Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology

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Operations on cohomology

Theorem

Let A be an algebra, $M, N \in {}_A\mathrm{Mod}$ and $d \geq 0$.
Let $\mathcal{O} = (\mathcal{O}^p)_{p\geq 0}$ be a sequence of natural transformations of functors of A-modules

$$\mathcal{O}^p : \mathrm{Ext}_A^p(N, -) \to \mathrm{Ext}_A^{p+d}(M, -).$$

Assume that, for each short exact sequence $P' \rightarrowtail P \twoheadrightarrow P''$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ext}_A^p(N, P'') & \xrightarrow{\partial} & \mathrm{Ext}_A^{p+1}(N, P') \\ \downarrow{\scriptstyle \mathcal{O}^p} & & \downarrow{\scriptstyle \mathcal{O}^{p+1}} \\ \mathrm{Ext}_A^{p+d}(M, P'') & \xrightarrow{\partial} & \mathrm{Ext}_A^{p+d+1}(M, P') \end{array}$$

Then there exists exactly one $Y(\mathcal{O}) \in \mathrm{Ext}_A^d(M, N)$ such that

$$\mathcal{O}^p(-) = (-) \circ Y(\mathcal{O}).$$

Operations on cohomology

Corollary

There is an isomorphism of bifunctors of A-modules

$$Y : \mathrm{sOp}_A^\bullet(-,-) \cong \mathrm{Ext}_A^\bullet(-,-).$$

Corollary

Let A be an algebra and $d \geq 0$. Let $\mathcal{O} = (\mathcal{O}^p)_{p\geq 0}$ be a sequence of natural transformations of functors of A-bimodules

$$\mathcal{O}^p : H^p(A,-) \to H^{p+d}(A,-)$$

which commutes with boundary maps. Then there exists exactly one $Y(\mathcal{O}) \in HH^d(A)$ such that

$$\mathcal{O}^p(-) = (-) \smile Y(\mathcal{O}).$$

Change of rings

Corollary

Let $\phi : A \to B$ *be a map of rings, and let* $M \in {}_A\mathrm{Mod}$*.*
For each $q \geq 1$ *there is a unique class*

$$\zeta^q \in \mathrm{Ext}^2_B(\mathrm{Tor}^A_{q-1}(B, M), \mathrm{Tor}^A_q(B, M))$$

such that the differential

$$d_2^{p,q} : \mathrm{Ext}^p_B(\mathrm{Tor}^A_q(B, M), -) \to \mathrm{Ext}^{p+2}_B(\mathrm{Tor}^A_{q-1}(B, M), -)$$

of the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_B(\mathrm{Tor}^A_q(B, M), -) \Rightarrow \mathrm{Ext}^\bullet_A(M, -)$$

is given on $\alpha \in \mathrm{Ext}^p_B(\mathrm{Tor}^A_q(B, M), -)$ *by*

$$d_2^{p,q}(\alpha) = \alpha \circ \zeta^q.$$

Change of rings

Theorem

Let $\phi : A \to B$ be an epimorphism of algebras. There exists a spectral sequence, functorial on B-bimodules,

$$E_2^{p,q} \cong \mathrm{Ext}^p_{B^e}(\mathrm{Tor}^A_q(B,B), -) \Rightarrow H^\bullet(A, -)$$

which has $E_2^{\bullet,0} \cong H^\bullet(B,-)$.
For each $q \geq 1$ there exists a unique class

$$\zeta^q \in \mathrm{Ext}^2_{B^e}(\mathrm{Tor}^A_{q-1}(B,B), \mathrm{Tor}^A_q(B,B))$$

such that $d_2^{p,q}(-) = (-) \circ \zeta^q$.
If ϕ is surjective and $I = \ker\phi$, $\mathrm{Tor}^A_1(B,B) \cong I/I^2$ and

$$\zeta^1 \in \mathrm{Ext}^2_{B^e}(B, I/I^2) = H^2(B, I/I^2)$$

is the class of the infinitesimal extension

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

Monogenic algebras

Theorem

Let k be a field and fix a monic $f = \sum_{i=0}^{N} a_i X^i \in k[X]$.
Let $d = (f, f')$, pick $q \in k[X]$ such that $f = qd$, and put

$$u = q^2 \sum_{i=0}^{N} a_i \frac{i(i-1)}{2} X^{i-2} = q^2 \Delta_2(f).$$

Let $A = k[X]/(f)$.
There is an isomorphism of graded commutative algebras

$$HH^\bullet(A) \cong \frac{k[x_0, \tau_1, \zeta_2]}{(f(x), d(x)\tau, f'(x)\zeta, \tau^2 - u(x)\zeta)}.$$

