On the modular automorphism of twisted Calabi–Yau algebras

Mariano Suárez-Álvarez

GTIIT — April 2024

Calabi–Yau manifolds

- Calabi–Yau manifolds

Calabi–Yau manifolds

- Calabi–Yau manifolds

- Algebra
- Differential forms
- Hochschild (co)homology and the Hochschild–Kostant–Rosenberg theorem

Calabi–Yau manifolds

- Calabi–Yau manifolds

- Algebra
- Differential forms
- Hochschild (co)homology and the Hochschild–Kostant–Rosenberg theorem

- Duality

Van den Bergh duality

Definition

An algebra A has Van den Bergh duality of dimension d if

- it is homologically smooth,
- for all $p \neq d$ we have $\mathrm{Ext}^p_{A^e}(A, A \otimes A) = 0$, and
- the A-bimodule $U = \mathrm{Ext}^p_{A^e}(A, A \otimes A)$ is invertible.

In that case the bimodule U is the dualizing bimodule of A.

Van den Bergh duality

Theorem

If A has Van den Bergh duality of dimension d with dualizing bimodule U, then for all A-bimodules M there is a natural isomorphism

$$H^{\bullet}(A, M) \cong H_{d-\bullet}(A, U \otimes_A M).$$

M. van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348.

R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, Invent. Math. **20** (1973), 103–124.

Calabi–Yau algebras

Definition

An algebra A is Calabi–Yau of dimension d if

- it has Van den Bergh duality in dimension d and
- the dualizing bimodule U of A is trivial: there is an isomorphism of A-bimodules

$$U \cong A.$$

V. Ginzburg, Calabi–Yau algebras, `arXiv:math/0612139`

Calabi–Yau algebras

Van den Bergh's Duality Theorem becomes:

Theorem

If A is Calabi–Yau of dimension d, then for all A-bimodules M there is a natural isomorphism

$$H^{\bullet}(A, M) \cong H_{d-\bullet}(A, M).$$

This means that the Calabi–Yau condition is a «higher version» of the Frobenius condition.

Calabi–Yau algebras: Examples

- Polynomial algebras $k[x_1, \ldots, x_d]$
- Coordinate rings of affine Calabi–Yau varieties (V. Ginzburg)
- If X is a connected and smooth projective variety and $\mathcal{E}$ is a tilting generator of $D^b(\operatorname{Coh} X)$, then

  $$\operatorname{End}_{D^b(\operatorname{Coh} X)}(\mathcal{E}) \text{ is Calabi–Yau} \iff X \text{ is Calabi–Yau.}$$

  (V. Ginzburg)

Calabi–Yau algebras: Examples

- Enveloping algebras of semisimple Lie algebras or, more generally, unimodular Lie algebras.
- Algebras of regular differential operators on affine Calabi–Yau varieties.

Calabi–Yau algebras: Examples

- If M is a connected compact orientable d manifold which is a $K(\pi, 1)$, then the group algebra $\mathbb{C}[\pi_1(M)]$ is Calabi–Yau of dimension d. (M. Kontsevich)
- If M is an orientable compact d-manifold and $C^*(M)$ is the differential graded algebra of k-valued singular cochains, then $C^*(M)$ is Calabi–Yau of dimension d. (P. Jørgensen)

Calabi–Yau algebras: Examples

- We know a simple description of all *graded* Calabi–Yau algebras of dimension 3 in terms of Jacobian algebras associated to a quiver and super-potentials. (Raf Bockland)

$$\frac{kQ}{(\partial_\alpha \Phi, \partial_\beta \Phi, \dots)}$$

- A similar description applies to *complete* Calabi-Yau algebras of arbitrary dimension. (M. Van den Bergh)

  The two results were conjectured by Ginzburg.

- Recognizing which Jacobian algebras are Calabi–Yau is difficult, but we have good «numerical» criteria when the algebra is (N-)Koszul. (R. Berger, A. Solotar)

Calabi–Yau algebras: Examples

- The (quadratic) *Sklyanin algebra* $A(a, b, c)$ freely generated by x, y and z subject to the relations

$$ayz + bzy + cx^2 = 0,$$
$$azx + bxz + cy^2 = 0,$$
$$axy + byx + cz^2 = 0,$$

  is Calabi–Yau for all choices of the parameters $(a, b, c) \in k^3$ except 12 involving cubic roots of unity.

