3-Calabi-Yau Algebras from Steiner Triple Systems

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To do:

• Asymptotic number of STSs
• The algebra is defined over the prime field, and over \(\mathbb{Z}\) even.
• Each bad orientation of \(\mathbb{P}^2\) is obtained by flipping one block in a good orientation, and this procedure gives each of them exactly once.

1 Steiner triple systems

1.1. A Steiner triple system is a pair \((E, S)\) in which \(E\) is a non-empty finite set of points, and \(S\) a set of 3-subsets of \(E\), the blocks, such that every 2-subset of \(E\) is contained in exactly one block. The number \(n = |E|\) is the order of the system; we will assume throughout that \(n \geq 3\). Abusing language a bit, we will refer to \(S\) itself as a Steiner triple system.

One can easily see that \(n\) must be odd and that \(|S| = n(n - 1)/6\): it follows that \(n(n - 1) \equiv 0 \mod 6\), so that \(n \equiv 1\) or \(3 \mod 6\). A celebrated theorem of Reverend Thomas Kirkman [Kir47] asserts that this necessary condition for the existence of a Steiner triple system of order \(n\) is also sufficient. There are 1, 1, 1, 2, 80, and 11 084 874 829 non-isomorphic Steiner triple systems of order 7, 9, 13, 15 and 19, respectively; see sequence A030129 in Sloane’s database [Slo08] and the references therein.

1.2. If \((E, S)\) is a Steiner triple system, there is a unique binary operation \(\star : E \times E \to E\) such that

\[
\{i, j, i \star j\} \in S \text{ for all } i, j \in E \text{ with } i \neq j, \quad \text{and}
\]

\(i \star i = i\) for all \(i \in E\).

It is clear, moreover, that

\(i \star j = j \star i\) for all \(i, j \in E\), and

\(i \star (i \star j) = j\) for all \(i, j \in E\).

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1 For example, if \(X = \{(p, b) \in (\mathbb{P}^2) \times S : p \subset b\}\), then the first projection \(X \to (\mathbb{P}^2)\) is a bijection, and the fibers of the second projection \(X \to S\) have exactly three elements, provided \(|E| \geq 3\).
These last three conditions tell us that \((E, \star)\) is an idempotent symmetric quasigroup; see [Man86]. Conversely, if \((E, \star)\) is an idempotent symmetric quasigroup, then

\[
\{\{i, j, i \star j\} : i, j \in E, i \neq j\}
\]

is the set of blocks for a Steiner triple system on \(E\).

1.3. An orientation of a system \((E, S)\) is an assignment of a cyclic order to each of its blocks. We will write \((i, j, k) \in S\) to mean that \(\{i, j, k\}\) is a block and \(i \rightarrow j \rightarrow k \rightarrow i\) is the corresponding cyclic order, and \(i \succ j\) to mean that there is a \(k \in E\) such that \((i, j, k) \in S\).

We can record an orientation in various ways. First, if \(i, j, k \in E\), we write

\[
\varepsilon_{i,j,k} = \begin{cases} 
1, & \text{if } (i, j, k) \in S; \\
-1, & \text{if } (j, i, k) \in S; \\
0, & \text{in any other case,}
\end{cases}
\]

and

\[
\varepsilon_{i,j} = \begin{cases} 
1, & \text{if } (i, j, i \star j) \in E; \\
-1, & \text{if } (j, i, i \star j) \in E; \\
0, & \text{if } i = j.
\end{cases}
\]

It is clear that \(\varepsilon_{i,j,k}\) and \(\varepsilon_{i,j}\) are totally anti-symmetric in their indices, and that either of them determines the orientation of \(S\).

2 Examples

2.1. If \(d \geq 1\), let \(E = \mathbb{P}^d(F_2)\) be the projective \(d\)-space over the Galois field \(F_2\) with two elements and let \(S\) be the set of lines in \(E\); we have \(|E| = \vert S \vert = 2^{d+1} - 1\). Each pair of distinct points in \(E\) is contained in exactly one line and each line has \(3\) points, so \(S\) is a Steiner triple system on \(E\).

The smallest example in this family, the projective plane \(S = \mathbb{P}^2(F_2)\), is traditionally called the Fano plane. It is, up to isomorphism, the unique Steiner triple system of order \(7\). We can label its seven points as \(\{1, 2, \ldots, 7\}\), so that the blocks of \(S\) are \(\{1, 2, 3\}, \{2, 4, 6\}, \{3, 6, 5\}, \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}\) and \(\{1, 4, 5\}\); with this labelling, the configuration of blocks is depicted in figure 1 on the following page. The automorphism group of \(S\) is \(G = \text{PGL}(3, 2)\), the simple group of \(168 = 2^3 \cdot 3 \cdot 7\) elements. There are \(2^7\) orientations, split into two orbits by the action of \(G\) and the operation of inverting the orientation of all blocks. One orbit contains 16 orientations, each fixed by the normalizer of one of the Sylow 7-subgroups of \(G\), while the other contains 112 orientations, each fixed by a subgroup of \(G\) of order 3. We call these good and bad orientations, respectively. Half of the good orientations are characterized by the following property:

\[
\text{for each } i \in I, \text{ the set } \{j \in I : i \succ j\} \text{ is a block of } S;
\]
the other 8 are the opposite of these. In each bad orientation, there are three points forming a block where this condition is not satisfied and if we reverse the orientation of that block, we get a good orientation; in fact, this procedure sets up a bijection between the bad orientations and the good orientations with a marked block — this explains the fact that \(112 = 7 \cdot 16\).

2.2. Let \(E = \mathbb{A}^d(\mathbb{F}_3)\) be the affine \(d\)-space over the Galois field \(\mathbb{F}_3\) with three elements, and let \(S\) be the set of affine subspaces of dimension 1 in \(E\); we have \(|E| = 3^d\) and \(|S| = 3^{d-2}(3^d - 1)/2\). It is easy to check that \(S\) is a Steiner triple system on \(E\).

When \(d = 2\), this is the affine plane, which is, up to isomorphism, the unique Steiner triple system on 9 points. This combinatorial structure first appeared in 1835 in Julius Plücker’s work on the inflection points of a non-singular plane cubic curve: he showed that such a curve has exactly 9 ordinary inflection points, that the line joining each pair of them intersects the curve in another inflection point, and that the configuration of inflection points is therefore a Steiner triple system of order 9.

The automorphism group of the affine plane is \(G = \text{GL}(2,\mathbb{F}_3) \rtimes \mathbb{F}_3^2\). Under the action of \(G\) and the operation of reversing orientation of all blocks, the \(2^{12}\) orientations are grouped in \(12\) orbits. The smallest of these orbits has 16 elements, each stabilized by a group isomorphic to \((C_3 \times C_3) \rtimes C_3\); orientations in other orbits have smaller symmetry groups. These sixteen good orientations have a simple description: they are determined by arbitrarily picking orientations for the four lines through a fixed point and then extending this choice to obtain a translation-invariant orientation of the lines in the plane.

2.3. A Fischer group \([\text{Fis64, Man86}]\) is a pair \((G, E)\) with \(G\) a group and \(E \subseteq G\) a generating set of \(G\) consisting of elements of order 2 such that \((xy)^3 = 1\) and \(xyx \in E\) whenever \(x, y \in E\); they are also known as 3-transposition groups of width 1, as in \([\text{Asc97}]\). Clearly \(E\) is then a conjugacy class in \(G\); moreover, according to a beautiful combinatorial argument of Theodore Bolis \([\text{Bol74}]\), \(G\) is finite as soon as \(E\) is.

If \((G, E)\) is a Fischer group, one can easily check that the set

\[S = \{\{x, y, xyx\} : x, y \in E, x \neq y\}\]

is a Steiner triple system on \(E\). The Steiner triple systems obtained in this way are precisely those where every 3-subset of non-collinear points generates an affine plane.
[Asc97, Section 18], and go under the name of Hall triple systems. In terms of the corresponding quasi-group, they are characterized by the self-distributive property: for all \( x, y, z \in S \) one has \( x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \). Such quasi-groups appear in the literature of knot-theory and Hopf algebras as commutative quandles, or as exponent 3 commutative Moufang loops.

3 The algebra \( A(S) \)

3.1. We fix an oriented Steiner triple system \((E, S)\) of order \( n \); we assume whenever it is convenient that \( E = \{1, \ldots, n\} \). We fix a ground field \( k \) of characteristic zero and consider the vector space \( V \) freely spanned by a set \( X = \{x_i : i \in E\} \) of formal variables indexed by the elements of \( E \), let \( T(V) \) be the free algebra on \( V \), and let

\[
\Phi = \Phi_S = \sum_{i,j,k \in E} \varepsilon_{i,j,k} x_i x_j x_k \in T^3(V).
\]

The main object studied in this work is the algebra \( A = A(S) \) which is the quotient of \( T(V) \) by the ideal \( I = I(S) \) generated by the scaled cyclic derivatives

\[
r_k = r_k^S = \frac{1}{3} \frac{\partial \Phi}{\partial x_k} = \sum_{i,j \in E} \varepsilon_{i,j,k} x_i x_j, \quad k \in E,
\]

as in [Kon93] or [Gin06]. We let \( R = R_S \subseteq T^2(V) \) be the subspace spanned by \( r_1, \ldots, r_n \), so that \( I = (R) \). The defining relations are homogeneous of degree 2, so \( A \) is an \( \mathbb{N}_0 \)-graded algebra.

3.2. For each \( k \in E \) we have

\[
r_k = \sum_{(i,j,k) \in S} [x_i, x_j],
\]

so that \( r_k \) is a Lie-polynomial in the elements of \( X \), and we can consider the free Lie algebra \( g = g(S) \) generated by the set \( X \) subject to the relations \( \{r_i : i \in E\} \). Then \( g \) is a \( \mathbb{N} \)-graded Lie algebra, and there is an obvious isomorphism \( A \cong \mathcal{U}(g) \) that extends the identity of \( V \).

**Proposition.** The algebra \( A \) is an integral domain isomorphic to its opposite algebra \( A^{\text{op}} \).

**Proof.** All enveloping algebras of Lie algebras have these properties. \( \square \)

3.3. In general, the algebra \( A(S) \) does depend on the orientation of \( S \) and not only on the underlying Steiner triple system, and the nature of this dependency seems complicated. We can make the following easy observations, though.

For each \( k \in E \), we let \( \mu_k S \) be the oriented Steiner system obtained from \( S \) by inverting the orientation of all blocks of \( S \) that contain \( k \). If \( \phi : T(V) \to T(V) \) is the morphism of algebras such that \( \phi(x_k) = -x_k \) and \( \phi(x_i) = x_i \) for all \( i \neq k \), then
\( \phi(r_k^S) = -r_k^S \) and \( \phi(r_i^S) = r_i^S \) for all \( i \neq k \). It follows that \( \phi(I_S) = I_{\mu_k S} \) and that \( \phi \) induces an isomorphism \( A(E, S) \rightarrow A(E, \mu_k S) \).

The oriented Steiner system \( S' = \mu_1 \cdots \mu_n S \) is obtained from \( S \) by inverting the orientation of each of the blocks —indeed, the operation \( \mu_1 \cdots \mu_n \) reverses the orientation of each block three times— so we see that \( A(E, S) \cong A(E, S') \) as graded algebras.

**Proposition.** Let \( a \in E \) and let \( B_a \) be the algebra freely generated by \( \{ x_i : i \in E \setminus \{ a \} \} \) subject to the single relation

\[
\sum_{(i,j,a) \in S} [x_i, x_j] = 0.
\]

Then \( B_a \) is a quadratic Koszul algebra of global dimension 2 whose Hilbert series is

\[
h_{B_a}(t) = \frac{1}{1 - (n-1)t + t^2}.
\]

Moreover, there is a unique derivation \( \delta : B_a \rightarrow B_a \) such that

\[
\delta(x_i) = \varepsilon_{i,a} \sum_{(u,v,i \neq a) \in S} [x_u, x_v]
\]

for each \( i \in E \setminus \{ a \} \).

**Proof.** Let \( V' \) be the vector space with basis \( \{ x_i : i \in E \setminus \{ a \} \} \) and let \( R' \subseteq V' \otimes V' \) be the vector space spanned by \( \rho = \sum_{i,j \in E} \varepsilon_{i,j,a} x_i \otimes x_j \), so that \( B_a = T(V'/R') \). It follows from [Ber09, Proposition 3.1] and the fact that \( \rho \) is a non-degenerate anti-symmetric element of \( V' \otimes V' \) that \( B_a \) is a Koszul algebra of global dimension 2 and, moreover, that the Koszul complex corresponding to \( B_a \) is of the form

\[
0 \rightarrow B_a \otimes R' \rightarrow B_a \otimes V' \rightarrow B_a \rightarrow k
\]

Since this is exact, we deduce that the Hilbert series \( h_B(t) \) of \( B \) is

\[
h_{B_a}(t) = (1 - (n-1)t + t^2)^{-1}.
\]

It is obvious that the formula in the statement of the lemma defines a unique derivation \( \tilde{\delta}_a : T(V') \rightarrow B_a \), and to see that it descends to a derivation \( \delta : B_a \rightarrow B_a \) it is enough to check that \( \tilde{\delta}_a(\rho) = 0 \) in \( B_a \). To do this, we compute:

\[
\tilde{\delta}_a(\rho) = \tilde{\delta}_a \left( \sum_{(i,j,a) \in S} [x_i, x_j] \right)
\]

\[
= \sum_{(i,j,a) \in S} \tilde{\delta}_a(x_i), [x_j] + \sum_{(i,j,a) \in S} [x_i, \tilde{\delta}_a(x_j)]
\]

\[
= \sum_{(i,j,a) \in S} \varepsilon_{i,a} [x_u, x_v], [x_j] + \sum_{(i,j,a) \in S} \varepsilon_{j,a} [x_i, [x_u, x_v]],
\]
In the first sum, the coefficient $\varepsilon_{i,a}$ is equal to $-1$ in every term, and in the second sum, $\varepsilon_{j,a}$ is equal to 1 in every term. Using this, and interchanging the summation variables $i$ and $j$ in the second sum, we find that this is equal to

$$\sum_{(i,j,a)\in S} [x_j, [x_u, x_v]] + \sum_{(j,i,a)\in S} [x_j, [x_u, x_v]] = \sum_{(i,j,a)\in S} [x_j, [x_u, x_v]],$$

which can be rewritten as

$$\sum_{(i,j,a)\in S, i,j,k \neq a} \left( [x_j, [x_i, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] \right).$$

This is zero because commutators satisfy Jacobi’s identity.

