Contents

Chapter 1

Plane curves

§1.1. Curves

A ***curve*** is a function $\gamma : (a, b) \to \mathbb{R}^2$ defined on a non-empty open interval of $\mathbb{R}$. We say such a curve is ***smooth*** if it is a function of type C^2. Unless we say otherwise, we will assume throughout in all that follows that all the curves with which we deal are smooth.

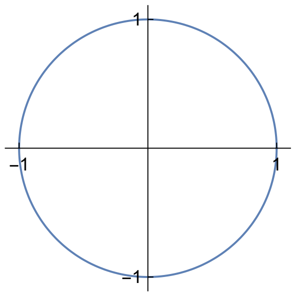
It is important to keep in mind that a curve $\gamma : (a, b) \to \mathbb{R}^2$ is a *function*. Its image

$$\{\gamma(t) : t \in (a, b)\}$$

is, on the other hand, a *subset* of the plane — we call it the ***trace*** of γ. There is more information in the function γ than in it its trace, but we usually depict graphically a curve by drawing its trace. For example, the curve

$$t \in \mathbb{R} \mapsto (\cos t, \sin t) \in \mathbb{R}^2 \tag{1.1}$$ {eq:ex-circle}

has as trace the unit circle



The trace of a curve does not determine it. For example, the curve $t \in \mathbb{R} \mapsto (\cos t^2, \sin t^2) \in \mathbb{R}^2$ has the same trace as that in (1.1).

If $\gamma : (a, b) \to \mathbb{R}^2$ is a curve, then for each $t \in (a, b)$ the point $\gamma(t) \in \mathbb{R}^2$ can be written in the form $(x(t), y(t))$ for a well-determined pair of real numbers $x(t)$ and $y(t)$, and in this way we obtain two real-valued functions $x, y : (a, b) \to \mathbb{R}$, which we call the ***components*** of γ. If γ is smooth, then the two functions x and y are themselves smooth. Conversely, if $x, y : (a, b) \to \mathbb{R}$ are two functions defined on a non-empty open interval of $\mathbb{R}$, then the function components

$$t \in (a, b) \mapsto (x(t), y(t)) \in \mathbb{R}^2$$

is a curve in $\mathbb{R}^2$ that has as components the functions x and y, and this curve is smooth if both functions x and y are. Using this we can easily construct curves, of course. For example, the functions

$t \in \mathbb{R} \mapsto (2\cos 3t, 3\sin 2t) \in \mathbb{R}^2,$	*a Lissajous curve*
$t \in \mathbb{R} \mapsto (t, t^2) \in \mathbb{R}^2,$	*a parabola*
$t \in \mathbb{R} \mapsto (t^2 - 1, t(t^2 - 1)) \in \mathbb{R}^2,$	*a nodal cubic*
$t \in \mathbb{R} \mapsto ((1 + 2\cos t)\sin t, (1 + 2\cos t)\sin t) \in \mathbb{R}^2,$	*a limaçon*
$t \in (0, +\infty) \mapsto (t\cos t, t\sin t) \in \mathbb{R}^2$	*a circular spiral*
$t \in \mathbb{R} \mapsto (t^3, t^2) \in \mathbb{R}^2$	*a cuspidal cubic*
$t \in \mathbb{R} \mapsto (\cos t + 2\sqrt{2}\cos t/2, \sin t) \in \mathbb{R}^2,$	*a fish curve*
$t \in \mathbb{R} \mapsto \left(t, \dfrac{1}{1 + e^{-4t}} - \dfrac{1}{2}\right) \in \mathbb{R}^2,$	*a sigmoid*

are all smooth curves. We have drawn their traces in Figure 1.1 on page 3.

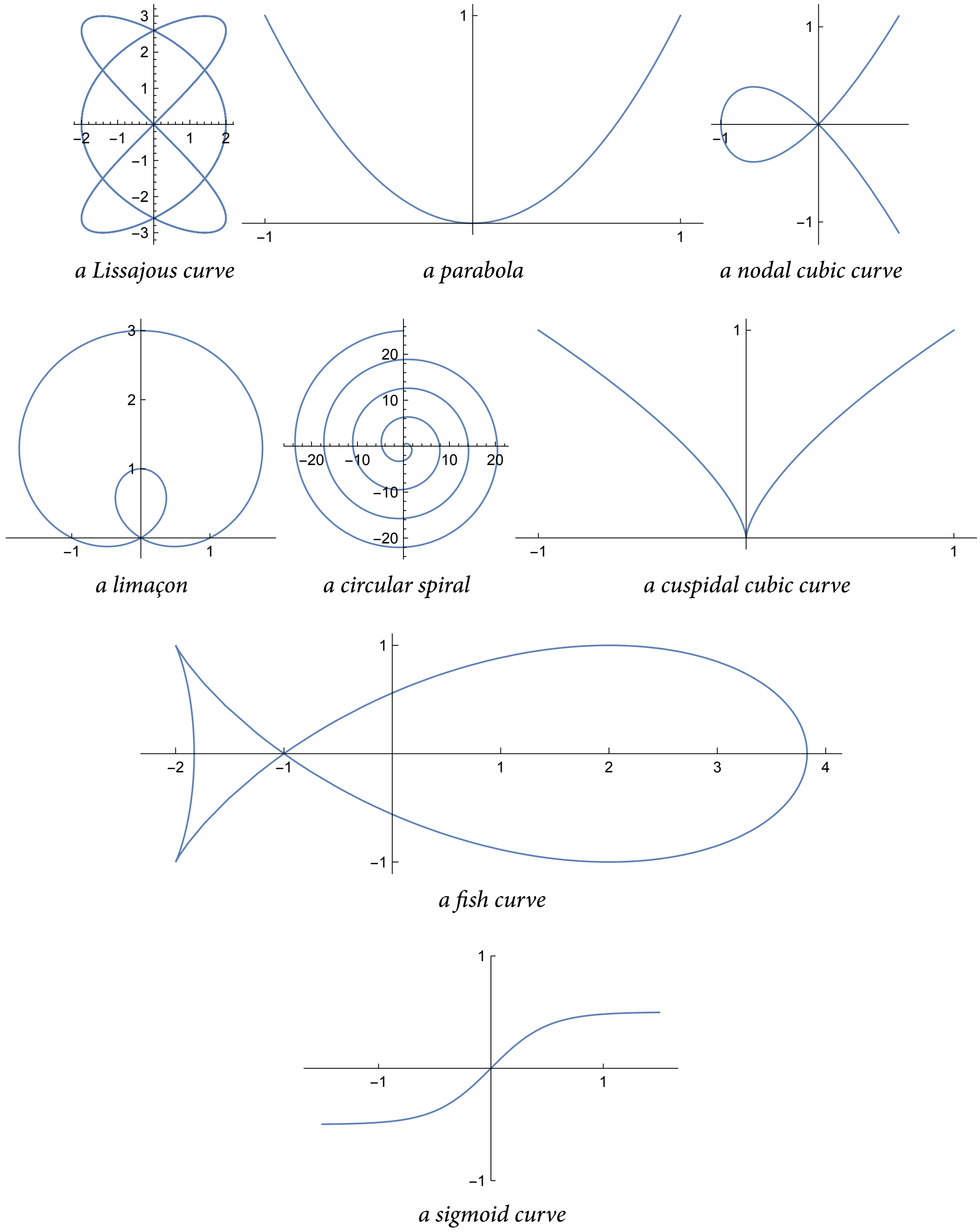
Example 1.1.1. If $f : (a, b) \to \mathbb{R}$ is a real-valued function, then we can construct a curve

$$\gamma : t \in (a, b) \mapsto (t, f(t)) \in \mathbb{R},$$

and this curve is smooth exactly when the function f is smooth. The trace of this curve is the set

$$C \coloneqq \{(t, f(t)) : t \in (a, b)\},$$

which is precisely the graph of the function f. Because of this, we will call the curve γ the ***standard parametrization of the graph of f***. the graph of a function



{fig:curves}

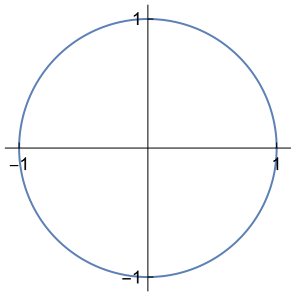
Figure 1.1. Some parametrized curves.