  It is the Jacobian algebra coming from the super-potential

$$\Phi = axyz + bxzy + \frac{c}{3}(x^3 + y^2 + z^3).$$

Calabi–Yau algebras: Examples

- Quantum enveloping algebras $\mathcal{U}_q(\mathfrak{g})$. (Sophie Chemla)
- The Jacobian algebras attached to ideal triangulations of Riemann surfaces with boundary and punctures. (Fomin, Shapiro, Thurston, Derksen, Weyman, Zelevinski, ...)

- Finally, the class of Calabi–Yau algebras is closed under polinomial extesions, Morita equivalence, crossed-products with appropriate groups or Hopf algebras, by certain special types of localizations, ...

Twisted Calabi–Yau algebras

Definition

An algebra A is twisted Calabi–Yau of dimension d if

- it has Van den Bergh duality of dimension d and
- the dualizing module U is isomorphic as a left and as a right A-module to A,

$$_{A}U \cong {}_{A}A, \qquad U_A \cong A_A.$$

Twisted Calabi–Yau algebras

Lemma

If U is an A-bimodule which is isomorphic to A as a left and as a right A-module, then there exists an algebra automorphism $\sigma : A \to A$ such that

$$U \cong A_\sigma.$$

The automorphism σ is determined by U up to an inner automorphism of A.

In other words, what we have is a class in $\mathrm{Out}(A) = \dfrac{\mathrm{Aut}(A)}{\mathrm{Inn}(A)}$.

Twisted Calabi–Yau algebras

Definition

An algebra A is twisted Calabi–Yau of dimension d if

- it has Van den Bergh duality of dimension d and
- there is an automorphism $\sigma : A \to A$ such that the dualizing module U is isomorphic as an A-bimodule to A_σ.

$$U \cong A_\sigma.$$

σ is the modular automorphism of A.

Twisted Calabi–Yau algebras: Examples

- Calabi–Yau algebras are twisted Calabi–Yau, with modular automorphism the identity.
- If a filtered algebra A is such that $\mathrm{gr}(A)$ is twisted Calabi–Yau, then A itself is Calabi–Yau. (M. Van den Bergh)
- Enveloping algebras of all Lie algebras (and not only unimodular ones) with modular automorphism related to the modular character.
- A graded Koszul algebra A of global dimension d such that $A^!$ is Frobenius is Calabi–Yau. (M. Van den Bergh)

Twisted Calabi–Yau algebras: Examples

- An Ore extension $A[X; \sigma, \delta]$ of a twisted Calabi–Yau algebra is twisted Calabi–Yau. (L. Liu, S. Wang, Q. Wu)
- Quantum coordinate algebras of affine spaces, euclidean spaces, symplectic spaces. Quantum nilpontent spaces.
- Quantum enveloping algebras $\mathcal{U}_q(\mathfrak{g}^+)$ of positive parts of semisimple Lie algebras.
- Quantum coordinate algebras of Grassmanians and many other varieties important in representation theory.

Twisted Calabi–Yau algebras: Examples

- Enveloping algebras of Lie–Rinehart pairs (S, L) with S Calabi–Yau. (Th. Lambre, P. Le Meur) and their PBW-deformations.
- Algebras of differential operators tangent to free hyperplane arrangements (F. Kordon)
- Algebras of q-difference operators.
- Azumaya algebras with Calabi–Yau center.
- Finally, the class is closed under «good» deformations.

Our theme

A twisted Calabi–Yau algebra A comes *ex nihilo* with an automorphism $\sigma : A \to A$.

Theme

The automorphism σ is generally *not* the identity, ...

Our theme

A twisted Calabi–Yau algebra A comes *ex nihilo* with an automorphism $\sigma : A \to A$.

Theme

The automorphism σ is generally *not* the identity, ... but it wants to be.

Our theme

A twisted Calabi–Yau algebra A comes *ex nihilo* with an automorphism $\sigma : A \to A$.

