A little bit of extra functoriality for Ext and the computation of the Gerstenhaber bracket

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δ-operators

A an algebra
$\delta : A \to A$ a derivation

M a left A-module

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M a left A-module

A δ-operator on M is a linear map $f : M \to M$ such that

$$f(am) = \delta(a)m + af(m).$$

δ-operators

Lemma

If $f, f' : M \to M$ are δ-operators on M, then $f - f' : M \to M$ is a morphism of A-modules.

δ-operators

Lemma

If $\epsilon : P \to M$ is a surjective morphism of left A-modules with projective domain and $f : M \to M$ is a δ-operator, then there exists a δ-operator $\tilde{f} : P \to P$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\epsilon} & M \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{\epsilon} & M \end{array}$$

is commutative, $\tilde{f}(\ker \epsilon) \subseteq \ker \epsilon$ and the restriction $f|_{\ker \epsilon} : \ker \epsilon \to \ker \epsilon$ is a δ-operator.

δ-liftings

$f : M \to M$ a δ-operator on a module M
$\epsilon : P_\bullet \to M$ a projective resolution of M

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

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A δ-lifting of f to $P_\bullet$ is a sequence $f_\bullet = (f_i)_{i \geq 0}$ of δ-operators $f_i : P_i \to P_i$ rendering the diagram commutative

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\ & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\ \cdots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \end{array}$$

δ-liftings

Lemma

Let M be a left A-module and let $\epsilon : P_\bullet \to M$ be a projective resolution.

- *There exists a δ-lifting $f_\bullet : P_\bullet \to P_\bullet$ of f to $P_\bullet$.*
- *If $f_\bullet$, $f'_\bullet$ are δ-liftings of a δ-operator $f : M \to M$ to $P_\bullet$, then $f_\bullet$ and $f'_\bullet$ are A-linearly homotopic.*

δ-liftings

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- If $\phi \in \hom_A(P_i, M)$, then $f_i^\sharp(\phi) : P_i \to M$ given by

$$f_i^\sharp(\phi)(p) = f(\phi(p)) - \phi(f_i(p))$$

  for all $p \in P_i$ is a morphism of A-modules

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$f : M \to M$ a δ-operator on a module M
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- The maps

$$f_i^\sharp : \hom_A(P_i, M) \to \hom_A(P_i, M)$$

are linear and we have a morphism of complexes of vector spaces

$$f_\bullet^\sharp : \hom_A(P_\bullet, M) \to \hom_A(P_\bullet, M).$$

δ-liftings

Lemma

Let $\epsilon : P_\bullet \to M$ and $\epsilon' : P'_\bullet \to M$ be projective resolutions and let $f_\bullet : P_\bullet \to P_\bullet$ and $f'_\bullet : P'_\bullet \to P'_\bullet$ be δ-liftings of f to $P_\bullet$ and to $P'_\bullet$. If $\alpha_\bullet : P'_\bullet \to P_\bullet$ is a morphism of complexes of A-modules lifting $\mathrm{id}_M : M \to M$, then the diagram

$$\begin{array}{ccc} \hom_A(P_\bullet, M) & \xrightarrow{f_\bullet^\sharp} & \hom_A(P_\bullet, M) \\ {\scriptstyle \alpha_\bullet^*}\downarrow & & \downarrow{\scriptstyle \alpha_\bullet^*} \\ \hom_A(P'_\bullet, M) & \xrightarrow{f'^\sharp_\bullet} & \hom_A(P'_\bullet, M) \end{array}$$

commutes up to homotopy.

The main theorem

Theorem

If $f : M \to M$ is a δ-operator on a module M, there is a canonical morphism of graded vector spaces

$$\nabla_f^\bullet : \operatorname{Ext}_A^\bullet(M, M) \to \operatorname{Ext}_A^\bullet(M, M)$$

such that for each projective resolution $\epsilon : P_\bullet \to M$ and each δ-lifting $f_\bullet : P_\bullet \to P_\bullet$ of f to $P_\bullet$ the diagram

$$\begin{array}{ccc} H(\hom_A(P_\bullet, M)) & \xrightarrow{\nabla_{f,P_\bullet}^\bullet} & H(\hom_A(P_\bullet, M)) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{Ext}_A^\bullet(M, M) & \xrightarrow{\nabla_f^\bullet} & \operatorname{Ext}_A^\bullet(M, M) \end{array}$$

commutes.