§1.2. Implicit curves

Often we want to study subsets of the plane that are given *implicitly* by equations and not as the trace of curves parametrizating them, but that nontheless deserve to be called curves. For example, the set C of points (x, y) of $\mathbb{R}^2$ that satisfy the equation

$$x^2 + y^2 = 1$$

is well-known to be a circle of radius 1, centered at the origin,



In this case, it is easy to find a curve whose trace is the set C, namely the function

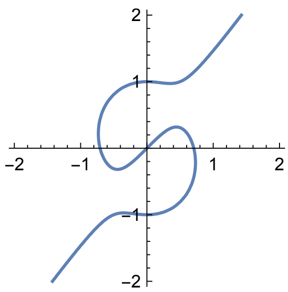
$$\gamma : t \in \mathbb{R} \mapsto (\cos t, \sin t) \in \mathbb{R}^2.$$

We call a curve whose trace is C a ***parametrization*** of the implicit curve C. As we observed, there are in fact many curves that parametrize this implicit curve. parametrization

Similarly, the set D of those points (x, y) of $\mathbb{R}^2$ that satisfy the equation

$$2x^3 - x^2 y + x y^2 - y^3 - x + y = 0$$

can be drawn as in the following picture



In this example it is much less obvious how to find a parametrization. Indeed, this is an example of what is called an ***elliptic curve*** and it can be shown that such curves cannot be parametrized using elementary functions. elliptic curve

It should be noted that not every equation determines a subset of the plane that deserves to be called a curve. For example, the equation $x^2 + y^2 = 0$ determines a subset of $\mathbb{R}^2$ consisting of exactly one point, the origin $(0,0)$, and we probably do not want to call this a curve. Similarly, the set determined by the equation $\sin^2 x + y^2 = 0$ determines in $\mathbb{R}^2$ the subset $\{(\pi k, 0) : k \in \mathbb{Z}\}$, which is countable and discrete — again, not something that we usually call a curve. It can also happen that an equation has no solutions at all, as it happens with the equation $x^2 + y^2 + 1 = 0$, so that the subset of $\mathbb{R}^2$ determined by it is empty. These examples show that some care is needed when studying implicitly defined curves to ensure that what we have is actually something that we want to call a curve. Similarly, we should care about ensuring some form of smoothness.

For us, the following result will be enough to deal with implicitly defined curves. It gives sufficient conditions on an equation in the plane that guarantee that the set it defines can be parametrized by a smooth curve — at least locally. Let us recall that the ***gradient*** of a function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^2$ at a point p of U is the vector

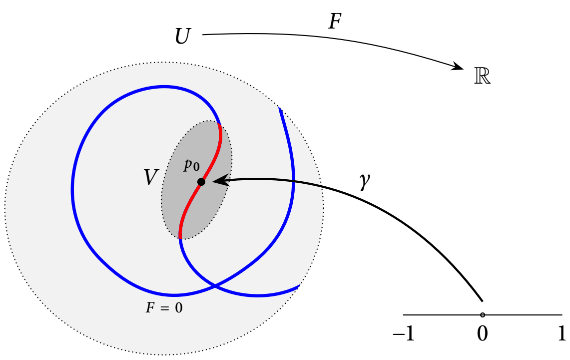
$$\nabla f(p) = \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right).$$

{prop:implicit}

Proposition 1.2.1. *Let k be a positive integer, let U be an open subset of $\mathbb{R}^2$ and let $F : U \to \mathbb{R}$ be a function that has k continuous derivatives at each point of its domain. If p_0 is a point of U such that $F(p_0) = 0$ and $\nabla F(p_0) \neq 0$, then there exists an open set V of $\mathbb{R}^2$ and a function $\gamma : (-1,1) \to V$ such that*

- *$p_0 \in V \subseteq U$,*
- *γ has k continuous derivatives,*
- *$\gamma'(t) \neq 0$ and $F(\gamma(t)) = 0$ for all $t \in (-1,1)$, and*
- *the trace of γ is exactly the set $V \cap \{p \in \mathbb{R}^2 : F(p) = 0\}$.* □

We will not prove this proposition here. It is essentially the so called *Implicit Function Theorem* from calculus. We refer the reader to the book [Mun91] where Munkres proves this as his Theorem 9.1. The following schematic drawing describes the situation of the proposition.



It should be noted that there is no uniqueness claim in the proposition. It is the case, in fact, that once we find an open set V and a function $\gamma : (-1,1) \to V$ satisfying the conditions listed there we can change both V and γ in many ways while preserving those conditions.

In practice, when we have a curve defined implicitly and the conditions of the proposition are satisfied, so that we know that in principle we can find parametrizations for it, doing so is often impracticable. This notwithstanding, we can generally do almost everything we want with such a curve. In particular, for our purposes — we are doing differential geometry, after all! — it is important that we obtain information about the derivatives of the curve, and this can be done.

The following are simple examples of curves given implicitly:

$x^3 + y^3 = 3xy,$	*Descartes' folium*
$x^{2/3} + y^{2/3} = 1,$	*astroid*
$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2),$	*cardioid*
$x^4 = (x^2 - y^2),$	*eight curve*
$x^6 + y^6 = x^2,$	*butterfly curve*
$(x^2 + y^2)^2 = x^2 - y^2,$	*Bernoulli's lemniscate*
$x^2 y + \frac{1}{2} y = x,$	*serpentine curve*

We have drawn them in Figure 1.2 on page 7.

Sometimes the following different version of Proposition 1.2.1, which is actually closer to the *Implicit Function Theorem*, is more useful:

{prop:implicit:graph}

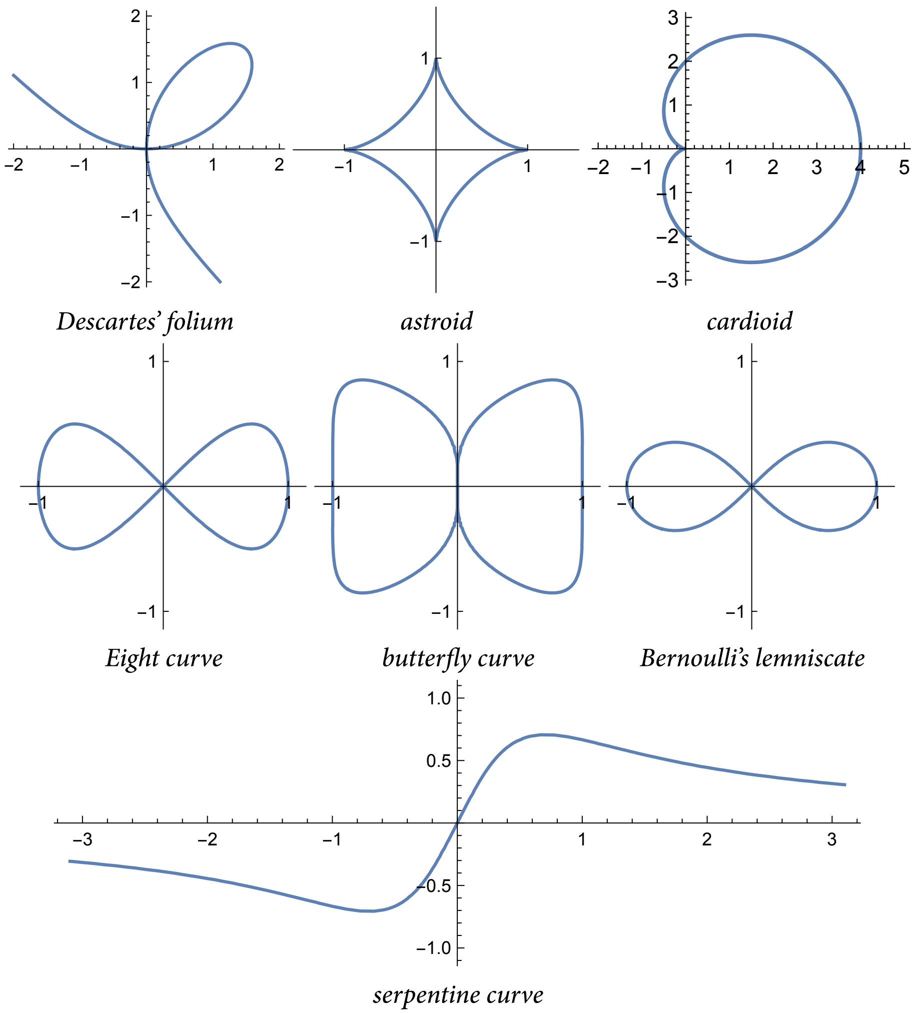
Proposition 1.2.2. *Let k be a positive integer, let U be an open subset of $\mathbb{R}^2$ and let $F : U \to \mathbb{R}$ be a function that has k continuous derivatives at each point of its domain. If p_0 is a point of U such that $F(p_0) = 0$ and $\frac{\partial F}{\partial x}(p) \neq 0$, then there exist*

- *open non-empty intervals (a,b) and (c,d) and*
- *a function $f : (a,b) \to (c,d)$*

such that

- *$p_0 \in (a,b) \times (c,d) \subseteq U$,*
- *f has k continuous derivatives,*
- *$F(t, f(t)) = 0$ for all $t \in (a,b)$,*
- *the graph of f is exactly the set $\{q \in (a,b) \times (c,d) : F(q) = 0\}$, and*
- *the derivative $\frac{\partial F}{\partial x}$ does not vanish on $(a,b) \times (c,d)$.* □

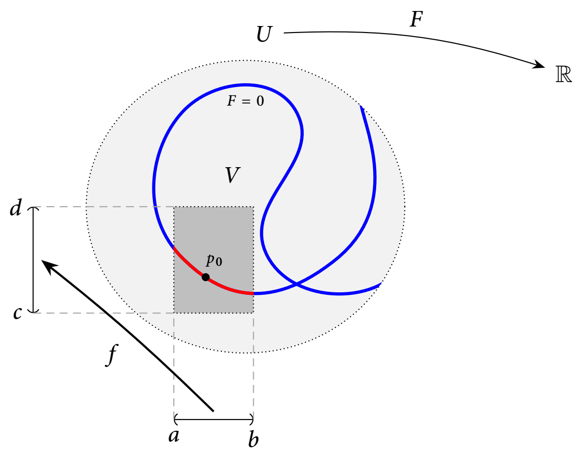
Here what we find is a function $f : (a,b) \to (c,d)$ such that the standard parametrization of its graph, $\gamma : t \in (a,b) \mapsto (t, f(t)) \in \mathbb{R}^2$, is a parametrization of a piece of the implicit curve determined by the equation $F(x,y) = 0$. Notice that in the situation of the proposition there is an



{fig:implicit-curves}

Figure 1.2. Some implicit curves.

element t_0 in (a, b) such that $p = (t_0, f(t_0))$. The situation is now as follows:



An advantage of this proposition over the previous one is that it is easier to obtain information about the derivatives of the parametrization γ, since they all depend in a simple way on the derivatives of f.