Theme

The automorphism σ is generally *not* the identity, ... but it wants to be.

What we mean here is that there are natural actions of σ on various objects attached to A and that action tends to be trivial.

The center

Theorem

If A is twisted Calabi–Yau, then the modular automorphism acts trivially on the center $Z(A)$.

The center

Theorem

If A is twisted Calabi–Yau, then the modular automorphism acts trivially on the center $Z(A)$.

Corollary

A twisted Calabi–Yau algebra which is commutative is Calabi–Yau.

The automorphism group

Theorem

The class of the modular automorphism $\sigma : A \to A$ of a twisted Calabi–Yau algebra is central in the outer automorphism group $\mathrm{Out}(A)$*.*

In other words, if $\alpha \in \mathrm{Aut}(A)$, then there exists a unit $u \in$ such that

$$\sigma\alpha(a) = u\alpha\sigma(a)u^{-1}, \text{for all } a \in A.$$

This is an existence theorem.

If all units of A are central, then σ is central in $\mathrm{Aut}(A)$.
This is a very strong condition!

A baby example

$A = k_q[x, y]$

$$yx = qxy$$

This is twisted Calabi–Yau with modular automorphism σ such that

$$\sigma(x) = qx, \qquad \sigma(y) = q^{-1}.$$

Since $\{x^i y^j : i, j \geq 0\}$ is a basis of A, the spectrum of σ is

$$\operatorname{spec}(\sigma) = \{q^r : r \in \mathbb{Z}\}$$

If q is not a root of unity, then the eigenspaces are

$$A(q^r) = \begin{cases} x^r k[xy] & \text{if } r \geq 0; \\ k[xy]y^{-r} & \text{if } r < 0. \end{cases}$$

A baby example

Now let $\alpha \in \mathrm{Aut}(A)$ and suppose that q is not a root of unity. Since the units of A are central, α and σ commute, α preserves the eigenspaces of σ and, in particular,

$$A(q) = xk[xy] \qquad A(1) = k[xy] \qquad A(q^{-1}) = k[xy]y$$

It follows that

$$\alpha(x) \in kx \qquad \alpha(y) \in ky.$$

A baby example

Corollary

- *If* $k_q[x, y] \cong k_{q'}[x, y]$*, then* $(q) = (q')$*.*
- *If* q *is not a root of unity, then neither is* q' *and* $q' \in \{q, q^{-1}\}$*.*

This is a theorem of J. Alev and F. Dumas.

J. Alev and F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, J. Algebra **170** (1994), no. 1, 229–265.

Locally ad-nilpotent elements

Proposition

Let A be twisted Calabi–Yau and suppose that all units of A are central and that the ground field is of characteristic zero.

- *If $x \in A$ is locally* ad*-nilpotent, then there exists a central element $z \in Z(A)$ such that $\sigma(x) = x + z$.*
- *If the center $Z(A)$ is a domain and A is a finitely generated module over $Z(A)$, then σ fixes all regular locally* ad*-nilpotent elements of A.*

Normal elements

Proposition

Let A be a twisted Calabi–Yau algebra. If θ is a regular normal element, then there exists a unit $u \in A$ such that

$$\sigma(\theta) = u\theta.$$

Derivations

If $\delta : A \to A$ is a derivation and $\alpha \in \mathrm{Aut}(A)$, then $\alpha\delta\alpha^{-1} : A \to A$ is also a derivation.

This produces an action of $\mathrm{Aut}(A)$ on $HH^1(A) = \dfrac{\mathrm{Der}(A)}{\mathrm{InnDer}(A)}$.

Theorem

If A is twisted Calabi–Yau with modular automorphism σ, then σ acts trivially on $HH^1(A)$.

This is again an existence theorem.

A conjecture

Conjecture

The modular automorphism of a twisted Calabi–Yau algebra acts trivially on the Hochschild cohomology of A.

