A Short Proof of the Cayley–Hamilton Theorem

Let M be an $n \times n$ matrix with real entries and let χ_M be its characteristic polynomial. The Cayley–Hamilton theorem states that

$$\chi_M(M) \textit{ is the zero matrix}$$

and to see this it is enough to show that $\chi_M(M) \cdot x_0 = 0$ for all $x_0 \in \mathbb{R}^n$.

Let us do this. Fix $x_0 \in \mathbb{R}^n$. According to the existence theorem for initial value problems for differential equations there is a smooth function $x : U \to \mathbb{R}^n$ defined on some open interval $U \subseteq \mathbb{R}$ containing 0 such that $x(0) = x_0$ and $x'(t) = M \cdot x(t)$ for all $t \in U$. We can write this last equation in the form $0 = (\partial I - M) \cdot x$, with I the identity matrix and ∂ the differentiation operator, and if we multiply by the adjugate matrix $\operatorname{adj}(\partial I - M)$ of $\partial I - M$ we find that

$$0 = \operatorname{adj}(\partial I - M) \cdot (\partial I - M) \cdot x = \det(\partial I - M) \cdot x = \chi_M(\partial) \cdot x.$$

Since $\partial^k \cdot x = M^k \cdot x$ for all $k \in \mathbb{N}_0$, this implies at once that

$$0 = \chi_M(\partial) \cdot x = \chi_M(M) \cdot x$$

and, in particular, if we evaluate the function $\chi_M(M) \cdot x$ at $t = 0$ we see that $\chi_M(M) \cdot x_0 = 0$, which is what we wanted to prove.

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