κ-DENSE TOTAL ORDERS

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Abstract. We show that for all cardinals κ there exist totally ordered sets S with the property that whenever $X, Y \subset S$ are non-empty subsets of cardinal at most κ such that $x < y$ for all $x \in X$ and all $y \in Y$ there exists an element $z \in S$ such that $x < z < y$ for all $x \in X$ and all $y \in Y$.

The construction is based on idea of starting with an arbitrary total order S and filling its gaps repeatedly, until we obtain a total order with the desired property.

1. Let S be a totally ordered set and let κ be a cardinal. We say that a pair (X, Y) of non-empty subsets $X, Y \subset S$ is a *κ-gap* the following conditions are satisfied:

- both X and Y are of cardinal at most κ;
- we have $x < y$ for all $x \in X$ and all $y \in Y$;
- there does not exist an element $z \in S$ such that $x < z < y$ for all $x \in X$ and all $y \in Y$.

If S does not admit any κ-gap, we say that S is *κ-dense*.

2. It is obvious that a totally ordered set of at most one element is κ-dense for all cardinals κ. We call these examples *trivial*.

3. If κ is a finite cardinal, it is clear that a set is κ-dense iff it is dense in the usual sense.

4. We want to show that more interesting examples exist:

Theorem. *Let κ be a cardinal. Then there exist non-trivial κ-dense total orders.*

The problem was suggested in a post [2] on the `sci.math` newsgroup by 'Marc'.

5. Let us fix a cardinal κ and a totally ordered set S. Let $G(S)$ be the set of all κ-gaps of S. Notice that $G(S)$ is empty iff S is κ-dense.

6. We put $S^+ = S \cup G(S)$ and consider the relation $\ll$ on the set S^+ which extends the relation $<$ and such that, given $z \in S$ and (X, Y), $(U, V) \in G(S)$,

- $z \ll (X, Y)$ iff for all $y \in Y$ we have $z < y$;
- $(X, Y) \ll z$ iff for all $x \in X$ we have $x < z$; and
- $(X, Y) \ll (U, V)$ iff there exist $y \in Y$ and $u \in U$ such that $y \leq u$.

7. Notice that whenever $z \in S$ and $(X, Y) \in G(S)$ are such that $z \ll (X, Y)$, there exists $y \in Y$ such that $y \leq z$, for otherwise z would separate X and Y, contradicting the third condition in definition **5**. There is, of course, a symmetric statement.

8. It is evident that $(X, Y) \in G(S)$ implies that we have $x \ll (X, Y) \ll y$ for all $x \in X$ and all $y \in Y$.

9. The relation $\ll$ on S^+ is anti-symmetric. To see this—since the restriction of $\ll$ to S is $<$, which is known to be anti-symmetric—we have to only consider the following two cases.

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- First, suppose there exist $z \in S$ and $(X, Y) \in G(S)$ such that $z \ll (X, Y)$ and $(X, Y) \ll z$. Then for all $y \in Y$ and all $x \in X$ we have $x < z < y$, contradicting the fact that $(X, Y) \in G(S)$.
- Second, suppose that (X, Y), $(U, V) \in G(S)$ are such that $(X, Y) \ll (U, V)$ and $(U, V) \ll (X, Y)$. Then there exist $y \in Y$, $u \in U$, $v \in V$ and $x \in X$ such that $y \leq u$ and $v \leq x$. Since $u < v$, this implies that $y < x$, which is impossible.

10. The relation $\ll$ is also transitive:

- If $x, y, z \in S$ are such that $x \ll y \ll z$, then clearly $x \ll y$.
- Let now $z, t \in S$ and $(X, Y) \in G(S)$. Assume first that $z \ll t \ll (X, Y)$. Then $z < t$ and for all $y \in Y$ we have $t < y$, so $z < y$ for all $y \in Y$, that is, $z \ll (X, Y)$. The case in which $(X, Y) \ll z \ll t$ is handled similarly.