Determinants for Calabi–Yau algebras

If $\alpha : A \to A$ is an automorphism, then there is a unique isomorphism $\omega : A_\sigma \to {}_{\alpha^{-1}}(A_\sigma)_{\alpha^{-1}}$ such that the square

$$\begin{array}{ccc}
\operatorname{Ext}^d_{A^e}(A, A \otimes A) & \xrightarrow{\operatorname{Ext}^d_{A^e}(\alpha, \alpha^{-1} \otimes \alpha^{-1})} & {}_{\alpha^{-1}} \operatorname{Ext}^e_{A^e}(A, A \otimes A)_{\alpha^{-1}} \\
\downarrow h & & \downarrow h \\
A_\sigma & \overset{\omega}{\dashrightarrow} & {}_{\alpha^{-1}}(A_\sigma)_{\alpha^{-1}}
\end{array}$$

Determinants for Calabi–Yau algebras

If $\alpha : A \to A$ is an automorphism, then there is a unique isomorphism $\omega : A_\sigma \to {}_{\alpha^{-1}}(A_\sigma)_{\alpha^{-1}}$ such that the square

$$\begin{array}{ccc} \mathrm{Ext}^d_{A^e}(A, A \otimes A) & \xrightarrow{\mathrm{Ext}^d_{A^e}(\alpha, \alpha^{-1} \otimes \alpha^{-1})} & {}_{\alpha^{-1}}\,\mathrm{Ext}^e_{A^e}(A, A \otimes A)_{\alpha^{-1}} \\ \downarrow h & & \downarrow h \\ A_\sigma & \overset{\omega}{\dashrightarrow} & {}_{\alpha^{-1}}(A_\sigma)_{\alpha^{-1}} \end{array}$$

and therefore there is a unique unit $u := \omega(1)$ in A such that

$$\omega(a) = \alpha^{-1}(a)u, \qquad \alpha^{-1}\sigma(a)u = u\sigma\alpha^{-1}(a)$$

for all $a \in A$. I call this unit the Calabi–Yau determinant of α,

$$\det_{\sigma,h}(\alpha).$$

Determinants for Calabi–Yau algebras

Proposition

The map $\det_{\sigma,h}(\alpha) : \mathrm{Aut}(A) \to A^{\times}$ *is such that*

- $\det_{\sigma,h}(\mathrm{id}_A) = 1$ *and*
- *whenever* α, $\beta : A \to A$ *are automorphisms we have that*

$$\det_{\sigma,h}(\alpha\beta) = \beta^{-1}(\det_{\sigma,h}(\alpha)) \cdot \det_{\sigma,h}(\beta).$$

Determinants for Calabi–Yau algebras

$\det_{\sigma,h}$ does depend on σ and h... but

Corollary

The map

$$\widetilde{\det}_{\sigma,h} : \alpha \in \mathrm{Aut}(A) \mapsto \alpha(\det_{\sigma,h}(\alpha)) \in A^\times$$

is a normalized non-abelian 1*-cocycle whose class* $\det_A$ *in* $H^1(\mathrm{Aut}(A), A^\times)$ *is independent of the choices of* σ *and* h*.*

This cohomology class is a true invariant of the algebra A.

Determinants for Calabi–Yau algebras

Proposition

Let $d \in \mathbb{N}$. The polynomial algebra $A = \Bbbk[x_1, \ldots, x_d]$ is Calabi–Yau of dimension d, so that $\mathrm{id} : A \to A$ is a modular automorphism. If $\alpha : A \to A$ is an automorphism, then

$$\det_{\mathrm{id}}(\alpha) = \mathrm{Jac}(\alpha).$$

In this case $A^\times = \Bbbk^\times$, abelian and with trivial $\mathrm{Aut}(A)$-action, so

$$H^1(\mathrm{Aut}(A), A^\times) = \hom_{\mathrm{Grp}}(\mathrm{Aut}(A), \Bbbk^\times).$$

Under this isomorphism the class $\det_A$ corresponds to the morphism

$$\alpha \in \mathrm{Aut}(A) \mapsto \mathrm{Jac}(\alpha) \in \Bbbk^\times,$$

which is not trivial.

Traces for Calabi–Yau algebras

Let $\delta : A \to A$ be a derivation.