**3.5. Proposition.** Let $(a, b, c) \in S$, let $B_a$ and $\delta_a$ be as in Proposition 3.4, and let $B_a[x_a; \delta_a]$ be the Ore extension of $B_a$ with respect to $\delta_a$.

(i) The set of non-commutative monomials on $\{x_i : i \in E \setminus \{a\}\}$ that do not contain $x_bx_c$ as a subword is a basis of $B$.

(ii) There is an isomorphism $A \cong B_a[x_a; \delta_a]$ of graded algebras, and the set of non-commutative monomials on $\{x_i : i \in E\}$ that do not contain $x_a x_i, \ i \in E \setminus \{a\}, \ x_b x_c$ as subwords is a basis of $A$.

(iii) The global dimension of $A$ is finite.

(iv) The Hilbert series of $A$ is

$$h_A(t) = \frac{1}{(1-t)(1-(n-1)t + t^2)} = \frac{1}{1 - nt + nt^2 - t^3}. \quad (2)$$

**Proof.** (i) The single relation defining $B_a$ can be written

$$x_b x_c = x_c x_b - \sum_{(i,j,a)\in S, i,j\neq c} [x_i, x_j].$$

By picking an arbitrary total order $\succ$ on the set $X' = \{x_i : i \in E \setminus \{a\}\}$ such that $b$ and $c$ are the two largest elements and $b \succ c$, and extending it degree-lexicographically to the set of words on $X'$, the claim follows immediately from Bergman’s Diamond Lemma [Ber78].

(ii) The existence of an isomorphism $A \cong B_a[x_a; \delta_a]$ is immediate. As a consequence, $A$ is a free left $B_a$-module on the set $\{x_a^i : i \in \mathbb{N}_0\}$. Taking into account the description of a basis of $B_a$ given in (i), we see at once that the set of monomials mentioned in the statement (ii) is a basis of $A$.

(iii) This follows from the fact that $A$ is an Ore extension of $B_a$, which has finite global dimension; see [MR87, Theorem 7.5.3 (i)].

(iv) As a consequence of the isomorphism in (ii), $h_A(t) = (1-t)^{-1}h_{B_a}(t)$, so the formula follows from that in Proposition 3.4. \qed
We could have proved (ii) in the same way as (i): the result follows from Bergman’s Diamond Lemma by picking any total order \( \succ \) on the generators of \( X \) such that \( a, b \) and \( c \) are the largest three elements and \( a \succ b \succ c \). We have avoided, though, the slightly messy calculation which proves that the monomial \( x_a x_b x_c \), which is the only ambiguity in the resulting rewriting system, is resolvable.

3.6. Corollary. The algebra \( A \) is neither left nor right noetherian and has infinite Gel’fand-Kirillov dimension.

Proof. Let \( i, j \in E \) be two distinct points, and consider the ideal

\[
I = \langle x_k : k \in E \setminus \{i, j\} \rangle \subset A.
\]

We have \( r_k \in I \) if \( k \in E \) is different from \( i \ast j \), and \( r_{i,j} \equiv [x_i, x_j] \mod I \). It follows that \( A/I \cong k[x_i, x_j] \), the polynomial ring in (the classes of) \( x_i \) and \( x_j \). We conclude that the Hilbert series of the left \( A \)-module \( A/I \) is \( h_{A/I}(t) = (1 - t)^{-2} \).

Suppose now that \( A \) is left noetherian. Since \( \text{gldim } A < \infty \), \( A/I \) has a free resolution of finite length by finitely generated graded modules and, in particular, there is a polynomial \( p \in \mathbb{Z}[t] \) such that \( h_{A/I} = h_{A^p} \). Looking at the order of the pole at 1 of the rational functions appearing in this last equality, we see that this is impossible. This, together the isomorphism \( A \cong A^{op} \), proves the first claim.

The second claim, that the Gel’fand-Kirillov dimension of \( A \) is infinite, follows from [SZ97, Corollary 2.2] and the fact that the Hilbert series of \( A \) computed in Proposition 3.5 has poles at points which are not roots of unity.

3.7. Though not noetherian, our algebra is not completely hopeless:

Proposition. The algebra \( A \) is left and right coherent.

Proof. By symmetry, it is enough to prove only right coherence, and without any loss of generality we may assume that \( (n-2, n-1, n) \in S \).

Let \( U_0 \) be the subspace of \( A \) spanned by \( x_1, \ldots, x_{n-2} \), and let \( U \) be the subspace spanned by \( U_0 \) and all the monomials \( x_1 x_{n-1}^\ell, \ldots, x_{n-3} x_{n-1}^\ell \) with \( \ell \geq 1 \). We assert that

the left ideal \( AU \) generated by \( U \) is in fact a bilateral ideal. (3)

To check this, we show that \( U x_i \subseteq AU \) for each \( i \in \{1, \ldots, n\} \).

This is obvious if \( i \leq n-2 \) because then \( U x_i \subseteq AU_0 \subseteq AU \). In view of the definition of \( U \), to show that \( U x_{n-1} \subseteq AU \) we need only prove that \( x_{n-2} x_{n-1} \in AU \), and this is easy: since \( r_n = 0 \) in \( A \), we have

\[
x_{n-2} x_{n-1} = x_{n-1} x_{n-2} + \sum_{\substack{(i,j,n) \in S \\{i,j\} \neq \{n-2,n-1\}}} [x_i, x_j] \in AU_0 \subseteq AU
\]

because in the sum the indices \( i \) and \( j \) can only take values different from \( n-1 \) and \( n \). Finally, we have to prove that \( U x_n \subseteq AU \).
• First, we fix \( k \in \{1, \ldots, n-3\} \) and show that \( x_k x_{n-1}^\ell x_n \in AU \) for all \( \ell \geq 0 \). Set \( u = (n-1) * (k * n) \), so that in particular \( u \notin \{n-1, n\} \). Since \( r_{k*(n-1)} = 0 \), we have

\[
x_k x_n = x_n x_k + \varepsilon_{n,k} \sum_{(i,j,k*n) \in S, \{i,j\} \neq \{k,n\}} [x_i, x_j]
\]

and, modulo \( AU_0 \) and possibly up to a sign, the right hand side is congruent to \( x_u x_{n-1} \). If \( u \leq n-3 \), then \( x_u x_{n-1} \in U \subseteq AU \); if instead \( u = n-2 \), from (4) we also have \( x_u x_{n-1} \in AU \). In any case, we see that \( x_k x_n \in AU \) and we can start an induction.

Let now \( \ell \geq 0 \) and suppose that \( x_k x_{\ell} x_{n-1} x_n \in AU \). What we have already done implies then that \( x_k x_{\ell} x_{n-1} x_n = AU x_{n-1} x_n \subseteq AU \), and since \( r_{n-2} = 0 \) we have

\[
x_{k+1} x_{n-1} = x_k x_{\ell} x_{n-1} x_n - x_k x_{\ell} x_{n-1} \sum_{(i,j,n-2) \in S, \{i,j\} \neq \{n-1,n\}} [x_i, x_j].
\]

As the sum in the second term of the right hand side is in \( AU_0 \), we see that \( x_k x_{\ell+1} x_n \) itself is in \( AU \). This completes the induction.

• Second and last, we have that \( r_{n-1} = 0 \) so

\[
x_{n-2} x_n = x_n x_{n-2} + \sum_{(i,j,n-1) \in S, \{i,j\} \neq \{n-2,n\}} [x_i, x_j] \in AU_0 \subseteq AU.
\]

This completes the proof of (3).

Let now \( I \) be the bilateral ideal generated by \( U_0 \) in \( A \). Since \( U_0 \subseteq U \subseteq I \) and \( AU \) is a bilateral ideal, we see that \( AU = I \): in other words, restricting the multiplication of \( A \) we obtain a surjective linear map \( \mu : A \otimes U \rightarrow I \). If we view \( U \) as a graded vector space with the grading induced from \( A \), its Hilbert series is

\[
h_U(t) = (n-2)t + (n-3) \sum_{i \geq 2} t^i = \frac{t(t-n+2)}{t-1}.
\]

On the other hand, the quotient \( A/I \) is isomorphic to the polynomial algebra \( k[x_{n-1}, x_n] \), so its Hilbert series is \( h_{A/I}(t) = (1-t)^{-2} \) and the Hilbert series of \( I \) is

\[
h_I(t) = h_A(t) - h_{A/I}(t) = \frac{1}{(1-t)(1-(n-1)t+t^2)} - \frac{1}{(1-t)^2}.
\]

A computation shows that \( h_{A \otimes U}(t) = h_A(t)h_U(t) \) is equal to \( h_I(t) \): this allows us to conclude that the surjection \( \mu \) is actually bijective. Since it is plainly a morphism of left \( A \)-modules, it follows that \( I \) is free as a left ideal. Since \( A/I \) is noetherian, we can therefore conclude that \( A \) is right coherent using the criterion given by Dmitri Piontkovski in [Pio08, Proposition 3.2]. \( \square \)
<table>
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Table 1. The dimensions of the first homogeneous components of $g$ for various values of $n$.

3.8. Proposition. Let $\xi$ be one of the roots of the polynomial $1 - (n - 1)t + t^2$. The Hilbert series of $g$ is

$$h_g(t) = \sum_{k \geq 1} \left( \frac{1}{k} \sum_{d \mid k} \mu\left(\frac{k}{d}\right)(1 + \xi^d + \xi^{-d}) \right)t^k.$$

Using this formula we can readily compute the dimensions of the homogeneous components of $g$ for small values of $n$; see table 1.

Proof. For each $k \geq 1$ let $g_k = \dim g_k$, so that $h_g(t) = \sum_{k \geq 1} g_k t^k$. The Poincaré-Birkhoff-Witt theorem implies that the Hilbert series of the enveloping algebra $\mathcal{U}(g)$ is then

$$h_{\mathcal{U}(g)}(t) = \prod_{k \geq 1} (1 - t^k)^{-g_k},$$

and the isomorphism $A \cong \mathcal{U}(g)$ implies that we have

$$\prod_{k \geq 1} \frac{1}{(1 - t^k)^{g_k}} = \frac{1}{(1 - t)(1 - (n - 1)t + t^2)}.$$

Taking logarithms, we see that

$$\sum_{k \geq 1} g_k \ln(1 - t^k) = \ln(1 - t) + \ln(1 - \xi t) + \ln(1 - \xi^{-1} t)$$

so

$$\sum_{k \geq 1} g_k \sum_{l \geq 1} \frac{t^{kl}}{l} = \sum_{k \geq 1} \frac{1}{k}(1 + \xi^k + \xi^{-k})t^k.$$

Looking at the coefficient of $t^k$ in both sides of this equality we find that

$$\sum_{d \mid k} d g_d = 1 + \xi^k + \xi^{-k}.$$

Using the classical Möbius inversion formula [Sta97, §3.7], then, we conclude that

$$g_k = \frac{1}{k} \sum_{d \mid k} \mu\left(\frac{k}{d}\right)(1 + \xi^d + \xi^{-d}),$$

for each $k \geq 1$, as we wanted. \qed
4 Homological properties

4.1. Proposition. The space

$$R_3 = (R \otimes V) \cap (V \otimes R) \subseteq T^3(V)$$

is 1-dimensional and generated by the element $\Phi$ defined in equation (1) on page 4, and

$$R_4 = (R \otimes V \otimes V) \cap (V \otimes R \otimes V) \cap (V \otimes V \otimes R) \subseteq T^4(V)$$

is the trivial subspace.

