how to compute these circuits and how to find the set of all sign vectors $\sigma(v) \in \{-1, 0, 1\}$ for $v \in V$.

Circuits

- We call circuit of a subspace V any nonzero $r \in V$ with minimal support (with respect to inclusion) among all nonzero vectors in V
- It is easy to see that two circuits of V have the same support, then they differ by a multiplicative constant (they lie on the same line).
- If $\dim(V) = d$, circuits are "expected" to have $d-1$ nonzero entries, but this is not always the case!
- Given an orthant $\mathcal{O}$ (or a vector v), we say that a circuit r is *conformal with* $\mathcal{O}$ (resp. with v) if for any $i \in supp(r), \sigma(r_i) = \mathcal{O}_i$ (resp. $\sigma(r_i) = sigma(v_i)$).

- Let V be the subspace generated by the rows of

$$C = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \end{pmatrix}.$$

Then, $(0, 2, 4, 3)$ is a circuit of V but $(1, 1, 2, 1)$ is not. For instance, $(-2, 0, 0, 1) = (0, 2, 4, 3) - 2.(1, 1, 2, 1)$ is also a circuit of V because its support $\{1, 4\}$ is minimal.

- Given the orthant

$$\mathcal{O}_1 = (-, +, +, +),$$

the circuits $r = (0, 2, 4, 3)$ and $r' = (-2, 0, 0, 1)$ are conformal with it. The support of $\mathcal{O}_1$ coincides with the union of the supports of the circuits conformal with it and any linear combination with positive coefficients of r and r' gives a vector $v \in V$ with $\sigma(v) \in \mathcal{O}_1$.

Theorem [Rockafellar'69]

Give $v \neq 0$ in a linear subspace $V \subset \mathbb{R}^n$, there exist $r_1, \ldots, r_m$ of V such that:

- r_i is a circuit conformal with v, $\forall i = 1, \ldots, m$, y
- $v = \sum_{i=1}^{m} \lambda_i \, r_i, \quad \lambda_i \in \mathbb{R}_{>0}, \forall i = 1, \ldots, m.$

Then, in order to find $\sigma(v)$ forall $v \in V$, it is enough to find $\sigma(r)$ for all circuits r of V.

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Let $C \in \mathbb{R}^{d \times n}$ with rank d and denote by V its row span. Then, all the circuits in C are obtained this way (with multiples and repetitions):

For any subset $J \subseteq \{1, \ldots, n\}$ of cardinal $d-1$ we define the circuits r_J:

$$(r_J)_k = \left((-1)^{\mu(k,J)}\, det\left(C_{\{k\}\cup J}\right)\right), \, k = 1, \ldots, n, \quad r_J \in \mathbb{R}^n,$$

where $C_{\{k\}\cup J}$ is the submatrix corresponding to the columns of C with indices in $\{k\} \cup J$ (if $k \in J$ we set the determinant equal to 0) and $\mu(k, J)$ is the number of transpositions we need to do to order the sequence k followed by J in increasing order, that is, the number of indices in J strictly smaller than k.

Exercise: Understand the statemente and prove it :-).

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We were considering the subspace V generated by the rows of

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In this case, all the circuits are listed below:

- para $J = \{1\}$ son: $r_{\{1\}} = (0, -2, -4, -3)$
- para $J = \{2\}$ los circuitos son $r_{\{2\}} = (2, 0, 0, -1)$;
- para $J = \{3\}$, $r_{\{3\}} = (4, 0, 0, -2)$;
- para $J = \{4\}$, $r_{\{4\}} = (3, 1, 2, 0)$.

Exercise: Compute all sign vectors of V. Compute all sign vectors in $\ker C$.

Two sign vectors $\sigma, \sigma' \in \{0, +1, -1\}^n$ are said to beorthogonal if either for all i we have that $\sigma_i \cdot \sigma'_i = 0$ o there exist i, j such that $\sigma_i \cdot \sigma'_i = 1$ y $\sigma_j \cdot \sigma'_j = -1$.

Theorem: A sign vector σ' is the sign vector or a vector in $\ker C$ if and only if σ' σ' is orthogonal to all sign vectors σ of circuits of the row span of C (and then to all $\sigma(v)$ for all v in the row span).

This is a basic result in the context of oriented matroids. Exercise: Use it to show that there exists a positive vector in $\ker C$ if and only if any circuit r in the row span of C has a positive and an negative coordinate. Why this is an effectively checkable condition?

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- One basic reference (available online) is:
  J. Richter-Gebert and G. Ziegler: *Oriented Matroids*, in: Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.
- There are implementations.