There is a unique right δ^e-operator $\nabla : A_\sigma \to A_\sigma$ that makes the following square commute:

$$\begin{array}{ccc} \mathrm{Ext}^d_{A^e}(A, A^e) & \xrightarrow{\nabla_{\delta,\delta^e}} & \mathrm{Ext}^d_{A^e}(A, A^e) \\ \downarrow h & & \downarrow h \\ A_\sigma & \overset{\nabla}{\dashrightarrow} & A_\sigma \end{array}$$

We then define the Calabi–Yau trace of δ to be

$$\mathrm{tr}_{\sigma,h}(\delta) := -\nabla(1),$$

Traces for Calabi–Yau algebras

The map $\mathrm{tr}_{\sigma,h} : \mathrm{Der}(A) \to A$ is such that

$$\mathrm{tr}_{\sigma,h}\big([\delta,\eta]\big) = \delta \cdot \mathrm{tr}_{\sigma,h}(\eta) - \eta \cdot \mathrm{tr}_{\sigma,h}(\delta) + [\mathrm{tr}_{\sigma,h}(\delta), \mathrm{tr}_{\sigma,h}(\eta)].$$

This means (if $2 \neq 0$) that $\mathrm{tr}_{\sigma,h}$ is a Maurer–Cartan element in the differential graded Lie algebra

$$\mathrm{CE}(\mathrm{Der}(A), A)$$

In the Calabi–Yau case, in fact, it takes values in $\mathrm{Z}(A)$ and is a Lie cocycle, and gives a well-defined class

$$\mathrm{tr}_A \in H^1(\mathrm{Der}(A), \mathrm{Z}(A)).$$

Traces for Calabi–Yau algebras

Proposition

Let $d \in \mathbb{N}$*. The polynomial algebra* $A = \mathbb{k}[x_1, \dots, x_d]$ *is Calabi–Yau of dimension* d*, so that* $\mathrm{id} : A \to A$ *is a modular automorphism. If* $\delta : A \to A$ *is a derivation, then*

$$\mathrm{tr}_{\mathrm{id}}(\delta) = \mathrm{div}(\alpha).$$

Traces for Calabi–Yau algebras

If $\delta : \mathbb{k}[x_1, \ldots, x_d] \to \mathbb{k}[x_1, \ldots, x_d]$ is a locally nilpotent derivation and $\chi(\mathbb{k}) = 0$, then

$$\operatorname{div}(\delta) = 0.$$

Gene Freudenburg, Algebraic theory of locally nilpotent derivations, 2nd ed., Encyclopaedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, 2017. Invariant Theory and Algebraic Transformation Groups, VII.

Leonid Makar-Limanov, On automorphisms of Weyl algebra, Bull. Soc. Math. France 112 (1984), no. 3, 359–363

Traces for Calabi–Yau algebras

Proposition

Suppose that the ground field $\Bbbk$ has characteristic zero. Let A be a Calabi–Yau algebra of dimension d whose center is a domain, and let $h : \mathrm{Ext}^d_{A^e}(A, A^e) \to A$ be an isomorphism of right A^e-modules.

The trace $\mathrm{tr}_{\mathrm{id},h}(\delta)$ of every locally nilpotent derivation $\delta : A \to A$ is zero.

What happens if A is only *twisted* Calabi–Yau? I don't know.

Traces for Calabi–Yau algebras

If $\delta : \mathbb{k}[x_1, \ldots, x_d] \to \mathbb{k}[x_1, \ldots, x_d]$ is a locally finite derivation and $\chi(\mathbb{k}) = 0$, then

$$\mathrm{div}(\delta) \in \mathbb{k}.$$

Hyman Bass and Gary Meisters, Polynomial flows in the plane, Adv. in Math. 55 (1985), no. 2, 173–208

Brian Coomes and Victor Zurkowski, Linearization of polynomial flows and spectra of derivations, J. Dynam. Differential Equations 3 (1991), no. 1, 29–66

Traces for Calabi–Yau algebras

Proposition

Suppose that the ground field $\mathbb{k}$ is algebraically closed of characteristic zero.
The trace of a locally finite derivation of a Calabi–Yau algebra whose center is an integral domain is a scalar.

Thanks!