§1.3. Regular curves and their tangent lines

Let $\gamma : (a, b) \to \mathbb{R}$ be a smooth curve in the plane, and let $x, y : (a, b) \to \mathbb{R}$ be its component functions. We say that γ is ***regular at a point*** t_0 of its domain (a, b) if $\gamma'(t_0) \neq 0$, and in that case the ***tangent vector*** to γ at t_0 is the unit vector

$$\mathbf{t}(t_0) \coloneqq \frac{\gamma'(t_0)}{\|\gamma'(t_0\|}.$$

If the curve is regular at each point of its domain we say that it is ***regular*** and call the function

$$\mathbf{t} : t \in (a, b) \mapsto \mathbf{t}(t) \in \mathbb{R}^2$$

the ***tangent vector field*** of γ. Notice that in this situation the function $\mathbf{t}$ is of class C^1.

It is important to keep in mind that regularity is a property of smooth curves that is not determined by their traces. For example, the two curves

$$t \in \mathbb{R} \mapsto (t, 0) \in \mathbb{R}, \qquad t \in \mathbb{R} \mapsto (t^3, 0) \in \mathbb{R}$$

have the same trace — namely, the horizontal axis of the plane — but the first one is regular while the second one is not, as its derivative at 0 vanishes.

Example 1.3.1. If $f : (a,b) \to \mathbb{R}$ is a smooth function, then the standard parametrization

$$\gamma : t \in (a,b) \mapsto (t, f(t)) \in \mathbb{R}^2$$

of the graph of f has derivative $\gamma'(t) = (1, f'(t))$ for all $t \in (a,b)$ and is therefore regular at each point of its domain. The corresponding tangent vector field $\mathbf{t} : (a,b) \to \mathbb{R}^2$ has

$$\mathbf{t}(t) = \left(\frac{1}{\sqrt{1+f'(t)^2}}, \frac{f'(t)}{\sqrt{1+f'(t)^2}} \right)$$

at each $t \in (a,b)$.

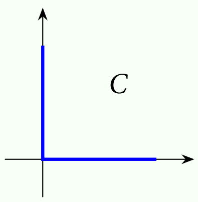
Example 1.3.2. In the situation of Proposition 1.2.1 the curve $\gamma : (-1,1) \to V \subseteq \mathbb{R}^2$ whose existence is asserted there is regular — indeed, this is part of the claim of the proposition. We thus see that every implicit curve $F(x,y) = 0$ determined by a smooth function F can be parametrized by a regular curve *near* a point p where $\nabla F(p) \neq 0$.

Some subsets of the plane that look like curves cannot be parametrized by regular curves. Let us look at two examples of this.

Example 1.3.3. Let us consider the set

$$C = \{(x,y) \in \mathbb{R}^2 : (x \geq 0 \wedge y = 0) \vee (x = 0 \wedge x \geq 0\},$$

which is the union of the positive semi-axes.



This set is the trace of smooth curves. For example, the function $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that

$$\gamma(t) = \begin{cases} (0, -te^{-1/t^2}) & \text{if } t < 0; \\ (0,0) & \text{if } t = 0; \\ (te^{-1/t^2}, 0) & \text{if } t > 0 \end{cases}$$

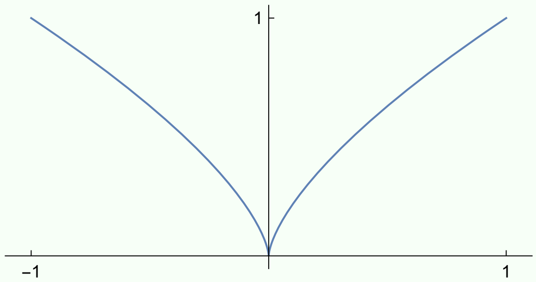
for all $t \in \mathbb{R}$ can be shown to be smooth, and its trace is clearly C. We claim that, on the other hand, there is no regular smooth curve $\gamma : (a,b) \to \mathbb{R}^2$ whose trace is contained in C and contains the origin. To verify this let us suppose that, on the contrary, such a curve does exist, and let t_0 be a point in its domain (a,b) such that $\gamma(t_0) = (0,0)$.

Let $x, y : (a,b) \to \mathbb{R}$ be the components of γ. If u and v are two positive numbers, then the function $h : t \in (a,b) \mapsto ux(t) + vy(t) \in \mathbb{R}$ is differentiable and, since $\gamma(t) \in C$ for all $t \in (a,b)$ and $\gamma(t_0) = (0,0)$, has $g(t) \geq 0$ for all $t \in (a,b)$ and $g(t_0) = 0$. This tells us that g has a local minimum at t_0, and therefore that $0 = g'(t_0) = ux'(t_0) + vy'(t_0)$. In particular, we have that $x'(t_0) + 2y'(t_0) = 0$ and that $2x'(t_0) + y'(t_0) = 0$, and it follows from these two equalities that in fact $x'(t_0) = y'(t_0) = 0$, that is, that $\gamma'(t_0) = 0$. This is absurd.

Example 1.3.4. Let C be the set of points (x,y) in the plane such that

$$x^2 = y^3,$$

which we called above a ***cuspidal cubic*** and which looks as follows:



The function

$$\gamma : t \in \mathbb{R} \mapsto (t^3, t^2) \in \mathbb{R}^2$$

is a smooth parametrization of C. Indeed, it is immediate that for every $t \in \mathbb{R}$ the point (t^3, t^2) is in C, as $(t^3)^2 = (t^2)^3$. On the other hand, if $p = (x,y)$ is a point in C, then $y^3 = x^2 \geq 0$, so that $y \geq 0$ and there is a unique non-negative real number s such that $y = s^2$. Now $x^2 = y^3 = s^6$, so that x is either s^3 or $-s^3$, and therefore p is either $\gamma(s)$ or $\gamma(-s)$. We thus see that the trace of γ is precisely the cuspidal cubic C.

Let us show that there is no regular smooth parametrization of C. Let $\sigma : (a,b) \to \mathbb{R}^2$ be a smooth curve whose trace is C, let t_0 be a point in (a,b) such that $\sigma(t_0) = (0,0)$, and let $x, y : (a,b) \to \mathbb{R}$ be the components of σ. The function y has a local minimum at t_0, since $y(t_0) = 0$ and $y(t) \geq 0$ for all $t \in (a,b)$, and is smooth, so we have $y'(t_0) = 0$.

Let us suppose that $x'(t_0) \neq 0$. There exists then a positive number ϵ such that $x(t) \neq 0$ for all t belonging to the set $T := (t_0 - \epsilon, t_0 + \epsilon) - \{t_0\}$. For all $t \in T$ we have that $x(t)^2 = y(t)^3$, so that $2x(t)x'(t) = 3y(t)^2y'(t)$. Squaring we deduce that also $4x(t)^2x'(t)^2 = 9y(t)^4y'(t)^2$ and, since $x(t) \neq 0$ because t is in T, that

$$x'(t)^2 = \frac{9y(t)^4}{4x(t)^2}y'(t) = \frac{9}{4}y(t)y'(t).$$

It follows from this that

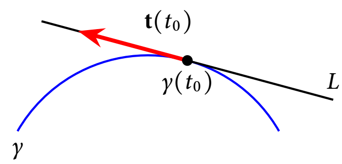
$$x'(t_0)^2 = \lim_{t\to t_0} x'(t)^2 = \lim_{t\to t_0} \frac{9}{4} y(t) y'(t) = 0,$$

because y is smooth and $y'(t_0) = 0$. Of course, this contradicts our hypothesis that $x'(t_0) \neq 0$. We can therefore conclude that no smooth parametrization of C is regular.

These two examples suggest that in general a curve given implicitly does not have a regular parametrization near a point at which it has «a kink», and that is indeed the case. We will not enter into details about this.

