On the modular automorphism of twisted Calabi–Yau algebras

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Van den Bergh duality

Definition

An algebra A has Van den Bergh duality of dimension d if

- it is homologically smooth,
- for all $p \neq d$ we have $\operatorname{Ext}^p_{A^e}(A, A \otimes A) = 0$, and
- the A-bimodule $U = \operatorname{Ext}^p_{A^e}(A, A \otimes A)$ is invertible.

In that case the bimodule U is the dualizing bimodule of A.

Van den Bergh duality

Theorem

If A has Van den Bergh duality of dimension d with dualizing bimodule U, then for all A-bimodules M there is a natural isomorphism

$$H^\bullet(A, M) \cong H_{d-\bullet}(A, U \otimes_A M).$$

M. van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348.

R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, Invent. Math. **20** (1973), 103–124.

Calabi–Yau algebras

Definition

An algebra A is Calabi–Yau of dimension d if

- it has Van den Bergh duality in dimension d and
- the dualizing bimodule U of A is trivial: there is an isomorphism of A-bimodules

$$U \cong A.$$

V. Ginzburg, Calabi–Yau algebras, `arXiv:math/0612139`

Calabi–Yau algebras

A compact complex manifold M of dimension n is Calabi–Yau if the canonical bundle $\Omega^n(M)$ of holomorphic n-forms is trivial, so that there exist a holomorphic volume form.

Calabi–Yau algebras

Van den Bergh's Duality Theorem becomes:

Theorem

If A is Calabi–Yau of dimension d, then for all A-bimodules M there is a natural isomorphism

$$H^\bullet(A, M) \cong H_{d-\bullet}(A, M).$$

This means that the Calabi–Yau condition is a «higher version» of the Frobenius condition.

Calabi–Yau algebras: Examples

- Polynomial algebras $k[x_1, \ldots, x_d]$
- Coordinate rings of affine Calabi–Yau varieties (V. Ginzburg)
- If X is a connected and smooth projective variety and $\mathcal{E}$ is a tilting generator of $D^b(\operatorname{Coh} X)$, then

  $$\operatorname{End}_{D^b(\operatorname{Coh} X)}(\mathcal{E}) \text{ is Calabi–Yau} \iff X \text{ is Calabi–Yau.}$$

  (V. Ginzburg)

Calabi–Yau algebras: Examples

- Enveloping algebras of semisimple Lie algebras or, more generally, unimodular Lie algebras.
- Algebras of regular differential operators on affine Calabi–Yau varieties.

Calabi–Yau algebras: Examples

- If M is a connected compact orientable d manifold which is a $K(\pi, 1)$, then the group algebra $\mathbb{C}[\pi_1(M)]$ is Calabi–Yau of dimension d. (M. Kontsevich)
- If M is an orientable compact d-manifold and $C^*(M)$ is the differential graded algebra of k-valued singular cochains, then $C^*(M)$ is Calabi–Yau of dimension d. (P. Jørgensen)

Calabi–Yau algebras: Examples

- We know a simple description of all *graded* Calabi–Yau algebras of dimension 3 in terms of Jacobian algebras associated to a quiver and super-potentials. (Raf Bockland)

$$\frac{kQ}{(\partial_\alpha \Phi, \partial_\beta \Phi, \ldots)}$$

- A similar description applies to *complete* Calabi-Yau algebras of arbitrary dimension. (M. Van den Bergh)

The two results were conjectured by Ginzburg.

- Recognizing which Jacobian algebras are Calabi–Yau is difficult, but we have good «numerical» criteria when the algebra is (N-)Koszul. (R. Berger, A. Solotar)

Calabi–Yau algebras: Examples

- The (quadratic) *Sklyanin algebra* $A(a, b, c)$ freely generated by x, y and z subject to the relations

$$ayz + bzy + cx^2 = 0,$$
$$azx + bxz + cy^2 = 0,$$
$$axy + byx + cz^2 = 0,$$

  is Calabi–Yau for all choices of the parameters $(a, b, c) \in k^3$ except 12 involving cubic roots of unity.