Proof. Let $\alpha = \sum_{i,j,k \in E} a_{i,j,k}x_i x_j x_k \in T^3(V)$ be an element of $R_3$, so that there exist scalars $u_{i,j}, v_{i,j} \in k$ for each $i, j \in E$, such that

$$\alpha = \sum_{i,j \in E} u_{i,j} r_i x_j \quad (5)$$

and

$$\alpha = \sum_{i,j \in E} v_{i,j} x_i r_j. \quad (6)$$

No monomial of the form $x_j^2$ appears in any $r_j$, so monomials of the form $x_i x_j^2$ and $x_i^2 x_j$ do not appear in $\alpha$, and

$$a_{i,i,j} = a_{i,j,j} = 0 \text{ for all } i, j \in E. \quad (7)$$

Since a block of $S$ is uniquely determined by any two of its elements, no monomial appears in two of $r_1, \ldots, r_n$. Looking at the coefficient of $x_i x_j x_k$ in (5) and of $x_k x_i x_j$ in (6) when $i \neq j$, then, we see that

$$a_{i,j,k} = -a_{j,i,k} \quad \text{and} \quad a_{k,i,j} = -a_{k,j,i} \text{ if } i, j, k \in E \text{ and } i \neq j. \quad (8)$$

It follows from (7) and (8) that

$$a_{i,j,k} \text{ is anti-symmetric in its three indices.} \quad (9)$$

We claim next that

$$\text{if } (i, j, l), (i', j', l) \in S \text{ and } k \in S, \text{ then } a_{i,j,k} = a_{i',j',k}. \quad (10)$$

Indeed, the monomials $x_i x_j x_k$ and $x_{i'} x_{j'} x_k$ appear with the same coefficient in the right hand side of (5), namely $u_{i,k}$. Using the anti-symmetry, we deduce also that

$$\text{if } (i, j, l), (i', j', l) \in S \text{ and } k \in S, \text{ then } a_{k,i,j} = a_{k,i',j'}. \quad (11)$$

Now, if $\{i, j, k\}$ a 3-subset of $E$ which is not a block of $S$, there exists an $l \in E$ such that $\{i, j, l\} \in S$, and also a $j' \in E$ such that $\{k, l, j'\} \in S$. In view of (10) and (9), this implies that $a_{i,j,k} = \pm a_{j,k,i} = 0$, and we conclude that

$$a_{i,j,k} = 0 \text{ if } \{i, j, k\} \text{ is not a block.}$$
Finally, suppose \((i,j,k), (i',j',k') \in S\) and that \(k \neq k'\), so that there exists an \(l \in E\) such that \((k',l,k) \in S\). Then
\[
\begin{align*}
a_{i,j,k} &= a_{k',l,k} \quad \text{using (10) and } (i,j,k), (k',l,k) \in S \\
a_{i',j',k'} &= a_{i',j',k'} \quad \text{using (11) and } (i,j,k), (i',j',k') \in S \\
a_{i',j',k'} &= a_{i',j',k'} \quad \text{because of (9)}.
\end{align*}
\]

It is at this point clear that \(\alpha\) is a scalar multiple of \(\Phi\), and this proves the first claim in the proposition, since \(\Phi \in R_3\).

Suppose now that \(\beta = \sum_{i,j,k,l} b_{i,j,k,l} x_i x_j x_k x_l \in R_4\). Proceeding as before, we can show that

\(b_{i,j,k,l}\) is anti-symmetric in its four indices

and that

\[
\text{if } (i,j,t), (i',j',t) \in S, \text{ then } b_{i,j,k,l} = b_{i',j',k,l}.
\]

Suppose now that \(\{i,j,k,l\}\) is a 4-subset of \(E\). We can suppose, up to renaming, that \(\{i,j,k\}\) is not a block, so there exists \(s \in E\) such that \(k \neq s\) and \((i,j,s) \in S\). Then there is \(t \in E\) such that \(\{k,t,s\}\) is a block, and

\[
b_{i,j,k,l} = \pm b_{k,t,k,l} = 0.
\]

We see that \(\beta = 0\). This proves the second claim in the proposition. \(\square\)

4.2. A connected graded algebra \(\Lambda\) is Gorenstein of dimension \(d\) and parameter \(\ell\) if \(\Ext^d_{\Lambda}(k,\Lambda) \cong k(\ell)\) and \(\Ext^p_{\Lambda}(k,\Lambda) = 0\) for all \(p \neq d\). In a similar vein, \(\Lambda\) is Calabi-Yau of dimension \(d\) if \(\Ext^d_{\Lambda}(\Lambda \otimes \Lambda) \cong \Lambda\) as a bimodule and \(\Ext^p_{\Lambda}(\Lambda \otimes \Lambda) = 0\) for all \(p \neq d\).

**Proposition.** The algebra \(A\) is a quadratic Koszul algebra of global dimension 3. It is a Gorenstein algebra of dimension 3 and parameter 3 and Calabi-Yau of dimension 3.

**Proof.** It follows from proposition 4.1 that the Koszul complex \([\text{Pri70}]\) for the quadratic algebra \(A\) is of the form

\[
\begin{array}{cccccc}
0 & \to & A \otimes R_3 & \xrightarrow{d_3} & A \otimes R & \xrightarrow{d_2} & A \otimes V & \xrightarrow{d_1} & A & \to & k \\
& & \downarrow d_1 & & \downarrow d_2 & & \downarrow d_3 & & & & \\
& & A \otimes R_3 & & A \otimes R & & A \otimes V & & A & & k
\end{array}
\]

with the canonical augmentation \(A \to k\) and differentials given by

\[
\begin{align*}
d_1(a \otimes x_k) &= ax_k, \\
d_2(a \otimes r_k) &= \sum_{(i,j,k) \in S} (a x_i \otimes x_j - a x_j \otimes x_i) = \sum_{i,j \in E} \varepsilon_{i,j,k} ax_i \otimes x_j, \\
d_3(a \otimes \Phi) &= \sum_{i \in E} ax_i \otimes r_i
\end{align*}
\]

(13)
for all \( a \in A \) and all \( k \in E \).

This complex is exact at \( k \), at \( A \) and at \( A \otimes V \). Since \( A \) is a domain, we see from (13) that the differential \( d_3 \) is an injective map, so the complex (12) is also exact at \( A \otimes R_3 \). Let \( \eta(t) \) be the Hilbert series for the cohomology space of (12) at \( A \otimes R \). Since that is the only possible non-zero cohomology space of the complex, and since the Euler characteristic does not change when passing to homology, we see that

\[
\eta(t) = 1 - n h_A(t) + nt^2 h_A(t) + t^3 h_A(t) = 1 - (1 - nt + nt^2 - t^3) h_A(t) = 0.
\]

In view of the expression (2) for \( h_A(t) \), we conclude that the complex (12) is also exact at \( A \otimes R \), so that \( A \) is a Koszul algebra, plainly of global dimension 3.

Let \( V^* \), \( R^* \) and \( R_3^* \) be the spaces dual to \( V \), \( R \) and \( R_3 \), and let \( \{ \hat{x}_1, \ldots, \hat{x}_n \} \), \( \{ \hat{r}_1, \ldots, \hat{r}_n \} \) and \( \{ \hat{\Phi} \} \) be the bases of these spaces which are dual to \( \{ x_1, \ldots, x_n \} \), \( \{ r_1, \ldots, r_n \} \) and \( \{ \Phi \} \), respectively. The complex obtained by applying the functor \( \text{hom}_A(-, A) \) to the Koszul complex (12) is, up to standard identifications,

\[
A \overset{d_1^*}{\longrightarrow} A \otimes V^* \overset{d_2^*}{\longrightarrow} A \otimes R^* \overset{d_3^*}{\longrightarrow} A \otimes R_3^* \tag{14}
\]

with differentials given by

\[
d_1^*(a) = \sum_{i \in E} x_i a \otimes \hat{x}_i,
\]

\[
d_2^*(a \otimes \hat{x}_k) = \sum_{i,j \in E} \xi_{i,j,k} x_i a \otimes \hat{r}_j,
\]

\[
d_3^*(a \otimes \hat{r}_k) = x_k a \otimes \hat{\Phi}
\]

for all \( a \in A \) and all \( k \in E \). Comparing these formulas with those of the differential in the Koszul complex we see at once that the complexes (12) and (14) are isomorphic up to a twist. It follows from this that

\[
\text{Ext}^p_A(k, A) \cong \begin{cases} 0, & \text{if } p \neq 3; \\ k(3), & \text{if } p = 3. \end{cases}
\]

By definition, then, \( A \) is Gorenstein of dimension 3 and parameter 3.

Let us now consider the diagram of \( A \)-bimodules

\[
0 \longrightarrow A \otimes R_3 \otimes A \overset{d_3}{\longrightarrow} A \otimes R \otimes A \overset{d_2}{\longrightarrow} A \otimes V \otimes A \overset{d_1}{\longrightarrow} A \otimes A \overset{\mu}{\longrightarrow} A \tag{15}
\]
where $\mu : A \otimes A \rightarrow A$ is the multiplication map, and
\[
d_1(a \otimes x_k \otimes b) = ax_k \otimes b - a \otimes x_kb, \\
d_2(a \otimes r_k \otimes b) = \sum_{(i,j,k) \in S} (ax_i \otimes x_j \otimes b - a \otimes x_i \otimes x_j b - ax_j \otimes x_i \otimes b + a \otimes x_i \otimes x_j)
\]
\[
= \sum_{i,j \in E} \varepsilon_{i,j,k}(ax_i \otimes x_j \otimes b + a \otimes x_i \otimes x_j b)
\]
\[
d_3(a \otimes \Phi \otimes b) = \sum_{i \in E} (ax_i \otimes r_i \otimes b - a \otimes r_i \otimes x_i b)
\]
for all $a, b \in A$ and all $k \in E$. A straightforward computation, which we leave to the reader, shows that this is a complex, which we write $A$-modules,

\[
\text{for all } a, b \in A \text{ and all } k \in E. 
\]

A straightforward computation, which we leave to the reader, shows that this is a complex, which we write $K$ and which we consider to be graded so that $A$ is in degree $-1$.

Since $K$ is bounded, there is a spectral sequence
\[
\tau \mathbb{E}^2_{p,q} \cong H_p(\text{Tor}^A_q(K, k)) \Rightarrow \text{Tor}^A_q(K, k)
\]
converging to the hyper-Tor; see [Wei94, §5.7]. Since $K$ is a complex of free right $A$-modules, $\tau \mathbb{E}^2_{p,q} = 0$ unless $q = 0$, and $\tau \mathbb{E}^2_{\bullet,0} = H_{\bullet}(K \otimes_A k)$. It is immediate, in view of the formulas for the complexes (12) and (15), than in fact $K \otimes_A k$ is isomorphic to the complex (12), so it is exact. It follows from this that $\text{Tor}^A_q(K, k) = 0$.

The second hyperhomology spectral sequence $\tau \mathbb{E}$ for $\text{Tor}^A_q(K, k)$ has
\[
\tau \mathbb{E}^2_{p,q} \cong \text{Tor}^A_p(H_q(K), k) \Rightarrow \text{Tor}^A_q(K, k).
\]

Suppose that
\[
q \in \mathbb{Z} \text{ is such that } H_r(K) = 0 \text{ for all } r < q. \tag{16}
\]

The location of zeros in $\tau \mathbb{E}^2$ implies that $\tau \mathbb{E}^2_{0,q} \cong H_q(K) \otimes_A k$ survives to $\tau \mathbb{E}^\infty = 0$, so $H_q(K) \otimes_A k = 0$. Since $H_q(K)$ is a non-negatively graded right $A$-module, it follows that $H_q(K) = 0$. By induction, starting from the trivial observation that (16) holds when $q = -1$, we conclude in this way that $K$ is exact.

We can now compute $\text{Ext}^A_\bullet(A, A \otimes A)$ from the complex obtained from (15) by applying the functor $\text{hom}_{A^\circ}(-, A \otimes A)$, which is, up to standard identifications and using the notations introduced above,

\[
A \otimes A \xrightarrow{d_1} A \otimes V^* \otimes A \xrightarrow{d_2} A \otimes R^* \otimes A \xrightarrow{d_3} A \otimes R_0^* \otimes A
\]

with differentials given by
\[
d_1(a \otimes b) = \sum_{i \in E} (ax_i \otimes \hat{x}_i \otimes b - a \otimes \hat{x}_i \otimes x_i b),
\]
\[
d_2(a \otimes \hat{x}_k \otimes b) = \sum_{i,j \in E} \varepsilon_{i,j,k}(x_ia \otimes \hat{x}_i \otimes b + a \otimes \hat{x}_j \otimes bx_i)
\]
\[
d_3(a \otimes \hat{r}_k \otimes b) = x_k a \otimes \hat{\Phi} \otimes b - a \otimes \hat{\Phi} \otimes bx_k
\]
for all $a, b \in A$ and all $k \in E$. Comparing these formulas with those for the differentials in the complex (15) we see immediately that there are $A$-bimodule isomorphisms

$$\text{Ext}_A^p(A, A \otimes A) \cong \begin{cases} 0, & \text{if } p \neq 3; \\ A, & \text{if } p = 3. \end{cases}$$

This tells us that $A$ is 3-Calabi-Yau. 

4.3. Using coherence and the resolutions constructed in the proof of the proposition above, we are able to describe the non-commutative projective geometry attached to the algebra $A$ in terms of “problems of linear algebra”.