The reason for which we are particularly interested in curves that are regular is that for them we can easily[1] define tangent lines. Let $\gamma : (a,b) \to \mathbb{R}^2$ be a smooth parametrized curve, and let t_0 be an element of (a,b) at which γ is regular. The ***tangent line*** to γ at t_0 is the line L that passes through $\gamma(t_0)$ with direction given by the tangent vector $\mathbf{t}(t_0)$, namely the set

$$L \coloneqq \{\gamma(t_0) + u \cdot \mathbf{t}(t_0) : u \in \mathbb{R}\}.$$



We should notice that this set is indeed a line because the tangent vector $\mathbf{t}(t_0)$ is non-zero.

As the tangent vector $\mathbf{t}(t_0)$ is a non-zero scalar multiple of the derivative $\gamma'(t_0)$, we can also describe the line L using the latter: we have that

$$L = \{\gamma(t_0) + u \cdot \gamma'(t_0) : u \in \mathbb{R}\}.$$

If x, $y : (a,b) \to \mathbb{R}$ are the coordinates of γ, then we have that $\gamma(t_0) = (x(t_0), y(t_0))$ and $\gamma'(t_0) = (x'(t_0), y'(y_0))$, and therefore we can write even more explicitly

$$L = \{(x(t_0) + ux'(t_0), y(t_0) + uy'(t_0)) : u \in \mathbb{R}\}.$$

We can read off this equality a parametrization of the tangent line: it is the trace of the curve

$$u \in \mathbb{R} \mapsto (x(t_0) + ux'(t_0), y(t_0) + uy'(t_0)) \in \mathbb{R}^2.$$

On the other hand, it is easy to check that a point $p = (x, y)$ belongs to this tangent line L exactly when its components are such that

$$-y'(t_0) \cdot (x - x(t_0)) + x'(t_0) \cdot (y - y(t_0)) = 0,$$

so that this is an implicit equation for that line.

[1] One can define tangent lines in more general situations, but we will not do this here.

Example 1.3.5. Let $f : (a,b) \to \mathbb{R}$ be a smooth function and let $\gamma : t \in (a,b) \mapsto (t, f(t)) \in \mathbb{R}^2$ be the standard parametrization of its graph. If t_0 is a point in (a,b), then $\gamma'(t_0) = (1, f'(t_0))$, and therefore the line tangent to γ at t_0 is

$$L = \{(x(t_0) + u, y(t_0) + uf'(t_0)) : u \in \mathbb{R}$$

and

$$-f'(t_0) \cdot (x - t_0) + (y - f(t_0)) = 0$$

is an implicit equation for it. Notice that we can rewrite this equation in the form

$$y = f'(t_0) \cdot (x - t_0) + f(t_0),$$

which is the familiar equation to the tangent line to the graph of f.

Example 1.3.6. Let U be an open subset of $\mathbb{R}^2$, let $F : U \to \mathbb{R}$ be a smooth function, and let $p = (x_0, y_0) \in U$ be a point such that $F(p) = 0$ and $\nabla F(p) \neq 0$. We know from Proposition 1.2.1 that there exists a regular smooth parametrized curve $\gamma : (-1,1) \to U$ such that $\gamma(0) = p$ and $F(\gamma(t)) = 0$ for all $t \in (-1,1)$. If $x, y : (-1,1) \to \mathbb{R}$ are the components of γ, then this last equation means that $F(x(t), y(t)) = 0$ for all $t \in (-1,1)$ and therefore differentiating with respect to t and taking $t = 0$ we see that

$$F_x(p) \cdot x'(0) + F_y(p) \cdot y'(0) = 0.$$

On the other hand, a point (x, y) of $\mathbb{R}^2$ belongs to the tangent line to the curve γ at 0 exactly when

$$y'(0) \cdot (x - x_0) - x'(0) \cdot (y - y_0) = 0.$$

It follows from this that the point (x, y) is in the tangent line exactly when the system of equations

$$\begin{cases} F_x(p) \cdot X + F_y(p) \cdot Y = 0 \\ -(y - y_0) \cdot X + (x - x_0) \cdot Y = 0 \end{cases}$$

is compatible, and therefore the determinant of its matrix of coefficients is zero,

$$F_x(p) \cdot (x - x_0) + F_y(p) \cdot (y - y_0) = \begin{vmatrix} F_x(p) & F_y(p) \\ -(y - y_0) & (x - x_0) \end{vmatrix} = 0.$$

We can therefore conclude that the tangent line to the parametrized curve γ at 0 has equation

$$F_x(p) \cdot (x - x_0) + F_y(p) \cdot (y - y_0) = 0. \tag{1.2}$$

{eq:imptg}

If we let q be the point (x, y) then we can rewrite this equation in the form

$$\langle \nabla F(p), q - p \rangle = 0,$$

with $\langle -, - \rangle$ the usual inner product of $\mathbb{R}^2$: this tells us that a point q is in the tangent line to γ at 0 exactly when the difference $q - p$ is orthogonal to $\nabla F(p)$.

Notice that the equation (1.3.6) does not depend on knowledge about the curve γ, but only on the function F that defines it implicitly. This is important because, as we observed above, in general we cannot determine γ explicitly.

§1.4. Reparametrizations

A function $u : (c, d) \to (a, b)$ from a non-empty open interval of the line $\mathbb{R}$ to another is a ***change of parameter*** if it is bijective and smooth, and its inverse function is also smooth. When that is the case the inverse function $u^{-1} : (a, b) \to (c, d)$ is also a change of parameter. change of parameter

Example 1.4.1. Let (c, d) and (a, b) be two non-empty open intervals of finite length. The function

$$u : s \in (c, d) \mapsto \frac{b - a}{d - c}(s - c) + a \in (a, b)$$

is a change of parameter.

Example 1.4.2. The function $u : s \in \mathbb{R} \mapsto s^3 \in \mathbb{R}$ is bijective and smooth, but its inverse function, $t \in \mathbb{R} \mapsto t^{1/3} \in \mathbb{R}$, is not smooth: the function u is therefore *not* a change of parameter.

If $u : (c, d) \to (a, b)$ is a change of parameter and $v : (a, b) \to (c, d)$ is its inverse function, we have that $v(u(s)) = s$ for all $s \in (c, d)$, and differentiating we see that also $v'(u(s)) \cdot u'(t) = 1$ for all $s \in (c, d)$: this implies that $u'(s) \neq 0$ for all $s \in (c, d)$ and, since u' is a continuous function, we see that u' is either strictly positive or strictly negative on the whole interval (c, d). In the first case we say that u ***preserves the orientation*** and in the second case that it ***reverses the orientation*** — it should be observed that in the first case the inverse function v also preserves the orientation while in the second one it reverses it.

As we see, having a nowhere-zero derivative is a necessary condition for a smooth bijective map to be a change of parameter. In fact, it is also sufficient:

{lemma:inv}

Lemma 1.4.3. *A smooth bijective function $u : (c, d) \to (a, b)$ is a change of parameter if and only if its derivative u' is nowhere zero on (c, d).*

This is a special case of the *Inverse Function Theorem*. We will prove it, as the general theorem found in calculus textbooks is considerably more complicated to obtain, even though this is really just a result of calculus.

Proof. We have already noted that the condition of the lemma is necessary for u to be a change of parameter, so we will only show here that it is also sufficient.

Let $u : (c, d) \to (a, b)$ be a smooth bijective function with nowhere-zero derivative, and let $v : (a, b) \to (c, d)$ be its inverse function. As u' is nowhere-zero and continuous, it is either everywhere positive or everywhere negative. We will suppose that it is everywhere positive, leaving the consideration of the other possibility to the responsibility of the reader.

As $u'(s) > 0$ for all $s \in (c, d)$, the function u is strictly increasing, and this implies that its inverse v is also strictly increasing. Indeed, let t_1 and t_2 are two elements of (a, b) such that $t_1 < t_2$: if we had that $v(t_1) \geq v(t_2)$, then the fact that u is increasing would tell us that $t_1 = u(v(t_1)) \geq u(v(t_2)) = t_2$, and this is absurd.

Our first objective is to prove that

the function v is continuous.

Let t_0 be an element of (a, b), and let us consider the set $L \coloneqq \{v(t) : t \in (a, t_0)\}$. This set is obviously not empty, and since the function v is increasing the number $v(t_0)$ is a upper bound for L: this implies that we can consider the number $l \coloneqq \sup L$ and that $l \leq v(t_0)$. In fact, we have that $l = v(t_0)$. To do this, let us suppose that, on the contrary, $l < v(t_0)$ and that therefore we can choose a number s such that $c < l < s < v(t_0) < d$. As the function v is surjective, there is then an element t_1 of (a, b) such that $v(t_1) = s$ and, since v is strictly increasing, we have that $a < t_1 < t_0$: this is absurd, since in that case $\sup L = l < s = v(t_1) \in L$. This proves that $l = v(t_0)$, as we wanted.