  It is the Jacobian algebra coming from the super-potential

$$\Phi = axyz + bxzy + \frac{c}{3}(x^3 + y^2 + z^3).$$

Calabi–Yau algebras: Examples

- Quantum enveloping algebras $\mathcal{U}_q(\mathfrak{g})$. (Sophie Chemla)
- The Jacobian algebras attached to ideal triangulations of Riemann surfaces with boundary and punctures. (Fomin, Shapiro, Thurston, Derksen, Weyman, Zelevinski, ...)

- Finally, the class of Calabi–Yau algebras is closed under polinomial extesions, Morita equivalence, crossed-products with appropriate groups or Hopf algebras, by certain special types of localizations, ...

Twisted Calabi–Yau algebras

Definition

An algebra A is twisted Calabi–Yau of dimension d if

- it has Van den Bergh duality of dimension d and
- the dualizing module U is isomorphic as a left and as a right A-module to A,

$$ {}_A U \cong {}_A A, \qquad U_A \cong A_A. $$

Twisted Calabi–Yau algebras

Lemma

If U is an A-bimodule which is isomorphic to A as a left and as a right A-module, then there exists an algebra automorphism $\sigma : A \to A$ such that

$$U \cong A_\sigma.$$

The automorphism σ is determined by U up to an inner automorphism of A.

In other words, what we have is a class in $\mathrm{Out}(A) = \dfrac{\mathrm{Aut}(A)}{\mathrm{Inn}(A)}$.

Twisted Calabi–Yau algebras

Definition

An algebra A is twisted Calabi–Yau of dimension d if

- it has Van den Bergh duality of dimension d and
- there is an automorphism $\sigma : A \to A$ such that the dualizing module U is isomorphic as an A-bimodule to A_σ.

$$U \cong A_\sigma.$$

σ is the modular automorphism of A.

Twisted Calabi–Yau algebras: Examples

- Calabi–Yau algebras are twisted Calabi–Yau, with modular automorphism the identity.
- If a filtered algebra A is such that $\mathrm{gr}(A)$ is twisted Calabi–Yau, then A itself is Calabi–Yau. (M. Van den Bergh)
- Enveloping algebras of all Lie algebras (and not only unimodular ones) with modular automorphism related to the modular character.
- A graded Koszul algebra A of global dimension d such that $A^!$ is Frobenius is Calabi–Yau. (M. Vam den Bergh)

Twisted Calabi–Yau algebras: Examples

- An Ore extension $A[X;\sigma,\delta]$ of a twisted Calabi–Yau algebra is twisted Calabi–Yau. (L. Liu, S. Wang, Q. Wu)
- Quantum coordinate algebras of affine spaces, euclidean spaces, symplectic spaces. Quantum nilpontent spaces.
- Quantum enveloping algebras $\mathcal{U}_q(\mathfrak{g}^+)$ of positive parts of semisimple Lie algebras.
- Quantum coordinate algebras of Grassmanians.

Twisted Calabi–Yau algebras: Examples

- Enveloping algebras of Lie–Rinehart pairs (S, L) with S Calabi–Yau. (Th. Lambre, P. Le Meur) and their PBW-deformations.
- Algebras of differential operators tangent to free hyperplane arrangements (F. Kordon)
- Algebras of q-difference operators.
- Azumaya algebras with Calabi–Yau center.
- Finally, the class is closed under «good» deformations.

Our theme

A twisted Calabi–Yau algebra A comes *ex nihilo* with an automorphism $\sigma : A \to A$.

Theme

The automorphism σ is generally *not* the identity, ...

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Theme

The automorphism σ is generally *not* the identity, ... but it wants to be.

What we mean here is that there are natural actions of σ on various objects attached to A and that action tends to be trivial.

The center

Theorem

If A is twisted Calabi–Yau, then the modular automorphism acts trivially on the center $Z(A)$.

The center

Theorem

If A is twisted Calabi–Yau, then the modular automorphism acts trivially on the center $Z(A)$.

Corollary

A twisted Calabi–Yau algebra which is commutative is Calabi–Yau.

The automorphism group

Theorem

The class of the modular automorphism $\sigma : A \to A$ of a twisted Calabi–Yau algebra is central in the outer automorphism group $\mathrm{Out}(A)$.

In other words, if $\alpha \in \mathrm{Aut}(A)$, then there exists a unit $u \in$ such that

$$\sigma\alpha(a) = u\alpha\sigma(a)u^{-1}, \text{for all } a \in A.$$

This is an existence theorem.