We let $\text{GrMod} A$ be the category of graded left $A$-modules, $\text{tors} A$ the full subcategory of $\text{GrMod} A$ of finite dimensional modules, and $\text{Tors} A$ the full subcategory of $\text{GrMod} A$ of modules which are colimits of objects of $\text{tors} A$; these categories are abelian. Finally, we let $\text{grmod} A$ be the full subcategory of $\text{GrMod} A$ spanned by the modules of finite presentation; this is an abelian category precisely because $A$ is coherent, and $\text{tors} A$ is contained in it because $A$ is locally finite. The subcategories $\text{Tors} A \subseteq \text{GrMod} A$ and $\text{tors} A \subseteq \text{grmod} A$ are Serre subcategories, so we can consider the quotients

$$\text{qcoh Proj} A := \frac{\text{GrMod} A}{\text{Tors} A}, \quad \text{coh Proj} A := \frac{\text{grmod} A}{\text{tors} A},$$

which are again abelian categories; since $\text{tors} A = \text{Tors} A \cap \text{grmod} A$, we can identify $\text{coh Proj} A$ with the full subcategory of $\text{qcoh Proj} A$ spanned by the image of $\text{grmod} A$ under the quotient $\text{GrMod} A \to \text{qcoh Proj} A$.

We let $\pi : \text{coh} A \to \text{coh Proj} A$ be the quotient functor and write $O = \pi(A)$. If $F \in \text{qcoh Proj} A$ we write $F(1)$ the image of $F$ under the self-equivalence on $\text{qcoh Proj} A$ induced by the shift functor of $\text{GrMod} A$, and similarly for $F(n)$ with $n \in \mathbb{Z}$.

**Proposition.** Consider the object $E = O \oplus O(1) \oplus O(2) \in \text{coh Proj} A$, denote $\text{End}(E)$ its algebra of endomorphisms and let $\text{mod End}(E)$ and $\text{Mod End}(E)$ be the categories of finitely presented $\text{End}(E)$-modules and of all $\text{End}(E)$-modules, respectively. There are equivalences of triangulated categories

$$D(\text{qcoh Proj} A) \cong D(\text{Mod End}(E)).$$

and

$$D^b(\text{coh Proj} A) \cong D^b(\text{mod End}(E)).$$

The algebra $\text{End}(E)$ is finite dimensional, and isomorphic to the quotient of the path algebra of the quiver with three vertices and $2n$ arrows

\[ \bullet \rightarrow \bullet \rightarrow \bullet \]

\[ \bullet \rightarrow \bullet \rightarrow \bullet \]

\[ \bullet \rightarrow \bullet \rightarrow \bullet \]

\[ \bullet \rightarrow \bullet \rightarrow \bullet \]

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by the ideal generated by the elements
\[ \rho_k = \sum_{(i,j,k) \in S} (v_i u_j - v_j u_i), \quad k \in E. \]

Proof. This follows from [MM11, Theorems 4.12 and 4.14] because the algebra \( A \) is AS-regular of Gorenstein parameter 3 and graded-coherent. \( \square \)

5 Normal elements and the center

5.1 Proposition. Let \( n \geq 5 \). If \( a \in E \), the only normal elements in the algebra \( B_a \) of proposition 3.4 are the scalars.

Proof. Let \( m = (n - 1)/2; \) this is an integer because \( n \) is congruent to 1 or 3 modulo 6. Renaming the elements of \( E \), we can assume that \( E = \{1, \ldots, n\} \), that \( a = n \), and that \((1, m + 1, n), (2, m + 2, n), \ldots, (m, 2m, n)\) are the oriented blocks of \( S \) that contain \( n \). Then \( B = B_a \) is the free algebra generated by variables \( x_1, \ldots, x_{2m} \) subject to the relation

\[ [x_1, x_{m+1}] + [x_2, x_{m+2}] + \cdots + [x_m, x_{2m}] = 0. \]

Notice that \( m \geq 2 \) because \( n \geq 5 \). It is clear that \( B \) is a graded algebra for the grading that has all the generating variables in degree 1, and it is a domain because it is the enveloping algebra of a Lie algebra. Moreover, it is evident from Bergman’s diamond lemma [Ber78] that the set of words in the variables \( x_1, \ldots, x_{2m} \) that do not contain the subword \( x_{2m} x_m \) is a basis of \( B \); let us call these words admissible. In particular, we remark that \( x_1 \neq x_m \) in \( B \).

For each \( d \geq 0 \) let \( F_d \subseteq B \) be the subspace spanned by all words of the form \( wx_{2m}^i \) with \( i \leq d \) and \( w \) a word not ending in \( x_{2m} \); notice such a word is admissible iff \( w \) is. We have \( F_d \subseteq F_{d+1} \) for all \( d \geq 0 \) and clearly \( B = \bigcup_{d \geq 0} F_d \).

Let \( u \in B \) be an homogeneous normal element of degree \( k \geq 1 \). There exist \( d \geq 0 \) and homogeneous elements \( u_0, \ldots, u_d \in B \), each a linear combination of admissible words not ending in \( x_{2m} \), such that \( u = \sum_{i=0}^d u_i x_{2m}^i \) and \( u_d \neq 0 \). As \( u \) is normal, there exist \( \alpha_1, \ldots, \alpha_{2m} \in k \) such that \( x_1 u = \sum_{j=1}^{2m} \alpha_j u x_j \). Let us consider the elements

\[ s = x_1 u = \sum_{i=0}^d x_1 u_i x_{2m}^i. \]
and
\[
    t = \sum_{j=1}^{2m} \alpha_j u x_j = \sum_{j=1}^{2m} \sum_{i=0}^d \alpha_j u^i x_{2m}^j x_j
\]
\[
    = \alpha_m u_d x_{2m}^d x_m + \alpha_{2m} u_{d-1} x_{2m}^{d-1} + \alpha_{2m} u_d x_{2m}^{d+1}
\]
\[
    + \sum_{j \neq m, 2m}^{2m} \sum_{i=0}^d \alpha_j u^i x_{2m}^j x_j + \alpha_m \sum_{i=0}^{d-1} u_i x_{2m}^i x_m + \alpha_{2m} \sum_{i=0}^{d-2} u_i x_{2m}^{i+1}.
\]

We must have \( \alpha_{2m} = 0 \): if that were not the case, we would have \( F_d \supseteq s = t \in F_{d+1} \setminus F_d \), which is absurd. Using this, we see that
\[
    t = \alpha_m u_d x_{2m}^d x_m \equiv \alpha_m u_d x_m x_{2m}^d \pmod{F_{d-1}}
\]
while \( s \equiv x_1 u_d x_{2m}^d \pmod{F_{d-1}} \), so in fact \( (\alpha_m u_d x_m - x_1 u_d) x_{2m}^d \in F_{d-1} \). This is only possible if \( \alpha_m u_d x_m = x_1 u_d \), so that \( \alpha_m \neq 0 \) and, moreover, \( x_1 = x_m \). This is a contradiction. It follows that there are no homogeneous normal elements of positive degree in \( B \). Since elements of degree 0 are normal, the lemma is proved. \( \square \)

5.2. Lemma. Let \( \Lambda \) be an algebra that is a domain and whose only normal elements are the scalars. If \( \delta : \Lambda \to \Lambda \) is a derivation that is not inner, then the only normal elements in the Ore extension \( \Lambda[x; \delta] \) are the scalars.

Proof. Every non-zero element \( u \in \Lambda[x; \delta] \) can be written in a unique way as a sum \( u = \sum_{i=0}^d \lambda_i x^i \) with \( d \geq 0 \), \( \lambda_0, \ldots, \lambda_d \in \Lambda \) and \( \lambda_d \neq 0 \). Let us say that the weight of \( u \) is then \( w(u) = d \). Since \( \Lambda \) is a domain, we have \( uv \neq 0 \) and \( w(uv) = w(u) + w(v) \) whenever \( u, v \in \Lambda[x; \delta] \setminus 0 \).

Suppose now that \( u = \sum_{i=0}^d \lambda_i x^i \), with \( d \geq 0 \), \( \lambda_0, \ldots, \lambda_d \in \Lambda \) and \( \lambda_d \neq 0 \), is a non-zero normal element in \( \Lambda[x; \delta] \). Since \( u \) is normal and \( \Lambda \) is a domain, there is an automorphism \( \phi : \Lambda[x; \delta] \to \Lambda[x; \delta] \) of algebras such that \( uv = u \phi(v) \) for all \( v \in \Lambda[x; \delta] \). It follows from this that \( w(\phi(u)) = w(u) \) for all \( v \in \Lambda[x; \delta] \). In particular, \( \phi \) restricts to an isomorphism of algebras \( \phi : \Lambda \to \Lambda \).

If \( d = 0 \), then \( u \in \Lambda \) is a normal element in \( \Lambda \) and therefore a scalar. We may then suppose that \( d > 0 \). Let \( \mu \in \Lambda \). We have that
\[
    \mu u = \sum_{i=0}^d \mu \lambda_i x^i = \mu \lambda_d x^d + \mu \lambda_{d-1} x^{d-1} + \cdots
\]
and, since \( w(\phi(u)) = w(u) = 0 \) so that \( \phi(\mu) \in \Lambda \),
\[
    u \phi(\mu) = \sum_{i=0}^d \lambda_i x^i \phi(\mu) = \lambda_d \phi(\mu) x^d + (d \lambda_d \delta(\phi(\mu)) + \lambda_{d-1} \phi(\mu)) x^{d-1} + \cdots,
\]
where the omitted terms all involve powers of \( x \) smaller than \( d - 1 \).
It follows from this that \( \mu \lambda_d = \lambda_d \phi(\mu) \) for all \( \mu \in \Lambda \), so that \( \lambda_d \) is a normal element of \( \Lambda \). The hypothesis, then, implies that \( \lambda_d \) is a scalar. By eventually substituting \( u \) by \( \lambda_d^{-1} u \), we can assume that \( \lambda_d = 1 \). Then, in fact, we see that \( \phi(\mu) = \mu \) for all \( \mu \in \Lambda \) and, looking at the coefficient of \( x^{d-1} \) in \( \mu u \) and in \( u \phi(\mu) \), that \( \mu \lambda_{d-1} = d \delta(\mu) + \lambda_{d-1} \mu \) for all \( \mu \in \Lambda \) or, equivalently, that
\[
\delta(\mu) = \frac{1}{d}[\mu, \lambda_{d-1}], \quad \forall \mu \in \Lambda.
\]
This is impossible, because \( \delta \) is not an inner derivation.

5.3. Proposition. The only normal elements in \( A \) are the scalars and, in particular, the center \( Z(A) \) is spanned by 1.

Proof. If \( B \) and \( \delta : B \to B \) are as in proposition 3.4, the algebra \( A \) can be identified with the Ore extension \( B[x_n; \delta] \). In view of lemma 5.2, to prove the proposition it is enough to show that \( \delta \) is not an inner derivation.

Let us suppose otherwise, so that there exists \( \xi \in B \) such that \( \delta = [\xi, -] \). Since \( \delta \) is homogeneous of degree 1, we can assume that \( \xi \) itself is of degree 1, and then there exist \( \xi_1, \ldots, \xi_{n-1} \in k \) such that \( \xi = \sum_{i \in E'} \xi_i x_i \).

Let \( i \in E' \). Since \( \delta = [\xi, -] \), in \( B \) we have that
\[
\delta(x_i) = \varepsilon_{i,n} \sum_{(u,v,i \neq n) \in S} [x_u, x_v]
\]
is the same as
\[
[\xi, x_i] = \sum_{k \in E'} \xi_k [x_k, x_i],
\]
so there exists a scalar \( \lambda \in k \) such that
\[
\varepsilon_{i,n} \sum_{(u,v,i \neq n) \in S} [x_u, x_v] - \sum_{k \in E'} \xi_k [x_k, x_i] = \lambda \sum_{(u,v,n) \in S} [x_u, x_v].
\]
in the free algebra \( T(V') \). We can rewrite this equality as
\[
\varepsilon_{i,n} \sum_{(u,v,i \neq n) \in S} [x_u, x_v] - \sum_{k \in E'} \xi_k [x_k, x_i] = \lambda \sum_{(u,v,n) \in S} [x_u, x_v] + \xi_{i,n} [x_{i,n}, x_i].
\]
On the left we have a sum of monomials \( x_s x_t \) with \( s \neq t \), while on the right only monomials \( x_s x_t \) with \( s \neq t = n \) appear. It follows that both sides of the equality vanish, so that
\[
\lambda \sum_{(u,v,n) \in S} [x_u, x_v] + \xi_{i,n} [x_{i,n}, x_i] = 0.
\]
If \( j \in E \setminus \{n, i, i \ast n\} \), then the coefficient of \( x_j x_{j \ast n} \) on the left hand side of this last equation is \( \pm \lambda \), so we see that \( \lambda = 0 \) and, as a consequence, that \( \xi_{i,n} = 0 \).

Every element of \( E' \) is of the form \( i \ast n \) for some \( i \in E' \), so we have shown that \( \xi = 0 \). This is absurd, because \( \delta \neq 0 \).
6 Derivations

6.1. Let \( \mathfrak{gl}(V) \) be the Lie algebra of endomorphisms of \( V \), which we identify with \( \mathfrak{gl}(n,k) \) by fixing the basis \( X = \{ x_i : i \in E \} \) of \( V \). Considering \( \Lambda^3 V \) as a \( \mathfrak{gl}(V) \)-module with its natural diagonal action, we let

\[
\mathfrak{s} = \mathfrak{s}(S) = \{ g \in \mathfrak{gl}(V) : g \cdot \Phi = 0 \}.
\]

This is a Lie subalgebra of \( \mathfrak{gl}(V) \).