We claim now that the limit

$$\lim_{t \nearrow t_0} v(t) \tag{1.3}$$

{eq:llim}

exists and is equal to $v(t_0)$. To see this, let ϵ be a positive number. As $v(t_0)$ is the supremum of the set L, there exists an element in L strictly greater than $v(t_0) - \epsilon$ and therefore there is an element t_2 in (a, t_0) such that $v(t_0) - \epsilon < v(t_2)$. Let δ be the number $t_0 - t_2$, which is positive and such that $(t_0 - \delta, t_0) = (t_2, t_0) \subseteq (a, b)$. If t is any element of $(t_0 - \delta, t_0)$, then $t_2 = t_0 - \delta < t < t_0$ and, as v is strictly increasing, also $v(t_0) - \epsilon < v(t_2) < v(t) < v(t_0)$: it follows from this that

$$-\epsilon < v(t) - v(t_0) < 0 < \epsilon,$$

so that $|v(t) - v(t_0)| < \epsilon$. This shows that the limit (1.4) exists and is equal to l, as we claimed.

In a similar way we can show that the limit

$$\lim_{t \searrow t_0} v(t)$$ {eq:rlim}

exists and that its value is $v(t_0)$, so that the function v is continuous at t_0. As this is true for all elements t_0 of the domain (a, b) of v, we can conclude, as we wanted, that the function v is continuous.

Next, we will show that the inverse function v is smooth, so that u is a change of parameter. Let t_0 be an element of (a, b), let us put $s_0 \coloneqq v(t_0)$, and let ϵ be a positive number. The function u is differentiable at s_0, so the limit

$$\lim_{k \to 0} \frac{u(s_0 + k) - u(s_0)}{k}$$

exists and its value is $u'(s_0)$. As this is a non-zero number, this implies that also

$$\lim_{k \to 0} \frac{k}{u(s_0 + k) - u(s_0)} = \frac{1}{u'(s_0)}.$$

In particular, there is a positive number η such that whenever k is a number such that $|k| < \eta$ and $s_0 + k \in (c, d)$ we have

$$\left| \frac{k}{u(s_0 + k) - u(s_0)} - \frac{1}{u'(s_0)} \right| < \epsilon.$$

The function v is continuous at t_0, so there is also a positive number δ such that for all $h \in (-\delta, \delta)$ we have $t_0 + h \in (a, b)$ and $|v(t_0 + h) - v(t_0)| < \eta$.

Let now h be any element of $(-\delta, \delta)$ and let us put $k \coloneqq v(t_0 + h) - v(t_0)$. The way we chose δ implies that $|k| < \eta$, and the way we chose η, in turn, that

$$\left| \frac{k}{u(s_0 + k) - u(s_0)} - \frac{1}{u'(s_0)} \right| < \epsilon. \tag{1.4}$$ {eq:inv:1}

As

$$u(s_0 + k) - u(s_0) = u\big(s_0 + v(t_0 + h) - v(t_0)\big) - u(s_0) = (t_0 + h) - t_0,$$

we have that

$$\frac{k}{u(s_0 + k) - u(s_0)} = \frac{v(t_0 + h) - v(t_0)}{h},$$

and therefore the inequality (1.4) tells us that

$$\left| \frac{v(t_0 + h) - v(t_0)}{h} - \frac{1}{u'(s_0)} \right| < \epsilon.$$

All this shows that the limit

$$\lim_{h\to 0} \frac{v(t_0+h)-v(t_0)}{h}$$

exists and has value $1/u'(s_0)$, which is equal to $1/u'(v(t_0))$.

The conclusion of this is that the function v is differentiable on (a,b), and that for all $t \in (a,b)$ it has derivative

$$v'(t) = \frac{1}{u'(v(t))}. \tag{1.5}$$ {eq:inv:2}

We claim that, in fact, the function v is smooth. As u is smooth, the derivative u' is continuous, and we already saw that v is continuous, so the equality (1.4) tells us that v' is continuous and therefore that v is of class C^1. Suppose now that l is a positive integer and that we know that v is of class C^l: as u is smooth, it follows from the equality (1.4) that v' is of class C^l and this that v is of class C^{l+1}. The smoothness of v thus follows by induction, and the proof of the lemma is complete. □

This lemma gives us an extremely convenient way to check whether a smooth bijection is a change of parameter or not.

Example 1.4.4. The functions $s \in (0,1) \mapsto s^2 \in (0,1)$ and $s \in (-1,0) \mapsto s^2 \in (0,1)$ are changes of parameter: the two functions are bijective and have nowhere-zero derivatives. The first of them preserves the orientation while the second reverses it.

If $\gamma : (a,b) \to \mathbb{R}^2$ is a curve and $u : (c,d) \to (b,d)$ then the composition

$$\eta \coloneqq \gamma \circ u : s \in (c,d) \mapsto \gamma(u(s)) \in \mathbb{R}^2$$

is also a curve, and we say that η is obtained from γ by ***reparametrization*** using the change of parameter u. Since the composition of smooth functions is smooth, it is clear that the curve η is smooth if the curve γ is smooth. On the other hand, if $s_0 \in (c,d)$ we have that reparametrization

$$\eta'(s_0) = \gamma'(u(s_0)) \cdot u'(s_0).$$

It follows from this that if γ is regular at $u(s_0)$ then η' is regular at s_0, because the number $u'(s_0)$ is not zero, and in turn this implies that the curve η is regular when γ is regular. Putting everything together, we have proved the following:

Lemma 1.4.5. *A reparametrization of a regular smooth curve is a regular smooth curve.* □

If $\gamma : (a,b) \to \mathbb{R}^2$ is a curve in the plane and $u : (c,d) \to (a,b)$ is a change of parameter, then the trace of the reparametrization $\eta \coloneqq \gamma \circ \eta : (c,d) \to \mathbb{R}^2$ coincides with that of γ: this is a consequence of the bijectivity of the function u. Indeed, if p is a point in the trace of γ, then there exists an element t of (a, b) such that $p = \gamma(t)$, and since the function u is surjective there is also an element s in (c, d) such that $u(s) = t$: we then have that $p = \gamma(t) = \gamma(u(s)) = \eta(s)$ and therefore that p belongs to the trace of η. Conversely, if q is a point in the trace of η there is an element s of (c, d) such that $q = \eta(s)$, and then $q = \eta(s) = \gamma(u(s))$ is clearly also in the trace of γ.

Exercise 1.4.6. Let us say that that two curves $\gamma : (a, b) \to \mathbb{R}^n$ and $\eta : (c, d) \to \mathbb{R}^n$ are ***equivalent*** if there is a change of parameter $u : (c, d) \to (a, b)$ such that $\eta = \gamma \circ u$ and in that case let us write $\gamma \mathscr{R} \eta$. In this we obtain a relation $\mathscr{R}$ on the set of all curves in $\mathbb{R}^n$. Show that it is an *equivalence* relation on that set.

§1.5. Unit speed curves

We say that a smooth curve $\gamma : (a, b) \to \mathbb{R}^2$ has ***unit speed*** if $\|\gamma'(t)\| = 1$ for all $t \in (a, b)$. As we will see later, working with unit-speed curves is much more convenient than working with arbitrary curves. Of course, not all curves have unit speed, but it is very useful to know that every regular curve has a reparametrization which is unit speed. unit speed

{lemma:unit-reparam}

Lemma 1.5.1. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be a smooth curve in the plane that is regular. There is a change of parameter $u : (c, d) \to (a, b)$ such that the reparametrization $\eta := \gamma \circ u : (c, d) \to \mathbb{R}^2$ has unit speed.*

Proof. Let us fix an element t_0 in (a, b). The function

$$t \in (a, b) \mapsto \|\gamma'(t)\| \in \mathbb{R}$$

is continuous and, in fact, actually smooth, since γ is smooth and $\gamma'(t) \neq 0$ for all $t \in (a, b)$ because we are supposing that γ is regular. In particular, we can consider the function

$$\sigma : t \in (a, b) \mapsto \int_{t_0}^{t} \|\gamma'(\xi)\| \, d\xi,$$

as the integral makes sense for all t in (a, b). This function σ is, according to the *Fundamental Theorem of Calculus*, continuous and differentiable, and its derivative is

$$\sigma'(t) = \|\gamma'(t)\|$$

at all points t of (a, b). Since the curve γ is regular, this tells us that the derivative σ' is strictly positive on (a, b) and therefore that the function σ is strictly increasing on that interval.

We want to check now that the image of σ, the set

$$I \coloneqq \{\sigma(t) : t \in (a,b)\},$$

is a (possibly infinite) interval of $\mathbb{R}$. Since the set I is clearly non-empty, we can put $c \coloneqq \inf I$ and $d \coloneqq \sup I$, keeping in mind that c is either a real number or $-\infty$, and that d is a real number or $+\infty$. It is clear that $c \leq d$ and, since the function σ is not constant, in fact $c < d$. We will show that the image of σ is precisely the interval (c,d).

- Let t be an element of (a,b). There are then elements t_1 and t_2 in (a,b) such that $a < t_1 < t < t_2 < b$, and the fact that σ is strictly increasing implies that

  $$c = \inf I \leq \sigma(t_1) < \sigma(t) < \sigma(t_2) \leq \sup I = d.$$

  We thus see that $\sigma(t) \in (c,d)$, and therefore that the image of σ is contained in (c,d).
- Let now s be an element of (c,d). Since $s > c = \inf I$, there exists an element x of I such that $x < s$, and this means that there is an element t_1 of (a,b) such that $\sigma(t_1) = x < s$. Similarly, since $s < d = \sup I$, there exists a $y \in I$ such that $s < y$, and thus there is an t_2 in (a,b) such that $s < y = \sigma(t_1)$. Now, since $s \in [\sigma(t_1), \sigma(t_2)]$ and the function σ is continuous, the *Intermediate Value Theorem* tells us that there is a $t \in [t_1, t_2]$ such that $\sigma(t) = s$ and, therefore, that s belongs to the image of s.