  Assume now that $z \ll (X, Y) \ll t$. This second inequality implies that there exists $y \in Y$ such that $y \geq t$, and the first one implies that $z < y$ so, in fact, $z \ll t$.
- Let next $z \in S$ and (X, Y), $(U, V) \in G(S)$. Suppose $z \ll (X, Y) \ll (U, V)$. Let $v \in V$. The second inequality implies that there exist $y \in Y$ and $u \in U$ such that $y \leq u$, and the first inequality implies, in turn, that $z < y$, so that in fact we have $z < y \leq u < v$. We see thus that $z \ll (U, V)$. If we had $(X, Y) \ll (U, V) \ll z$ instead we would reason in a similar way to show that $(X, Y) \ll z$.

  Suppose now that $(X, Y) \ll z \ll (U, V)$. The first inequality tells us that there exists $y \in Y$ such that $y \leq z$ and the first one, that there exists $u \in U$ such that $z \leq u$. We see that $y \leq u$, so $(X, Y) \ll (U, V)$.
- Finally, suppose that (X, Y), (U, V), $(S, T) \in G(S)$ are such that we have $(X, Y) \ll (U, V) \ll (W, Z)$. Then there exist $y \in Y$, $u \in U$, $v \in V$ and $w \in W$ such that $y \leq u$ and $v \leq w$. Since $u < v$, this implies that $y < w$, so $(X, Y) \ll (W, Z)$.

11. Since $\ll$ is anti-symmetric and transitive, there exists a total order $\lll$ on S^+ which extends $\ll$; we remark that, in general, $\ll$ itself is not a total order. From now on, we consider S^+ endowed with such an order $\lll$, and we will write it simply $<$ as this should be cause of no confusion. We are interested in S^+ because $S \subset S^+$ and

> whenever X, $Y \subset S$ are non-empty subsets of cardinal at most κ with $x < y$ for all $x \in X$ and all $y \in Y$, there exists $z \in S^+$ such that $x < z < y$ for all $x \in X$ and all $y \in Y$.

12. We define a transfinite sequence of totally ordered sets as follows: we put $S_0 = S$ and, for an ordinal α,

- if α is a successor ordinal, so that there exists an ordinal β such that $\alpha = \beta + 1$, we put $S_\alpha = (S_\beta)^+$, and
- if α is a limit ordinal, we put $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$, endowed with the unique total order which extends those of the S_β.

13. We need two definitions and a result from the theory of ordinals.

- An ordinal η is *initial* if there exists no ordinal γ which is equipotent to η and such that $\gamma < \eta$; *cf.* [1, Chapter 7, §1].
- An infinite initial ordinal η is *regular* if whenever θ is an ordinal such that $\theta < \eta$ and $(\alpha_\nu)_{\nu<\theta}$ is a transfinite increasing sequence of ordinals of length θ such that $\alpha_\nu < \eta$ for all $\nu < \theta$, we have that $\sup\{\alpha_\nu : \nu < \theta\} < \eta$; *cf.* [1, Chapter 9, §2].

The result we need is that that there exist arbitrarily large regular initial ordinals, *cf.* [1, Chapter 9, Theorem 2.4].

14. Let us fix a regular initial ordinal η which is strictly larger than κ. We claim that the totally ordered set S_η is κ-dense. Indeed, suppose $X, Y \subset S_\eta$ are two non-empty subsets of cardinality at most κ such that for all $x \in X$ and all $y \in Y$ we have $x < y$. Since η is a limit ordinal, $S_\eta = \bigcup_{\alpha<\eta} S_\alpha$. Moreover, since $X \cup Y$ has cardinal at most κ and η is regular, there exists an ordinal ϕ such that $\phi < \eta$ and $X \cup Y \subset S_\phi$, and we see that there exists a $z \in (S_\phi)^+ = S_{\phi+1} \subset S_\eta$ such that $x < z < y$ for all $x \in X$ and all $y \in Y$.

15. We can now prove the theorem. If S is an arbitrary total order, we have shown in **14** that there exist an ordinal η such that S_η is κ-dense. If S has more than one element, then of course S_η does also, so we are done.

References

[1] K. Hrbacek and T. Jech, *Introduction to set theory*, 3rd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 220, Marcel Dekker Inc., New York, 1999. MR1697766 (2000c:03001) ↑2, 3

[2] 'Marc', *Super densely ordered sets*, Usenet post on the `sci.math` newsgroup (February 15, 2008), available at `http://groups.google.com/group/sci.math/msg/7bf3d2f92061d2cd`. ↑1

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