6.2. Proposition. Suppose that \( n > 3 \).

(i) The matrix \( g = (g_{i,j})_{i,j \in E} \in \mathfrak{gl}(V) \) is in \( \mathfrak{s} \) iff

\[
\epsilon_{j,k} g_{j \cdot k,i} + \epsilon_{k,i} g_{k \cdot i,j} + \epsilon_{i,j} g_{i \cdot j,k} = 0
\]

for all choices of distinct \( i, j, k \in E \). If that is the case, then \( \text{tr} g = 0 \).

(ii) If we let \( \mathfrak{gl}(V) \) act on \( V \otimes V \) diagonally, then \( \mathfrak{s} \oplus k \text{id} \) is precisely the subalgebra of \( \mathfrak{gl}(V) \) of elements that preserve the subspace \( R \) spanned by \( \{ r_1, \ldots, r_n \} \) and \( \mathfrak{s} = (\mathfrak{s} \oplus k \text{id}) \cap \mathfrak{s}(V) \). The \( \mathfrak{s} \)-modules \( R \) and \( V^* \) are isomorphic and \( \mathfrak{s} \) is an algebraic Lie subalgebra of \( \mathfrak{gl}(V) \).

(iii) The Lie algebra \( \mathfrak{s} \oplus k \text{id} \) acts faithfully by homogeneous derivations of degree \( 0 \) on \( A \), and this action provides an isomorphism \( \mathfrak{s} \oplus k \text{id} \to \text{Der}^0(A) \) to the space of all such derivations.

Proof. (i) We can compute

\[
g \cdot \Phi = \sum_{i,j,k \in E} \left( \epsilon_{i,j,k} g_{i \cdot k,j} x_j x_k + \epsilon_{i,j,k} g_{j \cdot j,i} x_i x_k + \epsilon_{i,j,k} g_{k \cdot i,j} x_i x_j \right)
\]

\[
= \sum_{i,j,k \in E} \left( \epsilon_{i,j,k} g_{i \cdot k,j} x_j x_k + \epsilon_{i,j,k} g_{l \cdot l,j} x_i x_k + \epsilon_{i,j,k} g_{l \cdot i,k} x_i x_j \right)
\]

so \( g \in \mathfrak{s} \) iff for all \( i, j, k \in E \) we have

\[
\sum_{l \in E} \left( \epsilon_{i,j,k} g_{l \cdot i} + \epsilon_{i,l,k} g_{l \cdot j} + \epsilon_{i,j,l} g_{l \cdot k} \right) x_i x_j x_k = 0.
\]

The statement follows from this since, for example,

\[
\sum_{l \in E} \epsilon_{i,j,k} g_{l \cdot i} = \epsilon_{j \cdot k,j,k} g_{i \cdot i,j}.
\]

and the fact that equation (19) holds trivially when \( |\{i,j,k\}| < 3 \).

Let us suppose now that \( g \in \mathfrak{s} \). If \( \{i,j,k\} \in S \), the condition (E\( i,j,k \)) tells us that \( g_{i,i} + g_{j,j} + g_{k,k} = 0 \), so summing over all blocks and using the fact that there are \( \frac{1}{2}(n-1) \) of them, we see that \( \frac{1}{2}(n-1) \text{tr} g = 0 \).

(ii) At this point it is clear that \( \mathfrak{s} \cap k \text{id} = 0 \), and it is obvious that \( k \text{id} \) preserves \( R \). Let \( g = (g_{i,j})_{i,j \in E} \in \mathfrak{s} \). To see that \( g \) preserves \( R \), it is enough to show that for all \( k \in E \) we have

\[
g \cdot r_k = -\sum_{l \in E} g_{l,k} r_l.
\]
One readily computes that $g \cdot r_{k} = \sum_{i,j \in E} (\varepsilon_{i,j,k} gl_{i} + \varepsilon_{i,j,k} gl_{j}) x_{i} x_{j}$, so

$$g \cdot r_{k} + \sum_{l \in E} g_{l} k r_{l} = \sum_{i,j \in E} \left( \sum_{l \in E} (\varepsilon_{i,j,k} gl_{i} + \varepsilon_{i,j,k} gl_{j} + \varepsilon_{i,j,l} gl_{k}) \right) x_{i} x_{j}.$$

If $i, j \in E$, then the coefficient of $x_{i} x_{j}$ in this sum is

$$\sum_{l \in E} (\varepsilon_{i,j,k} gl_{i} + \varepsilon_{i,j,k} gl_{j} + \varepsilon_{i,j,l} gl_{k}) = \varepsilon_{j,k} g_{j+1,i} + \varepsilon_{k,i} g_{k+1,i} + \varepsilon_{j,i} g_{j+1,k} = 0$$

in view of $(i)$. That equation $(20)$ holds follows from this, and we see that the algebra $\mathfrak{s} \oplus \mathbb{k} \text{id}$ preserves $R$.

Conversely, suppose that $g = (g_{i,j})_{i,j \in E} \in \mathfrak{gl}(V)$ preserves $R$. Then $g$ preserves the subspaces $V \otimes R$ and $R \otimes V$ of $V \otimes S$ and, as a consequence, also their intersection $R_{3} = (V \otimes R) \cap (R \otimes V)$, which we know is spanned by $\Phi$. It follows that $g \cdot \Phi = \lambda \Phi$ for some scalar $\lambda \in \mathbb{k}$. From $(18)$, then, we see that

$$\varepsilon_{i,j,k} \lambda = \varepsilon_{j,k} g_{j+1,i} + \varepsilon_{k,i} g_{k+1,i} + \varepsilon_{j,i} g_{j+1,k}$$

for all $i, j, k \in E$. In particular, we have

$$(i,j,k) \in S \implies \lambda = g_{i,i} + g_{j,j} + g_{k,k}. \quad (21)$$

Let us now fix $i \in E$. Using $(21)$ we see that

$$\frac{n-1}{2} \lambda = \sum_{(i,j,k) \in S} (g_{i,i} + g_{j,j} + g_{k,k})$$

because there are $(n-1)/2$ blocks which contain $i$ and, since every element of $E \setminus \{i\}$ belongs to a unique block of $S$ which contains $i$, this is

$$= \frac{n-1}{2} g_{i,i} + \sum_{j \in E \setminus \{i\}} g_{j,j} = \frac{n-3}{2} g_{i,i} + \text{tr} g.$$

It follows easily from this $g_{i,i}$ is really independent of $i$ and that, in fact, that $g_{i,i} = \lambda / 3$ for all $i \in E$. The matrix $g' = g - \frac{\lambda}{3} \text{id} \in \mathfrak{gl}(V)$, which has zero diagonal, also preserves $R$ and we have $g' \cdot \Phi = 0$, so that $g' \in \mathfrak{s}$ and therefore $g \in \mathfrak{s} \oplus \mathbb{k} \text{id}$.

That $\mathfrak{s} = (\mathfrak{s} \oplus \mathbb{k} \text{id}) \cap \mathfrak{sl}(V)$ is clear, because the elements of $\mathfrak{s}$ have zero diagonal. Since $\mathfrak{s} \oplus \mathbb{k} \text{id}$ is precisely the subalgebra of $\mathfrak{gl}(V)$ which preserves $R \subseteq V \otimes V$, the criterion given by Claude Chevalley in [Che47, Lemma 1] implies at once that $\mathfrak{s} \oplus \mathbb{k} \text{id}$ is an algebraic subalgebra of $\mathfrak{sl}(V)$ and therefore $\mathfrak{s}$, which is its intersection with $\mathfrak{sl}(V)$, is also algebraic.

$(iii)$ As usual, $\mathfrak{gl}(V)$ acts on the tensor algebra $T(V)$ by homogeneous derivations of degree $0$, and by restriction so does $\mathfrak{s}$. Since $\mathfrak{s} \oplus \mathbb{k} \text{id}$ preserves $R$, it preserves the ideal $I$ generated by $R$, and then there is an induced action of $\mathfrak{s}$ on the quotient $A = T(V)/I$, which is evidently homogeneous of degree $0$. This action is faithful, because its restriction to the degree one component $A_{1} = V$ of $V$ is the tautological representation.

Finally, suppose that $d : A \to A$ is an homogeneous derivation of degree $0$. Identifying the homogeneous component $A_{1}$ with $V$, we get by restriction a linear map $g = d|_{V} : V \to V$. The diagonal action of $g$ on $V \otimes V$ clearly preserves $R$, so $(ii)$ implies that $g \in \mathfrak{s} \oplus \mathbb{k} \text{id}$. This proves the last statement. \qed
6.3. It is to be expected, in view of the description given in Proposition 6.2, that the structure of the Lie algebra $\mathfrak{s}$ will strongly depend, in general, on the combinatorial minutiae of the oriented Steiner triple system under consideration. One possible way to approach the prodigious exuberance of possibilities we therefore encounter is to focus first on local combinatorial information. We present next an example of this.

A Pasch configuration is a set $\pi$ of six points of a Steiner triple system $S$ which contains four blocks. The six points can be labeled $a, b, i, j, k, l$ in such a way that

$$\{a, i, j\}, \{a, k, l\}, \{b, i, k\} \text{ and } \{b, j, l\}$$

are those blocks; see figure 2. The group of permutations of the points of $\pi$ which map blocks to blocks is then generated by $(ij)(kl)$ and $(jk)(ibla)$ and isomorphic to $S_4$.

**Lemma.** Suppose $\pi = \{a, b, i, j, k, l\} \subset E$ is a Pasch configuration with blocks as in (22). If $g = (g_{i,j})_{i,j \in E} \in \mathfrak{s}$, then

$$\begin{cases} 
\varepsilon_{i,j} g_{a,b} + \varepsilon_{j,b} g_{l,i} + \varepsilon_{b,i} g_{k,j} = 0 \\
\varepsilon_{k,l} g_{a,b} + \varepsilon_{b,k} g_{l,i} + \varepsilon_{b,i} g_{l,j} = 0 \\
\varepsilon_{i,k} g_{b,a} + \varepsilon_{k,a} g_{l,i} + \varepsilon_{l,a} g_{j,k} = 0 \\
\varepsilon_{j,l} g_{b,a} + \varepsilon_{a,j} g_{l,i} + \varepsilon_{l,a} g_{k,j} = 0
\end{cases}$$

and these four conditions are linearly independent. Each automorphism of the configuration maps this set of equations to itself, up to scalars, so it depends only on $\pi$ and not on the labeling in this sense.

Notice that $g_{u,v}$ appears in these equations exactly when $\{u, v\}$ is contained in $\pi$ but not in any block contained in $\pi$.

**Proof.** The 3-subsets of $\pi$ which are not blocks of $S$ and such that the blocks spanned by all of their 2-subsets are contained in $\pi$ are $\{b, i, j\}$, $\{b, k, l\}$, $\{a, i, k\}$ and $\{a, j, l\}$. The conditions $(E_{i,j,k})$ corresponding to them are precisely the equations listed in the statement of the lemma.

The determinant of the minor of the matrix of coefficients of these four equations corresponding to picking the first, second, third and fifth columns is

$$\varepsilon_{i,j}(\varepsilon_{a,i} \varepsilon_{b,k} \varepsilon_{j,l} + \varepsilon_{a,j} \varepsilon_{l,b} \varepsilon_{i,k}) = 2 \varepsilon_{b,k} \varepsilon_{j,l} \neq 0.$$
since \( \varepsilon_{l,b} = \varepsilon_{j,l} \), \( \varepsilon_{i,k} = -\varepsilon_{b,k} \) and \( \varepsilon_{a,j} = -\varepsilon_{a,i} = -\varepsilon_{i,j} \). This tells us that the equations are linearly independent.

The claim about the action of the automorphisms of the configuration on these equations can be checked by inspection.

This lemma allows us to get information on the Lie algebra \( \text{Der}^0(A) \) when there are many Pasch configurations in our Steiner system. An extreme example of this is the Fano plane:

6.4. Proposition. If \( S = \mathbb{P}^2(\mathbb{F}_2) \) is the Fano plane with some orientation, then the Lie algebra \( \mathfrak{s} \) has dimension 14.

Computer computations show immediately that it is not true that the dimension of the Lie algebra \( \mathfrak{s} \) depends in general only on the underlying Steiner triple system and not on the orientation.

Proof. In \( \mathbb{P}^2(\mathbb{F}_2) \) there are exactly seven Pasch configurations, the complements of each of its points. For each \( i \in E \) let \( \pi_i = E \setminus \{i\} \) be the corresponding configuration, let \( \Sigma_i \) be the set of four equations attached to \( \pi_i \) as in the previous lemma, and let \( s_i \) be the set of matrices \( g \in \text{gl}(V) \) which satisfy the equations of \( \Sigma_i \), which have zero diagonal and such that \( g_{u,v} = 0 \) when \( \{u,v,i\} \not\in S \)—from the lemma we know that \( \dim s_i = 2 \).

A matrix \( g \in \text{gl}(V) \) with vanishing diagonal is in \( \mathfrak{s} \) exactly when the equations \( (E_{i,j,k}) \), for all \( i, j, k \in E \) such that \( \{i,j,k\} \) is not a block of \( S \), hold. Now, if \( i, j, k \in E \) do not form a block and we denote \( u = i \star (j \star k) \), then the equation \( (E_{i,j,k}) \) is one of the four equations that appear in \( \Sigma_u \) and does not appear in \( \Sigma_v \) for any \( v \in E \setminus \{u\} \): it follows that \( g \) is in \( \mathfrak{s} \) iff it satisfies all the equations in \( \Sigma = \bigcup_{i \in E} \Sigma_i \).