Now we know that the image of the function σ is the interval (c,d), so we can view σ as a function $(a,b) \to (c,d)$. We know it is injective, because it is strictly increasing, and with its new codomain it is surjective, so it is bijective. Finally, since the derivative of σ is non-zero at all points of (a,b) our Lemma 1.4.3 tells us that $\sigma : (a,b) \to (c,d)$ is a change of parameter. Moreover, if we let $u : (c,d) \to (a,b)$ be the inverse function of σ, then u is also a change of parameter. As $\sigma(u(s)) = s$ for all $s \in (c,d)$, we have that $\sigma'(u(s)) \cdot u'(s) = 1$ for all such s, so that

$$u'(s) = \frac{1}{\sigma'(u(s))} = \frac{1}{\|\gamma'(u(s))\|}. \tag{1.6}$$ {eq:rpu}

Let us now consider the curve $\eta \coloneqq \gamma \circ u : (c,d) \to \mathbb{R}^2$, which is a reparametrization of γ. As we observed above, η is smooth and regular, because γ is. Additionally, if s is an element of (c,d) we have that

$$\|\eta'(s)\| = \|\gamma'(u(s)) \cdot u'(s)\| = \|\gamma'(u(s))\| \cdot \|u'(s)\| = \|\gamma'(u(s))\| \cdot \frac{1}{\|\gamma'(u(s))\|} = 1,$$

because of the equality (1.5), and thus η is a unit speed curve. This proves the lemma. □

Let us give a couple of examples in which we can carry out the procedure that we used to prove this lemma.

Example 1.5.2. Let $p = (x_0, y_0)$ be a point in $\mathbb{R}^2$ and let $v = (\alpha, \beta)$ be a non-zero vector in $\mathbb{R}^2$. The line through p with direction v can be parametrized with the curve

$$\gamma : t \in \mathbb{R} \mapsto p + tv \in \mathbb{R}^2.$$

It has speed $\|\gamma'(t)\| = \|v\|$ for all $t \in \mathbb{R}$: we thus see that γ is a unit-speed curve exactly when the vector v is a unit vector. Let us follow the procedure we used in the proof of the lemma to find a unit-speed reparametrization in the general case. Let us choose $t_0 = 0$. The function $\sigma : \mathbb{R} \to \mathbb{R}$ is given in this situation by

$$\sigma(t) = \int_{t_0}^{t} \|\gamma'(\tau)\| \,\mathrm{d}\tau = \int_0^t \|v\| \,\mathrm{d}\tau = t\,\|v\|$$

for all $t \in \mathbb{R}$. Its image is the whole line $\mathbb{R}$ and its inverse function is $u : s \in \mathbb{R} \mapsto s/\|v\| \in \mathbb{R}$. The unit-speed reparametrization constructed in the proof of the lemma for the curve γ is therefore

$$\eta : s \in \mathbb{R} \mapsto p + s \cdot \frac{v}{\|v\|} \in \mathbb{R}^2.$$

Example 1.5.3. Let R be a positive number. The function

$$\gamma : t \in \mathbb{R} \mapsto (R\cos t, R\sin t) \in \mathbb{R}^2$$

is a parametrization of the circle centered at the origin of radius R. For each $t \in \mathbb{R}$ we have that $\gamma'(t) = (-R\sin t, R\cos t)$ and $\|\gamma'(t)\| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = R$, so it is a unit-speed curve exactly when $R = 1$. In the general case the function σ that we constructed in the proof of Lemma 1.5.1, taking $t_0 = 0$, has

$$\sigma(t) = \int_{t_0}^{t} \|\gamma'(\tau)\| \,\mathrm{d}\tau = \int_0^t R \,\mathrm{d}\tau = Rt$$

for all $t \in \mathbb{R}$. The function inverse to σ is thus $r : t \in \mathbb{R} \mapsto t/R \in \mathbb{R}$ and the unit-speed reparametrization that we obtain is the curve

$$\eta : s \in \mathbb{R} \mapsto \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right) \in \mathbb{R}^2.$$

Very often, though, it is not possible to write down explicitly unit-speed reparametrizations of curves, as the following example shows.

Example 1.5.4. Let a and b be two positive numbers such that $b \geq a$. The function

$$\gamma : t \in \mathbb{R} \mapsto (a\cos t, b\sin t) \in \mathbb{R}^2$$

is a parametrization of the ellipse centered at the origin of semi-axes a and b. When the two semi-axes are equal this curve is a circle and in general we use the number

$$k \coloneqq \sqrt{1 - \frac{a^2}{b^2}},$$

called the ***eccentricity*** of the ellipse, to measure how far the curve is from being a circle; clearly we always have $0 \leq k < 1$, and k is 0 exactly when the curve is a circle. For each $t \in \mathbb{R}$ we have $\gamma'(t) = (-a \sin t, b \cos t)$ and $\|\gamma'(t)\| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$. In particular, $\|\gamma'(t)\|$ is not a constant function if $a \neq b$, so γ is certainly not a unit-speed curve in that case. The function σ that we constructed in the proof of Lemma 1.5.1, when we take $t_0 = 0$, has

$$\begin{aligned}
\sigma(t) &= \int_0^t \|\gamma'(\tau)\| \, d\tau \\
&= \int_0^t \sqrt{a^2 \sin^2 \tau + b^2 \cos^2 \tau} \, d\tau = \int_0^t \sqrt{a^2 \sin^2 \tau + b^2 (1 - \sin^2 \tau)} \, d\tau \\
&= \int_0^t \sqrt{b^2 - (b^2 - a^2) \sin^2 \tau} \, d\tau = b \int_0^t \sqrt{1 - k^2 \sin^2 \tau} \, d\tau.
\end{aligned}$$

This last integral is called a ***incomplete elliptic integral of the second kind*** and cannot be expressed in terms of elementary functions. The function inverse to σ is also of the same nature, so in practice we are not able to work with the unit-speed reparametrization of this curve.

The Lemma 1.5.1 tells us that we can reparametrize any regular curve so that the result is a unit-speed curve, and we can always do this in many ways. Indeed, if $\gamma : (a, b) \to \mathbb{R}^2$ is a curve and $u : (c, d) \to (a, b)$ is a change of parameter such that the curve $\gamma \circ u$ is a unit-speed curve, then for every real number z the function $v : s \in (c + z, d + z) \mapsto u(s - z) \in (a, b)$ is also a change of parameter such that the curve $\gamma \circ v$ has unit speed, as the reader can immediately check. The following lemma shows that, in fact, *all* unit-speed reparametrizations of γ that preserve the orientation can be obtained from u in this way.

Lemma 1.5.5. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular smooth curve in the plane. If $u_1 : (c_1, d_1) \to (a, b)$ and $u_2 : (c_2, d_2) \to (a, b)$ are changes of parameter that preserve orientations such that the curves $\eta_1 \coloneqq \gamma \circ u_1 : (c_1, d_1) \to \mathbb{R}^2$ and $\eta_2 \coloneqq \gamma \circ u_2 : (c_2, d_2) \to \mathbb{R}^2$ both have unit speed, then there exist a number $\delta \in \mathbb{R}$ such that* {lemma:two-params:1}

$$c_2 = c_1 + \delta, \qquad d_2 = d_1 + \delta, \qquad u_1(s) = u_2(s + \delta) \text{ for all } s \in (c_1, d_1). \tag{1.7}$$ {eq:rep:1}

If $c_1 = -\infty$ we have to interpret the first equality in (1.5.6) as saying that also $c_2 = -\infty$ and, similarly, if $d_1 = +\infty$ then the second equality there means that also $d_2 = +\infty$.

Proof. Let $u_1 : (c_1, d_1) \to (a, b)$ and $u_2 : (c_2, d_2) \to (a, b)$ be changes of parameter that preserve the orientation and such that the curves $\eta_1 \coloneqq \gamma \circ u_1 : (c_1, d_1) \to \mathbb{R}^2$ and $\eta_2 \coloneqq \gamma \circ u_2 : (c_2, d_2) \to \mathbb{R}^2$ both have unit speed. The function $v \coloneqq u_2^{-1} \circ u_1 : (c_1, d_1) \to (c_2, u_2)$ is also a change of parameter and we have that $\eta_1 = \eta_2 \circ v$. If $s \in (c_1, d_1)$, then

$$1 = |\eta_1'(s)| = |\eta_2'(v(s)) \cdot v'(s)| = |\eta_2'(v(s))| \cdot |v'(s)| = |v'(s)| \tag{1.8}$$ {eq:spt}

because both η_1 and η_2 are unit-speed curves. Since the functions u_1 and u_2 are strictly increasing, so is v, and therefore its derivative is non-negative: we can then conclude from (1.5) that $v'(s) = 1$ for all $s \in (c_1, d_1)$. We see that there is a number δ such that $v(s) = s + \delta$ for all $s \in (c_1, d_1)$. Since v is a bijection $(c_1, d_1) \to (c_2, d_2)$, clearly we have that $c_1 + \delta = c_2$ and that $d_1 + \delta = d_2$. Moreover, since $u_2^{-1}(u_1(s)) = v(s) = s + \delta$ for all $s \in (c_1, d_1)$, also $u_1(s) = u_2(s + \delta)$ for all such s. This proves the lemma. □

Lemma 1.5.5 describes the relationship there is between any two unit-speed reparametrizations of a curve when the two preserve orientations. The following exercise describes what happens in the more general situation in which this last condition is not satisfied.