Monogenic algebras

Proposition

Let

$$w = \sum_{i=0}^{N} \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} a_i \, \overline{(s+1)X^s q} \, X^t.$$

The Gerstenhaber Lie structure on $HH^\bullet(A)$ *is such that*

$$[\tau, x] = q(x),$$
$$[\zeta, \tau] = w(x)\zeta,$$
$$[x, x] = [\tau, \tau] = [\tau, \zeta] = [x, \zeta] = 0.$$

Nice morphisms

Theorem

Let $\phi : A \to B$ be an epimorphism of algebras. The following statements are equivalent:

a) $\phi : A \to B$ *is a homological epimorphism;*

b) $\mathrm{Tor}_{+}^{A}(B, M) = 0$ *for all* $M \in {}_{A}\mathrm{Mod}$*;*

c) $\mathrm{Tor}_{+}^{A}(B, B) = 0$*;*

d) $\phi^{e} : A^{e} \to B^{e}$ *is a homological epimorphism.*

When they hold, there is an isomorphism of functors of B-bimodules

$$H^{\bullet}(B, -) \xrightarrow{\cong} H^{\bullet}(A, -).$$

Nice ideals

Corollary

Let $\phi : A \to B$ *be a surjective homological epimorphism and let* $I = \ker \phi$*. There is a long exact sequence*

$$\cdots \to \operatorname{Ext}^p_{A^e}(A, I) \to HH^p(A) \to HH^p(B) \to \operatorname{Ext}^{p+1}_{A^e}(A, I) \to \cdots$$

Nice ideals

Proposition

Let $\phi : A \to B$ be a surjective homological epimorphism such that $I = \ker\phi$ is A-flat on one side. Then $H^0(B,-) \cong H^0(A,-)$ on ${}_B\mathrm{Mod}_B$ and there is a natural long exact sequence of functors of B-bimodules

$$\cdots \longrightarrow H^p(B,-) \longrightarrow H^p(A,-) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^{p-1}_{A^e}(I/I^2,-) \xrightarrow{\smile\zeta} H^{p+1}(B,-) \longrightarrow \cdots$$

with $\zeta \in H^2(B, I/I^2)$ the class of the infinitesimal extension

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

Nice ideals

Lemma

Let $\phi : A \to B$ be a surjective morphism of algebras and put $I = \ker\phi$. Then

$$\mathrm{Tor}_q^A(B,B) \cong \begin{cases} B, & \text{if } q = 0; \\ I/I^2, & \text{if } q = 1; \\ \ker\left(I \otimes_A I \xrightarrow{\mu} I\right), & \text{if } q = 2; \\ \mathrm{Tor}_{q-2}^A(I,I), & \text{if } q > 2. \end{cases}$$

Nice ideals: an example

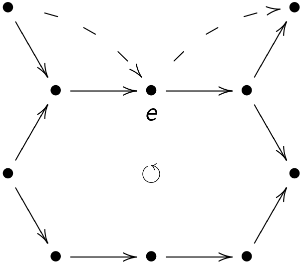
Let $A = kQ/J$ be an admissible quotient of the path algebra on a quiver Q and let $e \in Q_0$.
Assume

- Every minimal relation in J involving a path passing through e also involves a path not passing through e; and
- e is on no oriented cycle of Q.

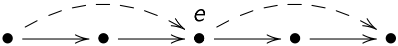
Then $I = AeA \lhd A$ is homological and, if $B = A/I$, there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^p_A(D(eA), Ae) \longrightarrow HH^p(A) \longrightarrow$$
$$\longrightarrow HH^p(B) \longrightarrow \mathrm{Ext}^{p+1}_A(D(eA), Ae) \longrightarrow \cdots$$

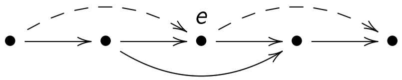
Nice morphisms



Nice morphisms



Nice morphisms



Nice morphisms