Each system \( \Sigma_i \) involves the six components \( g_{j,k} \) of \( g \) such that \( \{i,j,k\} \in S \) and none other. The subsystems \( \Sigma_i \) of \( \Sigma \) are therefore independent of each other: each imposes four independent conditions on the six components of \( g \) it involves, and therefore in all the system \( \Sigma \) imposes \( 7 \cdot 4 \) independent conditions on the \( 7 \cdot 6 \) non-diagonal components of the matrix \( g \). Substracting dimensions, we see that \( \dim \mathfrak{s} = 14 \) and that, in fact, \( \mathfrak{s} = \bigoplus_{i \in E} s_i \). \qed

6.5. Corollary. If \( k \) is algebraically closed, the isomorphism type of the graded algebra \( \mathbb{A}(\mathbb{P}^2(\mathbb{F}_2)) \) corresponding to the Fano plane endowed with some orientation does not in fact depend on the orientation.

Proof. The algebra \( \mathbb{A}(S) \) attached to an oriented Steiner triple system is the Jacobian algebra of the cubic form \( \Phi_S \in \Lambda^3 V \), and its isomorphism type depends only on the orbit of \( \Phi_S \) under the action of \( \text{GL}(V) \) on \( \Lambda^3 V \). To prove the corollary it is therefore sufficient that we show that for all orientations of \( S = \mathbb{P}^2(\mathbb{F}_2) \) the orbit of \( \Phi_S \) is an open subset of \( \Lambda^3 V \) in its Zariski topology: the hypothesis made on the field \( k \) implies that there is at most one such orbit.

Let \( S \) be the Fano plane with one of its orientations. We have shown that there is a 14-dimensional Lie algebra \( \mathfrak{s} \subseteq \text{gl}(V) \) whose elements annihilate \( \Phi \in \Lambda^3 V \). Since \( \mathfrak{s} \) is
an algebraic Lie subalgebra of \( \mathfrak{gl}(V) \), there exists then an irreducible closed algebraic subgroup \( G \subseteq \text{GL}(V) \) whose Lie algebra is precisely \( \mathfrak{s} \). The group \( G \) is contained in the stabilizer of \( \Phi \) in \( \text{GL}(V) \) and in consequence the \( \text{GL}(V) \)-orbit of \( \Phi \) in \( \Lambda^3 V \) has dimension

\[
\dim \text{GL}(V) \Phi \geq \dim \text{GL}(V) - \dim G = 7^2 - 14 = 35 = \dim \Lambda^3 V.
\]

This implies that the orbit \( \text{GL}(V) \Phi \) is indeed a Zariski open subset of \( \Lambda^3 V \), as we were to show. \( \square \)

6.6. Proposition. If \( S \) is the Fano plane with a good orientation, then the Lie algebra \( \mathfrak{s} \) is semisimple of type \( G_2 \).

**Proof.** For concreteness, we suppose that the orientation on \( S = \mathbb{F}^2(\mathbb{F}_2) \) is in fact the one depicted in figure 1 on page 3. We can also suppose for the purpose of this proof that \( k = \mathbb{Q} \), the field of rational numbers, as the conclusion of the proposition is stable under extension of scalars.

The automorphism group of the oriented Steiner triple system is the subgroup \( G \subseteq S_7 \) which is the semidirect product \( C_7 \rtimes C_3 \) of its normal Sylow 7-subgroup \( C_7 \) generated by \( \gamma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) \) by its Sylow 3-subgroup \( C_3 \) generated by \( \sigma = (1 \ 2 \ 4)(3 \ 6 \ 5) \); notice that we have \( \gamma \sigma = \gamma^2 \sigma \).

We need some information on the rational representation theory of \( G \). According to [Ser77, Corollary 13.1], the isomorphism classes of simple rational representations of \( G \) are as many as the conjugacy classes of cyclic subgroups of \( G \); using the Sylow theorems we see at once, then, that there are three of them. One is of course the trivial module \( \mathbb{Q} \) and a second one is obtained from the simple \( 2 \)-dimensional representation \( W_2 \) of \( C_3 \) by restriction along the quotient map \( G \to C_3 \). Since the defining permutation representation of \( G \) is transitive on 2-subsets of \( \{1, \ldots, 7\} \), the complement of the trivial submodule in its linear permutation representation on \( \mathbb{Q}^7 \) is irreducible over \( \mathbb{Q} \): this provides us with the remaining simple rational \( G \)-module \( W_6 \), which is of degree 6. Given a rational representation \( W \) of \( G \), we write \( I(W) = (\dim V, \dim V^\gamma, \dim V^\sigma) \in \mathbb{N}_0^3 \). It is immediate that \( I(\mathbb{Q}) = (1, 1, 1), I(W_2) = (2, 2, 0) \) and \( I(W_6) = (6, 0, 2) \), and —since these three vectors are linearly independent— a rational representation \( W \) of \( G \) is completely determined by \( I(W) \), and we can use this to compute. For example, since \( (W_6 \otimes W_2)^\gamma = 0 \), we have

\[
W_6 \otimes W_2 \cong 2W_6. \tag{23}
\]

The group \( G \) acts on \( \mathfrak{gl}(V) \) by Lie algebra automorphisms: if \( p \in G \) and \( g = (g_{i,j})_{i,j \in E} \), we have \( p \cdot g = (g'_{i,j})_{i,j \in E} \) with \( g'_{i,j} = g_{p(i),p(j)} \) for all \( i, j \in E \). Since \( G \) preserves the oriented triple system, it is clear that this action on \( \mathfrak{gl}(V) \) restricts to one on \( \mathfrak{s} \) and that we have, moreover

\[
p \cdot s_i = s_{p(i)} \text{ for all } p \in G \text{ and } i \in E, \tag{24}
\]

using the notation introduced in the proof of Proposition 6.4.
we consider the “reflection” map \( \tau_x : y \in S \mapsto x \ast y \in S \), which is an involution, and the translation map \( \tau_x : y \in S \mapsto y + x \in S \). We let \( W \) be the group of permutations of \( S \) generated by the set \( \Sigma = \{ \sigma_y : x \in S \} \) and \( T = \{ \tau_x : x \in S \} \) be the set of translations, which is of course an elementary abelian group of order \( 3^d \). One sees at once that if \( x, y \in S \), the composition \( \sigma_y \sigma_y \) is the translation \( \tau_{y-x} \) by the vector \( y - x \). It follows

Since \( \dim s_i = 2 \) for all \( i \in E \), the fact (24) implies that \( \dim s^? = 2 \) and that \( \dim(\bigoplus_{i \neq 7} s_i)^? = 4 \). The system of equations \( \Sigma_7 \) is

\[
\Sigma_7 : \begin{cases}
-g_{1,3} + g_{6,2} + g_{5,4} = 0,
g_{1,3} + g_{2,6} + g_{4,5} = 0,
-g_{3,1} - g_{6,2} + g_{4,5} = 0,
-g_{3,1} - g_{2,6} + g_{5,4} = 0,
\end{cases}
\]

and it is easy to see that it has no non-zero \( \sigma \)-invariant solutions. It follows that \( s^?_2 = 0 \), so \( I(s) = (14, 2, 4) \) and we can conclude that \( s \cong W_2 \oplus 2W_6 \) as \( G \)-modules.

Let \( h = s^? \), a two-dimensional subspace of \( s \). Since \( \gamma \) generates a normal subgroup of \( G \) and \( G \) acts on \( s \) by Lie algebra automorphisms, \( h \) is a \( G \)-invariant Lie subalgebra of \( s \) and \( h \cong W_2 \) as a \( G \)-module. The derived subalgebra \([h, h]\) is therefore a \( G \)-invariant subspace of \( h \) of dimension at most 1: since \( h \) is a simple \( G \)-module, we see that \( h \) is in fact an abelian Lie subalgebra.

The circulant matrix \( m \) whose first row is \((0, 0, 1, 1, -1, -1, 0)\) satisfies the equations in (25) and is invariant under \( \gamma \), so it belongs to \( h \). Its characteristic polynomial is \( t(t^6 + 14t^4 + 49t^2 + 7) \), and Eisenstein’s criterion at the prime 7 immediately shows that the second factor is irreducible. It follows that \( m \) has all its eigenvalues simple, so that its centralizer \( C_{gl(V)}(m) \) in \( gl(V) \) is of dimension 7 and generated as an associative algebra by \( m \) itself. In particular, \( \gamma \) acts trivially on all of \( C_{gl(V)}(m) \) so that the same is true on \( s \cap C_{gl(V)}(m) = C_s(m) \) and, in view of the \( G \)-module structure of \( s \), this implies that \( C_s(m) \subseteq h \). Since \( h \) is abelian, we actually have \( C_s(h) = h \).

Let \( N_s(h) \) be the normalizer of \( h \) in \( s \). This is a \( G \)-submodule of \( s \) and the map \( v \otimes h \in N_s(h) \otimes h \mapsto [v, h] \in h \) is \( G \)-equivariant. Since \( \hom_G(W_2 \otimes W_2, W_2) = 0 \) because of the isomorphism (23) this map must in fact be zero, and then \( N_s(h) = C_s(h) = h \). We conclude that \( h \) is a Cartan subalgebra in \( s \). In particular, \( h \) contains the center of \( s \) and then, since \( h \) is its own centralizer, that center is in fact trivial.

The tautological action \( s \otimes V \rightarrow V \) of \( s \) on \( V \) is of course faithful. It is also completely reducible: indeed, an \( s \)-submodule of \( V \) must be, in particular, stable under the action of the semisimple matrix \( m \) given above and the latter has exactly two invariant subspaces in \( V \) which are mutually complementary, so \( V \) is either simple as an \( s \)-module or a direct sum of two simple submodules. According to [Bon60, §6 Proposition 5], the algebra \( s \) is reductive and therefore, since it is centerless, semisimple of rank \( \dim h = 2 \). Since the dimension of \( s \) is 14, its type must be \( G_2 \).

6.7. We consider next the affine space \( S = \mathbb{A}^d(\mathbb{F}_3) \) of dimension \( d \). On \( S \) we have the operation \( \ast \) coming from the structure of Steiner triple system and a structure of \( \mathbb{F}_3 \)-vector space; these two are related by the identity \( x \ast y = 2(x + y) \). For each \( x \in S \) we consider the “reflection” map \( \sigma_x : y \in S \mapsto x \ast y \in S \), which is an involution, and the translation map \( \tau_x : y \in S \mapsto y + x \in S \). We let \( W \) be the group of permutations of \( S \) generated by the set \( \Sigma = \{ \sigma_x : x \in S \} \) and \( T = \{ \tau_x : x \in S \} \) be the set of translations, which is of course an elementary abelian group of order \( 3^d \). One sees at once that if \( x, y \in S \), the composition \( \sigma_x \sigma_y \) is the translation \( \tau_{y-x} \) by the vector \( y - x \). It follows
that \([\sigma_x, \sigma_y] = \tau_{x-y}\) and from this one can check easily that the derived subgroup \(W'\) of \(W\) is \(T\). Moreover, since \(\sigma_x, \sigma_y \in T\) for all \(x, y \in S\), we see that the quotient \(W/W'\) is of order at most two, being generated by the class of any \(\sigma_x\) with \(x \in S\); as \(\sigma_x \not\in T\) because \(\sigma_x\) fixes a point and no translation does that, we see that the abelianization \(W/W'\) is in fact of order exactly two. Noticing that \(\sigma_y \tau_x \sigma_y = \sigma_y \sigma_y \sigma_x \sigma_y = \tau_y - (x+y) = \tau_{-x}\) for all \(x, y \in S\), we can conclude that \(W \cong T \rtimes C_2\), which the non-trivial element of \(C_2\) acting by inversion on \(T\).

Suppose now \(o\) is an orientation of \(S\) which is invariant under translations, and for each block \(b\) of \(S\) denote \(o_b\) the corresponding orientation of \(b\). We claim that for all \(x \in S\) the reflection \(\sigma_x\) reverses the orientation of all blocks, that is, \(\sigma_x(o_b) \neq o_{\sigma_x(b)}\) for all blocks \(b\). Indeed, let us assume that this is not the case so there is an \(x \in S\) and a block \(b\) such that \(\sigma_x(o_b) = o_{\sigma_x(b)}\). If \(y \in \sigma_x(b)\) then, because \(\sigma_y \sigma_x\) is a translation,

\[
o_{\sigma_y \sigma_x(b)} = \sigma_y \sigma_x(o_b) = \sigma_y(o_{\sigma_x(b)}) = -o_{\sigma_y \sigma_x(b)},
\]

which is absurd; the last equality is a consequence of the fact that the reflection at a point inverts the orientation of every block which contains it. We have proved half of the following lemma:

**Lemma.** An orientation of an affine space \(A^d(\mathbb{F}_3)\) is invariant under translation iff the reflection at each point reverses the orientations of all blocks.

**Proof.** We are left with showing the sufficiency of the condition, but this is clear since we have proved that every translation is the product of two reflections. □

**6.8. Proposition.** Let \(S = A^2(\mathbb{F}_3)\) be the affine plane over \(\mathbb{F}_3\) endowed with a translation invariant orientation. Then \(s\) is a four dimensional abelian Lie algebra.