Exercise 1.5.6. Let $\gamma : (a, b) \to \mathbb{R}^2$ be a smooth parametrized curve in the plane that is regular. If $u_1 : (c_1, d_1) \to (a, b)$ and $u_2 : (c_2, d_2) \to (a, b)$ are changes of parameter such that the curves $\eta_1 \coloneqq \gamma \circ u_1 : (c_1, d_1) \to \mathbb{R}^2$ and $\eta_2 \coloneqq \gamma \circ u_2 : (c_2, d_2) \to \mathbb{R}^2$ both have unit speed, then there exist numbers $\delta \in \mathbb{R}$ and $\epsilon \in \{1, -1\}$ such that

$$c_2 = c_1 + \delta, \qquad d_2 = d_1 + \delta, \qquad u_1(s) = u_2(\epsilon s + \delta) \text{ for all } s \in (c_1, d_1), \tag{1.9}$$ {eq:rep:1}

and $|d_1 - c_1|$ and $|d_2 - c_2|$ both coincide with the length of γ.

§1.6. The length of curves

Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular smooth curve in the plane and let t_0 be an element of (a, b). During the proof of Lemma 1.5.1 we showed that there is a smooth function $\sigma : (a, b) \to \mathbb{R}$ that on each $t \in (a, b)$ takes the value

$$\sigma(t) = \int_{t_0}^{t} \|\gamma'(\tau)\| \, d\tau.$$

We call σ the ***arc-length function*** of γ. If t_1 and t_2 are two elements of (a,b) such that $t_1 < t_2$, then we have that

$$\int_{t_1}^{t_2} \|\gamma'(\tau)\| \,\mathrm{d}\tau = \sigma(t_2) - \sigma(t_1).$$

The value of this integral is the ***length*** of the segment of γ determined by the interval $[t_1, t_2]$. In particular, if the curve γ is a unit-speed curve, then $\|\gamma'(t)\| = 1$ for all $t \in (a,b)$, and then the length of the segment of γ determined by the interval $[t_1, t_2]$ is simply

$$\int_{t_1}^{t_2} \|\gamma'(\tau)\| \,\mathrm{d}\tau = \int_{t_1}^{t_2} \mathrm{d}\tau = t_2 - t_1.$$

A key fact is that the length of segments of a curve is invariant under reparametrizations of the curve, in the following precise sense:

Lemma 1.6.1. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular smooth curve, and let t_1 and t_2 be two elements of (a,b) such that $t_1 < t_2$. If $u : (c,d) \to (a,b)$ is a change of parameter that preserves the orientation, $\eta := \gamma \circ u : (c,d) \to \mathbb{R}^2$ is the reparametrization of γ corresponding to u, and s_1 and s_2 are the elements of (c,d) such that $u(s_1) = t_1$ and $u(s_2) = t_1$, then*

$$\int_{t_1}^{t_2} \|\gamma'(\tau)\| \,\mathrm{d}\tau = \int_{s_1}^{s_2} \|\eta'(\xi)\| \,\mathrm{d}\xi.$$

In words, this tells us that the length of the segment of γ determined by the interval $[t_1, t_2]$ coincides with the length of the segment of η determined by the interval $[s_1, s_2]$.

Proof. Let $u : (c,d) \to (a,b)$ be a change of parameter that preserves the orientation, let $\eta := \gamma \circ u$, and let s_1 and s_2 be the elements of (c,d) such that $u(s_1) = t_1$ and $u(s_2) = t_1$. We have that $\eta'(s) = \gamma'(u(s)) \cdot u'(s)$ for all $s \in (c,d)$, so that $\|\eta'(s)\| = \|\gamma'(u(s))\| \cdot u'(s)$, since the function u' is positive on (c,d). Changing variables in the integral, then, we see that

$$\int_{s_1}^{s_2} \|\eta'(\xi)\| \,\mathrm{d}\xi = \int_{s_1}^{s_2} \|\gamma'(u(\xi))\| \cdot u'(\xi) \,\mathrm{d}\xi = \int_{t_1}^{t_2} \|\gamma'(\tau)\| \,\mathrm{d}\tau,$$

and this is the claim of the lemma. □

A useful consequence of Lemma 1.5.5 is the following:

Corollary 1.6.2. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve in the plane. There is an element L of the set $(0, +\infty]$ such that whenever $u : (c,d) \to (a,b)$ is a change of parameter that preserves orientation such that the curve $\gamma \circ u : (c,d) \to \mathbb{R}^2$ has unit speed we have $d - c = L$.*

We call L the ***length*** of the parametrized curve γ. When computing the difference $d - c$ that appears in this statement we follow the usual conventions for dealing with infinities: if d is $+\infty$ or c is $-\infty$, then the value of the difference is $+\infty$.

Proof. If $u_1 : (c_1, d_1) \to (a, b)$ and $u_2 : (c_2, d_2) \to (a, b)$ are changes of parameter that preserve orientations such that the curves $\eta_1 \coloneqq \gamma \circ u_1 : (c_1, d_1) \to \mathbb{R}^2$ and $\eta_2 \coloneqq \gamma \circ u_2 : (c_2, d_2) \to \mathbb{R}^2$ both have unit speed, the lemma tells us that there is a number δ such that $c_2 = c_1 + \delta$ and $d_2 = d_1 + \delta$, and therefore $d_1 - c_1 = d_2 - c_2$. The claim of the corollary follows immediately from this. □

The reader should notice the difference between the definition of the length of a segment of a curve $\gamma : (a, b) \to \mathbb{R}^2$ and the length of the whole curve. This is due to the fact that under our current definitions we insist that curves be defined on open intervals. The two notions are connected, though — this is the point of the following exercise.

Exercise 1.6.3. Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular curve. Show that the length of γ coincides with the value of the possibly *improper* integral

$$\int_a^b \|\gamma'(\tau)\| \, d\tau.$$

This integral can be improper because either the interval (a, b) is not bounded or because the function $t \in (a, b) \mapsto \|\gamma'(\tau)\| \in \mathbb{R}$ is not bounded — in any case, the integral always converges, because its integrand is a positive function.

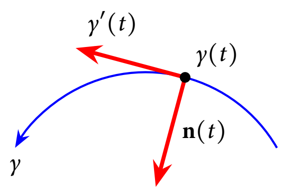
§1.7. The curvature of plane curves

Let $\gamma : (a, b) \to \mathbb{R}^2$ be a unit-speed curve in the plane. At each $t \in (a, b)$ the vector $\gamma'(t)$ determines the direction of tangent line to γ at the point $\gamma(t)$, and therefore the vector $\gamma''(t)$ gives us information about how the direction of that line changes as we move along the curve. We will define an invariant of the curve, its curvature, that measures this in a very concrete way.

Let $x, y : (a, b) \to \mathbb{R}$ be the components of γ, so that $\gamma(t) = (x(t), y(t))$ for all $t \in (a, b)$. Of course, then $\gamma'(t) = (x'(t), y'(t))$ for all $t \in (a, b)$ and, since the curve has unit speed, this is a vector of norm 1. For each $t \in (a, b)$ we consider the vector

$$\mathbf{n}(t) \coloneqq (-y'(t), x'(t))$$

that is obtained from $\gamma'(t)$ by rotating it 90° in the positive direction.



The function $\mathbf{n} : (a, b) \to \mathbb{R}^2$ that we obtain in this way is the ***normal field*** of γ.

The vectors $\gamma'(t)$ and $\mathbf{n}(t)$ are mutually orthogonal, so that $(\gamma'(t), \mathbf{n}(t))$ is a positive[2] orthonormal basis of $\mathbb{R}^2$ and, in particular, $\mathbf{n}(t)$ spans the subspace of $\mathbb{R}^2$ orthogonal to $\gamma'(t)$. For all $t \in (a, b)$ we have that

$$1 = \langle \gamma'(t), \gamma'(t) \rangle$$

because γ is a unit-speed curve, and therefore we have that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma'(t), \gamma'(t) \rangle = 2 \langle \gamma''(t), \gamma'(t) \rangle.$$

This tells us that the vector $\gamma''(t)$ is also orthogonal to $\gamma'(t)$: there exists, therefore, a unique scalar $\kappa(t) \in \mathbb{R}$ such that

$$\gamma''(t) = \kappa(t) \cdot \mathbf{n}(t).$$

We obtain in this way a function $\kappa : (a, b) \to \mathbb{R}$, uniquely determined by γ, which we call the ***signed curvature*** of γ. Since $\mathbf{n}(t)$ is a unit vector, for all $t \in (a, b)$ we have that signed curvature

$$\langle \gamma''(t), \mathbf{n}(t) \rangle = \langle \kappa(t) \cdot \mathbf{n}(t), \mathbf{n}(t) \rangle = \kappa(t) \cdot \langle \mathbf{n}(t), \mathbf{n}(t) \rangle = \kappa(t).$$

As both γ'' and $\mathbf{n}$ are smooth functions, this implies that the signed curvature κ is also a smooth function on the interval (a, b). Let us record the above formula for the curvature as a lemma.