If all units of A are central, then σ is central in $\mathrm{Aut}(A)$.
This is a very strong condition!

A baby example

$A = k_q[x, y]$

$$yx = qxy$$

This is twisted Calabi–Yau with modular automorphism σ such that

$$\sigma(x) = qx, \qquad \sigma(y) = q^{-1}.$$

Since $\{x^i y^j : i, j \geq 0\}$ is a basis of A, the spectrum of σ is

$$\mathrm{spec}(\sigma) = \{q^r : r \in \mathbb{Z}\}$$

If q is not a toor of unity, then the eigenspaces are

$$A(q^r) = \begin{cases} x^r k[xy] & \text{if } r \geq 0; \\ k[xy]y^{-r} & \text{if } r < 0. \end{cases}$$

A baby example

Now let $\alpha \in \mathrm{Aut}(A)$ and suppose that q is not a root of unity. Since the units of A are central, α and σ commute, α preserves the eigenspaces of σ and, in particular,

$$A(q) = xk[xy] \qquad A(1) = k[xy] \qquad A(q^{-1}) = k[xy]y$$

It follows that

$$\alpha(x) \in kx \qquad \alpha(y) \in ky.$$

A baby example

Corollary

- *If* $k_q[x, y] \cong k_{q'}[x, y]$*, then* $(q) = (q')$*.*
- *If* q *is not a root of unity, then neither is* q' *and* $q' \in \{q, q^{-1}\}$*.*

This is a theorem of J. Alev and F. Dumas.

J. Alev and F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, J. Algebra **170** (1994), no. 1, 229–265.

Locally ad-nilpotent elements

Proposition

Let A be twisted Calabi–Yau and suppose that all units of A are central and that the ground field is of characteristic zero.

- *If $x \in A$ is locally* ad*-nilpotent, then there exists a central element $z \in Z(A)$ such that $\sigma(x) = x + z$.*
- *If the center $Z(A)$ is a domain and A is a finitely generated module over $Z(A)$, then σ fixes all regular locally* ad*-nilpotent elements of A.*

Derivations

If $\delta : A \to A$ is a derivation and $\alpha \in \mathrm{Aut}(A)$, then $\alpha\delta\alpha^{-1} : A \to A$ is also a derivation.

This produces an action of $\mathrm{Aut}(A)$ on $HH^1(A) = \dfrac{\mathrm{Der}(A)}{\mathrm{InnDer}(A)}$.

Theorem

If A is twisted Calabi–Yau with modular automorphism σ, then σ acts trivially on $HH^1(A)$.

This is again an existence theorem.

A conjecture

Conjecture

The modular automorphism of a twisted Calabi–Yau algebra acts trivially on the Hochschild cohomology of A.

Introducing a twist

Proposition

Let A be a twisted Calabi–Yau algebra and let $\alpha \in \mathsf{Aut}(A)$.

- *There is a natural action of σ on $H^\bullet(A, A_\alpha)$.*
- *The action is trivial on $H^0(A, A_\alpha)$ and on $H^1(A, A_\alpha)$.*

Normal elements

Proposition

Let A be a twisted Calabi–Yau algebra. If θ is a regular normal element, then there exists a unit $u \in A$ such that

$$\sigma(\theta) = u\theta.$$

Localizations

Theorem

Let A be a twisted Calabi–Yau algebra with modular automorphism σ and let $f : A \to B$ be an epimorphism of algebras such that B is flat on both sides.

If σ extends to an automorphism $\tilde{\sigma}$ of B, then B is twisted Calabi–Yau with modular automorphism $\tilde{\sigma}$.

M. Farinati, Hochschild duality, localization, and smash products, J. Algebra **284** (2005), no. 1, 415–434.

C. Năstăsescu and N. Popescu, On the localization ring of a ring, J. Algebra **15** (1970), 41–56.

Localizations

Theorem

Let A be a twisted Calabi–Yau algebra. If $S \subseteq A$ is a multiplicatively closed subset of normal zero divisors, then the localization A_S is twisted Calabi–Yau of the same dimension as A.

Localizations

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Let A be a twisted Calabi–Yau algebra. If $S \subseteq A$ is a multiplicatively closed subset of normal zero divisors, then the localization A_S is twisted Calabi–Yau of the same dimension as A.

Conjecture

The same conclusion holds if S is a general Ore set.