**Proof.** Let us fix a translation invariant orientation of \(S\). Let \(o \in S\) and let us consider the set \(N = \{x \in S : \varepsilon_{o,x} = 1\}\) of the four points which follow \(o\) according to the orientation in the four blocks which contain it.

Suppose that \(N\) does not contain a block. Let \(x, y, x', y'\) in \(N\) be such that \(x \neq y\) and \(x' \neq y'\) and suppose that \(x \ast y = x' \ast y'\). Then \(x \ast y \neq o\) because of the definition of \(N\), and the hypothesis made on \(N\) implies that \(x \ast y \not\in N\), so that \(o \ast (x \ast y) \in N\); on the other hand, \(o \ast (x \ast y)\) is different to \(x\) and to \(y\); for example, if we had \(o \ast (x \ast y) = x\), then we would have \(x \ast y = o \ast x\), which is only possible if \(y = o\), which is not the case. A similar reasoning applies to \(x' \ast y'\), so the five elements

\[
x, \quad y, \quad o \ast (x \ast y), \quad x', \quad y'
\]

are in \(N\), the first three are distinct, and so are the last three. Since \(N\) has only four elements, we see that one of \(x, y\) is equal to one of \(x', y'\). This and the hypothesis that \(x \ast y = x' \ast y'\) imply at once that \(\{x, y\} = \{x', y'\}\). It follows that the set \(M = \{x \ast y : x, y \in N, x \neq y\}\) has \(\binom{4}{2} = 6\) elements. Since we are supposing that \(N\) does not contain blocks, \(N \cap M = \emptyset\), and this is impossible because \(|M \cup N| = 10\) and \(S\) has only 9 elements. Our hypothesis is therefore untenable, and \(N\) must contain a block;
moverover, since two blocks contained in $N$ have two points of intersection, $N$ contains
in fact exactly one block.

So suppose $b = \{x, y, z\}$ is a block contained in $N$ and let $w$ be the remaining element
of $N$. There is a unique automorphism $\alpha$ of the affine plane which maps the points $\sigma, x, w$ to $\sigma, y, w$, because these two triples of points are triangles. This map $\alpha$ preserves the
orientation of the blocks incident to $\sigma$ and that of the block $b$.

\[ \square \]

### 7 A non-associative algebra

#### 7.1. Let us endow the vector space $O = O_S = k1 \oplus V$, with 1 a new symbol, with the
bilinear multiplication $O \times O \to O$ for which 1 is a unit element and

\[
x_i \cdot x_j = \begin{cases} 
-1, & \text{if } i = j; \\
\varepsilon_{i,j}x_{i\star j}, & \text{in any other case.}
\end{cases}
\]

If $x = \lambda + v$ with $\lambda \in k$ and $v \in V$, we say that $v$ is the imaginary part of $x$ and write it
im $x$, and if $\lambda = 0$ we say that $x$ is purely imaginary; this defines a linear map
im : $O \to V$.

There is an involutive algebra anti-automorphism $(-)^* : O_S \to O_S$ such that $1^* = 1$ and $x_i^* = -x_i$ for all $i \in E$. If for each $x = \lambda + \sum_{i=1}^n \lambda_i x_i \in O$ we set $t(x) = 2\lambda$ and
$N(x) = \lambda^2 + \sum_{i=1}^n \lambda_i^2$, we obtain a linear form $t : O \to k$ and a quadratic form $N : O \to k$ such that $xx^* = N(x)$ and

\[
x^2 - t(x)x + N(x) = 0 \tag{26}
\]

for all $x \in O$. In particular, the algebra $O$ is quadratic.

#### 7.2. If $S$ has three points, then $O$ is isomorphic to the algebra $\mathbb{H}$ of standard quaternions
over $k$, and if $S$ is the Fano plane with a good orientation, then $O$ is isomorphic to the
Cayley-Dickson-Graves algebra $O$ of octonions over $k$—see [Bae02, Bae05]. These two
algebras are alternative, that is, the associator $(a, b, c) = (ab)c - a(bc)$ is an alternating
function of its three arguments. In fact, these two are the only alternative algebras we
get in this way:

**Proposition.** The algebra $O$ is alternative iff $n = 3$ or $S$ is the Fano plane endowed
with a good orientation.

**Proof.** According to the observations made above, we need only check the necessity of
the condition. Let us suppose that $O$ is an alternative algebra, so that the associator is
alternating. If $i, j, k \in E$ form a triangle of $S$, then

\[
(x_i, x_j, x_k) + (x_j, x_i, x_k) = \varepsilon_{j,k}x_{i\star (j\star k)} + \xi_{i,k}x_{j\star (i\star k)} = 0,
\]

so that

\[
i \star (k \star j) = (i \star k) \star j \tag{27}
\]
and

\[ \varepsilon_{j,k} \varepsilon_{i,j} + \varepsilon_{k,i} \varepsilon_{j,i} + k = 1. \]  

(28)

Let \( E^+ = E \cup \{0\} \), with 0 a new symbol, and let \(+ : E^+ \times E^+ \to E^+\) the operation on the set \( E^+ \) which has 0 as an identity element and such that \( i + 0 = i \) for all \( i \in E \) and \( i + j = i \star j \) for all \( i, j \in E \) such that \( i \neq j \). Using (27), it is easy to check that \((E^+,+)\) is an finite abelian group of exponent two, so that we can view it as a vector space over \( \mathbb{F}_2 \), and that in fact the Steiner system \( S \) can be identified with the projectivisation \( \mathbb{P}(E^+) \).

We thus see that \( S \) is necessarily a projective space.

We want to show that the dimension of \( S \) is at most two and to do so, it will be enough to show that when \( S = \mathbb{P}^2(\mathbb{F}_2) \) the algebra \( O \) we reach a contradiction; let us write \( xyzu \) instead of \( (x : y : z : u) \in S \). The sets

\[
\begin{align*}
1000, 0001, 0011, & \quad 0100, 0010, 1010, & \quad 0100, 0001, 1111, \\
1100, 0001, 1111, & \quad 1000, 1111, 0011
\end{align*}
\]

are triangles of \( S \), and to these five triangles there correspond five equations like in (28), namely

\[
\begin{align*}
\varepsilon_{0001,0011} \cdot \varepsilon_{1000,0010} \cdot \varepsilon_{0011,1000} \cdot \varepsilon_{0001,1011} &= 1, \\
\varepsilon_{0010,1010} \cdot \varepsilon_{0100,1000} \cdot \varepsilon_{1010,0100} \cdot \varepsilon_{0010,1110} &= 1, \\
\varepsilon_{0001,1111} \cdot \varepsilon_{0100,1110} \cdot \varepsilon_{1111,0100} \cdot \varepsilon_{0001,1011} &= 1, \\
\varepsilon_{0001,1111} \cdot \varepsilon_{1100,1110} \cdot \varepsilon_{1111,1100} \cdot \varepsilon_{0001,0011} &= 1, \\
\varepsilon_{1111,0011} \cdot \varepsilon_{1000,1100} \cdot \varepsilon_{0011,1000} \cdot \varepsilon_{1111,1111} &= 1.
\end{align*}
\]

(29)

The product of the left hand sides of these equations is

\[
\begin{align*}
\varepsilon_{0001,0011}^2 \cdot \varepsilon_{0011,1000}^2 \cdot \varepsilon_{0001,1011}^2 \cdot \varepsilon_{0001,1111}^2 \\
\cdot (\varepsilon_{1010,0100} \cdot \varepsilon_{0100,1110}) \cdot (\varepsilon_{1000,0010} \cdot \varepsilon_{0010,1010}) \cdot (\varepsilon_{0100,1000} \cdot \varepsilon_{1000,1100}) \\
\cdot (\varepsilon_{1111,0100} \cdot \varepsilon_{1111,1111}) \cdot (\varepsilon_{1111,1100} \cdot \varepsilon_{1111,0011}) \cdot (\varepsilon_{1100,1110} \cdot \varepsilon_{0010,1110})
\end{align*}
\]

which evaluates to \(-1\); indeed, the squares in the first line are obviously equal to 1, the three factors in the second line are all equal to 1, and the three factors of the third row are equal to \(-1\). This is absurd, and this proves what we wanted\(^2\).

If \( n = 3 \) there is nothing to prove, so let us suppose that \( S = \mathbb{P}^2(\mathbb{F}_2) \). Since the action of the group \( \text{PGL}(2,3) \) and the reversal of all orientations does not change the isomorphism class of \( O \), we can assume that the orientation of \( S \) is one of the two in figure 1 on page 3. For the good one, we know that \( O \) is the algebra of octonions, which is alternative; on the other hand, if the orientation is the bad one, computing we see that \( (x_1, x_3, x_2) + (x_3, x_1, x_3) = 2x_6 \neq 0 \), so that \( O \) is not alternative in that case.

\(^2\)Each triangle in \( \mathbb{P}^3(\mathbb{F}_2) \) provides us with an equation of the form (28) on the entries of the matrix \((\varepsilon_{i,j})\).

A computer calculation shows that these 420 equations together with the equations that reflect the fact that this matrix is antisymmetric are incompatible. A brute force search of a small subset of these set of equations which is still incompatible yielded the system (29) given above. It would be desirable to replace this horrendous argument with a more conceptual one.

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7.3. A possibly non-associative algebra $\Lambda$ is **flexible** if

$$x(yx) = (xy)x, \quad \forall x, y \in \Lambda,$$

and a flexible algebra $\Lambda$ is a **non-commutative Jordan algebra** if

$$(xy)x^2 = x(yx^2), \quad \forall x, y \in \Lambda,$$

A commutative algebra is automatically flexible and a commutative non-commutative Jordan algebra (!) is classically called simply a **Jordan algebra**.

An algebra is **power-associative** if the subalgebra generated by any of its elements is associative, and **strictly power associative** if it remains power-associative after extension of scalars to any field containing the base field. It is a rather non-trivial theorem of Albert [Alb48b] that, over a base field of characteristic zero such as ours, an algebra is strictly power-associative as soon as it satisfies the identities $xx^2 = x^2x$ and $x^2x^2 = (x^2x)x$.

**Proposition.** The algebra $O$ is a power-associative non-commutative Jordan algebra.

**Proof.** To show flexibility, it is enough to show that for all $x, y, z \in O$ we have

$$(x, y, z) + (z, y, x) = 0$$

This is the condition obtained from equation (30) by the standard process of linearization —see [Art00, §2.1]. Since it is linear in each of the three variables, it is enough to check it when $x, y$ and $z$ are taken from the set $\{1\} \cup \{x_i : i \in E\}$, and since it holds automatically if one of the three variables is set to 1, we need only consider the case where $x = x_i$, $y = x_j$ and $z = x_k$ for some $i, j, k \in E$. If either $|\{i, j, k\}| < 3$ or $\{i, j, k\} \in S$, then the equality (32) can be checked immediately. If instead $\{i, j, k\}$ is a triangle of $S$, then the six points $i, j, k, i \star j, j \star k$ and $k \star i$ are distinct and in evaluating the left hand side of (32) we have to consider five cases, one for each of the possible ways in which elements of the set $\{i \star (j \star k), j \star (k \star i), k \star (i \star j)\}$ can coincide; the corresponding local pieces of the Steiner triple system are depicted in figure 3 on the next page. For example, if we are in the third case depicted in that figure, in which $i \star (j \star k) = j \star (i \star k) \neq k \star (i \star j)$, the left hand side of (32) evaulates to

$$(\varepsilon_{j,k} \varepsilon_{i,j \star k} - \varepsilon_{k,j \star k} \varepsilon_{i,j,i}) x_{i \star (j \star k)} + (\varepsilon_{i,j} \varepsilon_{k,j \star i} + \varepsilon_{i,j} \varepsilon_{i \star j,k}) x_{i \star (j \star k)} = 0,$$

and the rest of the cases are similar.

The non-commutative Jordan identity (31) and power-associativity follow immediately from the flexible identity (30) and equation (26), which shows that $O$ is a quadratic algebra.  

\[\square\]
7.4. Let $\Lambda$ be a possibly non-associative algebra. We say that $\Lambda$ is simple if it does not have any non-trivial ideal, as usual.

If $\lambda \in \Lambda$, the maps $L_{\lambda} : x \in \Lambda \mapsto \lambda x$ and $R_{\lambda} : x \in \Lambda \mapsto x\lambda \in \Lambda$ are linear, and the multiplication algebra of $\Lambda$ is the unital associative subalgebra $M(\Lambda)$ of $\text{End}_k(\Lambda)$ generated by the set $\{L_\lambda : \lambda \in \Lambda\} \cup \{R_\lambda : \lambda \in \Lambda\}$. The centroid $Z(\Lambda)$ of $\Lambda$ is the centralizer of $M(\Lambda)$ in $\text{End}_k(\Lambda)$ and, if the centroid of $\Lambda$ is equal to $k$, we say that $\Lambda$ is central. If $\Lambda$ happens to be associative, this condition is equivalent to the usual one that the center of $\Lambda$ be $k$.

An algebra is central simple iff it remains simple after extending scalars to arbitrary field extensions of the base field; this follows from results in [Jac62, Chapter X, §1].

**Proposition.** The algebra $\mathcal{O}$ is a central simple algebra.

**Proof.** Let $(-, -) : \mathcal{O} \times \mathcal{O} \to k$ be the bilinear form such that $(x, y) = t(xy)$ for all $x, y \in \mathcal{O}$. One can see at once that this is a non-degenerate symmetric form, and it is moreover associative, in that $(xy, z) = (x, yz)$ for all $x, y, z \in \mathcal{O}$. If $e \in \mathcal{O}$ is a non-zero idempotent, then $e \notin k1$ and from (26) we have $(e, e) = t(e) = 1 \neq 0$. Finally, if $x, y \in \mathcal{O}$ are such that $z = \frac{1}{2}(xy + yx)$ is nilpotent, then from (26) we see that $t(z) = 0$ and then $(x, y) = t(z) = 0$. It follows from a theorem of Albert [Alb49] (see [Sch66, Chapter V, §3, Theorem 5.4] for a modern exposition) that $\mathcal{O}$ is a direct product of simple algebras.

Let $Z(\mathcal{O}) \subseteq \text{End}_k(\mathcal{O})$ be the centroid of $\mathcal{O}$ and let $\phi \in Z(\mathcal{O})$. There are scalars $\lambda, \lambda_i, \mu_i, \mu_{i,i} \in k$ such that $\phi(1) = \lambda + \sum_{i \in E} \lambda_i x_i$ and $\phi(x_i) = \mu_i + \sum_{i \in E} \mu_{i,i} x_i$. If $s \in E$, we have $L_{x_s} \circ \phi = \phi \circ L_{x_s}$; evaluating both sides of this equality at 1, we see that

$$
\mu_s = -\lambda_s, \quad \mu_{s,s} = \lambda_s, \quad \mu_{s,i} = -\epsilon_{s,i} \lambda_{s,s}i
$$

for all $s, i \in E$ such that $s \neq i$, and these equations show that $\phi$ is completely determined.

---

3That a central simple algebra remains simple under extension of scalars is the first part of Theorem 3 there. Conversely, suppose $\Lambda$ has that property. Of course, it is then simple, and then Theorem 1 tells us that its centroid $Z(\Lambda)$ is a field. The hypothesis implies that $Z(\Lambda) \otimes_k \Lambda$ is simple, and we see that $\Lambda$ is central using the second part of Theorem 3.
by its value \( \phi(1) \). Also, since \( \phi \) is in \( Z(O) \), for all \( s \in E \) we have

\[
x_s \cdot \phi(1) = \phi(x_s \cdot 1) = \phi(x_s) = \phi(1 \cdot x_s) = \phi(1) \cdot x_s,
\]
and writing out the first and the last members of this chain of equalities and comparing, we see that \( \lambda_i = 0 \) for all \( i \in E \). It follows that the centroid \( Z(O) \) is at most one-dimensional. Since the scalar multiples of the identity \( id_O \) is in \( Z(O) \), we see that \( \dim Z(O) = 1 \).

It is clear that the number of factors appearing in a decomposition of \( O \) as a direct product of simple subalgebras bounds from below the dimension of the centroid \( Z(O) \), so we can conclude that \( O \) is in fact central and simple.

7.5. Proposition. If \( d : O \to O \) is a derivation, then \( d(V) \subseteq V \) and restriction to \( V \) gives an injective Lie algebra map \( \text{Der}(O) \to \text{End}_k(V) \) whose image is exactly \( \mathfrak{s} \cap \mathfrak{s}(V) \).

Here \( \mathfrak{s}(V) \) is the special orthogonal Lie algebra corresponding to the quadratic form obtained from the quadratic form \( N \) by restriction to \( V \); in other words, this is the set of endomorphisms whose matrix with respect to the basis \( \{ x_i : i \in E \} \) is anti-symmetric.

Proof. Let \( d : O \to O \) be a derivation. As usual, \( d(1) = 0 \) and there are scalars such that \( d(x_i) = \alpha_i + \sum_{j \in E} \alpha_{i,j} x_j \) for all \( i \in E \). If \( i \in E \), then

\[
0 = d(-1) = d(x_i^2) = x_i d(x_i) + d(x_i)x_i = 2\alpha_i x_i - 2\alpha_{i,i}.
\]

It follows that for all \( i \in E \) we have

\[
\alpha_i = \alpha_{i,i} = 0 \quad (33)
\]

In particular, \( d(V) \subseteq V \). Let now \( i, j \in E \) be such that \( i \neq j \). Then

\[
d(x_i x_j) = \sum_{k \in E} \varepsilon_{i,j} \alpha_{i,j,k} x_k
\]

and

\[
x_i d(x_i) + d(x_i)x_j = -(\alpha_i + \alpha_{i,i}) + \varepsilon_{i,j} \alpha_{i,j,i} x_i + \varepsilon_{i,j} \alpha_{i,j,j} x_j + \sum_{k \in E \setminus \{i,j\}} (\varepsilon_{i,k} \alpha_{i,j,k} + \varepsilon_{k,j} \alpha_{i,k,j}) x_k,
\]

and the equality of the left and sides in these equations is equivalent to the conditions

\[
\alpha_{i,i} + \alpha_{j,j} = 0, \quad (34)
\]

\[
\varepsilon_{i,j} \alpha_{i,j,i} = \varepsilon_{i,j} \alpha_{i,j,j}, \quad \varepsilon_{i,j} \alpha_{i,j,j} = \varepsilon_{i,j} \alpha_{i,j,i}, \quad (35)
\]

\[
\varepsilon_{i,j} \alpha_{i,j,k} + \varepsilon_{k,j} \alpha_{i,k,j} = \varepsilon_{i,j} \alpha_{i,j,k}, \quad \forall k \in E \setminus \{i,j\}. \quad (36)
\]

From equations (33) and (34) we see that the restriction of \( d \) to \( V \) is in \( \mathfrak{s}(V) \), and that the equations (35) do not impose any new conditions. Finally, the equation (36) can be now rewritten in the form

\[
\varepsilon_{j,k} \alpha_{k,j,i} + \varepsilon_{k,i} \alpha_{k,i,j} + \varepsilon_{j,i} \alpha_{i,j,k} = 0.
\]

Comparing this to the equation \( E_{i,j,k} \) appearing in Proposition 6.2 we see that the restriction \( d|_V \) belongs to \( \mathfrak{s} \cap \mathfrak{s}(V) \). The converse statement follows from the same computations; we omit the corresponding details. \( \square \)
A Some details

A.1 Flexible algebras, d’après [Alb48a, Chapter IV]

Let $\Lambda$ be an algebra over a field of characteristic different from 2, and for each $x \in \Lambda$ let $L_x, R_x : \Lambda \to \Lambda$ be the maps given by left and right multiplication by $x$. We let $\Lambda^+$ be the algebra with underlying vector space equal to $\Lambda$ and product $\circ$ given by $x \circ y = \frac{1}{2}(xy + yx)$; clearly, $\Lambda^+$ is commutative.

The algebra $\Lambda$ is flexible if the identity $x(yx) = (xy)x$ holds in it or, equivalently, if for all $x \in \Lambda$ the maps $L_x$ and $R_x$ commute; a commutative algebra is flexible.

If $\Lambda$ is flexible, then for all $x, y \in \Lambda$ we have

\[
0 = [L_x y, R_{x+y}] = [L_x, R_0] + [L_x, R_y] + [L_y, R_x] + [L_y, R_y]
\]

and therefore

\[
R_x L_y - L_y R_x = L_x R_y - R_y L_x.
\] (37)

It follows that we have identically

\[
(yz)x - y(zx) = x(zy) - (xz)y
\]

so that

\[
L_{yz} - L_y L_z = R_{zy} - R_y R_z.
\]

Taking $z$ and $y$ equal to $x$ here, we see that in particular

\[
L_{x^2} - L_x^2 = R_{x^2} - R_x^2.
\] (38)

Lemma. If $\Lambda$ is a flexible algebra, the following four identities are equivalent:

\[
x(y x^2) = (xy)x^2, \quad x^2(y x) = (x^2 y)x, \quad (yx) x^2 = (yx^2) y, \quad x^2(xy) = x(x^2 y).
\]

If these identities hold, then they also hold in the algebra $\Lambda^+$.

Notice that if the algebra is commutative, the equivalence of these four identities presented is evident.

Proof. The four identities are, respectively, equivalent to the following commutation relations:

\[
[R_x, R_{x^2}] = 0, \quad [R_x, L_{x^2}] = 0, \quad [L_x, R_{x^2}] = 0, \quad [L_x, L_{x^2}] = 0. \] (39)

Applying $[\cdot, R_x]$ to the equality (38), and using the fact that $[L_x, R_x] = 0$ because of flexibility and $[R_x, R_x] = 0$, we see that

\[
[L_{x^2}, R_x] = [R_{x^2}, R_x].
\] (40)
Similarly, applying \([L_x, -]\) to the same equality we see that

\[
[L_x, L_{x^2}] = [L_x, R_{x^2}].
\]

(41)

On the other hand, setting \(y = x^2\) in (37) we obtain that

\[
[R_x, L_{x^2}] = [L_x, R_{x^2}].
\]

(42)

The equalities (40), (41) and (42) show that the four equalities (39) hold simultaneously, and proves the first claim of the lemma.

Notice that \(x \ast x = xx\), so that squaring in \(\Lambda\) and in \(\Lambda^+\) are the same operation. Since

\[
4[x \circ (y \circ x^2) - (x \circ y) \circ x^2] = [x(yx^2) - (xy)x^2] + [x(x^2y) - x^2(xy)] + [(yx^2)x - (yx)x^2] + [(x^2y)x - x^2(xy)],
\]

we see that if \(\Lambda\) satisfies the four equations of the statement, so does \(\Lambda^+\).

A.2 The trace form

Let \(\Lambda\) be a non-commutative Jordan algebra, so that it is flexible and satisfies the identities of the Lemma in A.1, and let \(\Lambda^+\) be the algebra which has the same underlying vector space as \(\Lambda\) and product \(x \circ y = \frac{1}{2}(xy + yx)\). If \(x \in \Lambda\), let \(\Lambda^+_x : y \in \Lambda \mapsto x \circ y \in \Lambda\) be the map given by right multiplication by \(x\) in the algebra \(\Lambda^+\).

Lemma. The form \((-,-) : \Lambda \times \Lambda \to \mathbb{k}\) such that for all \(x, y \in \Lambda\) one has \((x,y) = \text{tr} \, R^+_x y\) is associative, that is,

\[
(xy,z) = (x,yz)
\]

for all \(x, y, z \in \Lambda\).

Proof. This is proved in [Sch66, Chapter IV, §1 and Chapter V, §3] but we repeat the argument because it is very beautiful. Let \((-,-)\) be the associator of \(\Lambda^+\). Since \(\Lambda^+\) is a Jordan algebra, the identity \((x,y,x^2)_o = 0\) is valid in \(\Lambda^+\); this was proved in A.1. Replacing \(x\) by \(x + \lambda z\) with \(\lambda \in \mathbb{k}\), the coefficient of \(\lambda^2\) must vanish, and therefore the identity \(2(z,y,x \circ z)_o + (x,y,z^2)_o = 0\). Replacing now \(z\) by \(z + \lambda w\), and looking at the coefficient of \(\lambda\), we conclude that

\[
(x,y,z \circ w)_o + (z,y,w \circ x)_o + (w,y,x \circ z)_o = 0
\]

is also an identity. Viewing this as a function of \(w\), we can rewrite it as

\[
R^+_x y R^+_z - R^+_x R^+_y R^+_z + R^+_z R^+_x y - R^+_x R^+_y R^+_z + R^+_x R^+_y + R^+_y R^+_x - R^+_y R^+_x = 0.
\]

Interchanging \(x\) and \(y\) in this equality and substracting the result, we see that

\[
[R^+_x, [R^+_x, R^+_y]] = R^+_y (x,z) - y (x,z) = 0.
\]

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It follows that
\[(x \circ y, z) - (x, y \circ z) = \text{tr } R^+_{(x \circ y)oz} - \text{tr } R^+_{xo(y)oz} = \text{tr } R^+_{(x,y,z)o} = 0,\]
and the form \((-,-\)) is associative on \(\Lambda^+\). Now \(\Lambda\) is a flexible algebra, so that (32) holds, and we can rewrite that equation as
\[L_x y - L_x L_y + R_x R_y - R_y x = 0.\]
Interchanging, as before, \(x\) and \(y\) and substracting, we obtain
\[L_{[x,y]} + R_{[x,y]} = [L_x, L_y] + [R_y, R_x],\]
which implies that
\[(1, [x, y]) = \text{tr } R^+_{[x,y]} = \frac{1}{2}(\text{tr } L_{[x,y]} + \text{tr } R_{[x,y]}) = 0.\]
Since \(xy = x \circ y + \frac{1}{2} [x, y]\), we see that \((1, xy) = (1, x \circ y)\) and, in particular,
\[(1, xy) = (1, yx) = (x, y). \tag{43}\]
Because the form is associative on \(\Lambda^+\), we have
\[0 = (x \circ y, z) - (x, y \circ z) = (1, (x,y,z)_o) = \frac{1}{2}(1, (xy)z + (yx)z + z(xy) + z(yx) - x(yz) - x(yz) - (yz)x - (zy)x) = \frac{1}{2}(1, (xy)z - x(yz)) = (1, (xy)z) = (1, (xy)z - (1, x(yz))),\]
and using (43) again, we conclude that \((xy, z) = (x, yz)\), as we wanted. \(\square\)
References

[Che47] Claude Chevalley, Algebraic Lie algebras, Ann. of Math. (2) 48 (1947), 91–100. MR0019603 (8,435d)