Lemma 1.7.1. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be a unit-speed plane curve and let $\mathbf{n} : (a, b) \to \mathbb{R}^2$ be the corresponding normal field. For each $t \in (a, b)$ the curvature of γ at t is*

$$\kappa(t) = \langle \gamma''(t), \mathbf{n}(t) \rangle. \qquad \square$$

The intention of the definition of the signed curvature of a curve is that it gives us information about how the curve curves. Let us see a couple of examples of this.

Example 1.7.2. Let $p = (x_0, y_0) \in \mathbb{R}^2$ be a point, let $v = (\alpha, \beta) \in \mathbb{R}^2$ be a unit vector, and consider the curve

$$\gamma : t \in \mathbb{R} \mapsto p + tv \in \mathbb{R}^2,$$

which is a unit-speed parametrization of the line through p that has direction v. Of course, we have that $\gamma(t) = (x_0 + t\alpha, y_0 + t\beta)$ for all $t \in \mathbb{R}$, so

$$\gamma'(t) = (\alpha, \beta), \qquad \mathbf{n}(t) = (-\beta, \alpha)$$

[2] An orthonormal basis (v, w) of $\mathbb{R}^2$ is ***positive*** if the 2×2 matrix that has v and w as columns, in that order, has positive determinant.

for all $t \in \mathbb{R}$. Since $\gamma''(t) = 0$ for all $t \in \mathbb{R}$ we have that

$$\kappa(t) = \langle \gamma''(t), \mathbf{n}(t) \rangle = \langle 0, \mathbf{n}(t) \rangle = 0.$$

We thus see that the curvature of our line is identically zero. This does make sense, as a line clearly does not curve at all.

Example 1.7.3. Let now r be a positive real number and let us consider the curve

$$\gamma : t \in \mathbb{R} \mapsto \left(R \cos \frac{t}{R}, R \sin \frac{t}{R} \right) \in \mathbb{R}^2,$$

which is a unit-speed parametrization of the circle of radius R centered at the origin. For all $t \in \mathbb{R}$ we have that

$$\gamma'(t) = \left(-\sin \frac{t}{R}, \cos \frac{t}{R} \right), \qquad \mathbf{n}(t) = \left(-\cos \frac{t}{R}, -\sin \frac{t}{R} \right),$$

and

$$\gamma''(t) = \left(-\frac{1}{R} \cos \frac{t}{R}, -\frac{1}{R} \sin \frac{t}{R} \right),$$

so that

$$\kappa(t) = \langle \gamma''(t), \mathbf{n}(t) \rangle = \frac{1}{R} \cos^2 \frac{t}{R} + \frac{1}{R} \sin^2 \frac{t}{R} = \frac{1}{R}.$$

The curvature of the circle is therefore constant — this makes sense, since the circle looks exactly the same at all its points — and takes the value $1/R$. If the circle has a small radius, then the curvature is large, and conversely: this is consistent with the idea that the curvature measures how fast the curve changes direction.

The signed curvature of a curve is, by its very definition, a real number that can be positive, zero or negative. We will now describe a simple interpretation of its sign. Suppose that $\gamma : (a,b) \to \mathbb{R}^2$ is a unit-speed plane curve, let $x, y : (a,b) \to \mathbb{R}$ be its components, let $\mathbf{n} : (a,b) \to \mathbb{R}^2$ be the corresponding normal field, and let us fix a point t_0 in (a,b). As we know, the tangent line $L(t_0)$ to γ at t_0 has equation

$$-y'(t_0) \cdot (x - x(t_0)) + x'(t_0) \cdot (y - y(t_0)) = 0.$$

This line determines two closed half-planes: the ***left closed half-plane*** $H_+(t_0)$, which is the one that contains the point $\gamma(t_0) + \mathbf{n}(t_0)$, and the ***right closed half-plane*** $H_-(t_0)$, which is the other one. We have

$$H_+(t_0) \coloneqq \{(x,y) \in \mathbb{R}^2 : -y'(t_0) \cdot (x - x(t_0)) + x'(t_0) \cdot (y - y(t_0)) \geq 0\}$$

and

$$H_-(t_0) \coloneqq \{(x,y) \in \mathbb{R}^2 : -y'(t_0)\cdot(x - x(t_0)) + x'(t_0)\cdot(y - y(t_0)) \le 0\}.$$

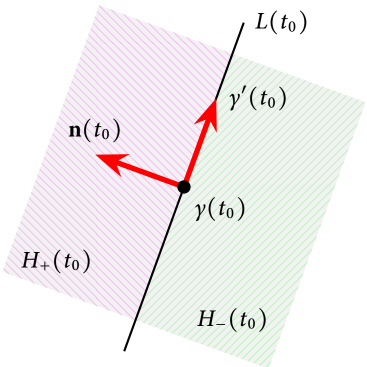
We also consider the corresponding *open* half-planes,

$$H_+^0(t_0) \coloneqq \{(x,y) \in \mathbb{R}^2 : -y'(t_0)\cdot(x - x(t_0)) + x'(t_0)\cdot(y - y(t_0)) > 0\}$$

and

$$H_-^0(t_0) \coloneqq \{(x,y) \in \mathbb{R}^2 : -y'(t_0)\cdot(x - x(t_0)) + x'(t_0)\cdot(y - y(t_0)) < 0\}.$$

The following picture hopefully helps in visualizing these two half-planes.



The sign of the signed curvature tells us the position of the curve with respect to these half-planes, at least when it is non-zero:

Proposition 1.7.4. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a unit-speed curve in the plane and let t_0 be a point in (a,b).*

(i) If $\kappa(t_0) > 0$, then there exists a positive number ϵ such that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq (a,b)$ and $\gamma(t) \in H_+^0(t_0)$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon) \setminus \{t_0\}$.

(ii) If instead $\kappa(t_0) < 0$, then there exists a positive number ϵ such that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq (a,b)$ and $\gamma(t) \in H_-^0(t_0)$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon) \setminus \{t_0\}$.

This proposition tells us that if the curvature of γ at a point t_0 is positive, then a little segment of the curve around t_0 is completely contained in the left open half-plane $H_+(t_0)$, for example. We have illustrated the claim of this proposition in Figure 1.3 on page 28.

Proof. Let $x, y : (a,b) \to \mathbb{R}$ be the components of γ and let $\mathbf{n} : (a,b) \to \mathbb{R}^2$ be the normal field of γ. We consider the function

$$h : p \in \mathbb{R}^2 \mapsto \langle \mathbf{n}(t_0), p - \gamma(t_0)\rangle \in \mathbb{R},$$

which is such that $H_+(t_0) = \{p \in \mathbb{R}^2 : h(p) \ge 0\}$ and $H_-(t_0) = \{p \in \mathbb{R}^2 : h(p) \le 0\}$. The derivative of the composition $h \circ \gamma : (a, b) \to \mathbb{R}$ is

$$(h \circ \gamma)'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{n}(t_0), \gamma(t) - \gamma(t_0)\rangle = \langle \mathbf{n}(t_0), \gamma'(t)\rangle$$

and, in particular, we have that

$$(h \circ \gamma)'(t_0) = \langle \mathbf{n}(t_0), \gamma'(t_0)\rangle = 0.$$

On the other hand, the second derivative of that composition is

$$(h \circ \gamma)''(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle \mathbf{n}(t_0), \gamma(t) - \gamma(t_0)\rangle = \langle \mathbf{n}(t_0), \gamma''(t)\rangle$$

and its value at t_0 is thus

$$(h \circ \gamma)''(t_0) = \langle \mathbf{n}(t_0), \gamma''(t_0)\rangle = \kappa(t_0).$$

Suppose now that γ has positive curvature at t_0, that is, that $\kappa(t_0) > 0$. In that case we have that $h \circ \gamma$ has $(h \circ \gamma)'(t_0) = 0$ and $(h \circ \gamma)''(t_0) > 0$, and we know from calculus that it follows from this that $h \circ \gamma$ has a strict local minimum at t_0. This means that there exists a positive number ϵ such that $(t_0 - \epsilon, t_0 + \epsilon)$ is contained in (a, b) and

$$h(\gamma(t)) = (h \circ \gamma)(t) > (h \circ \gamma)(t_0) = 0 \qquad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon) \setminus \{t_0\},$$

and this tells us that $\gamma(t)$ is in the left open half-plane $H^0_+(t_0)$ for all such t.

Suppose next that the curve γ has negative curvature at t_0. The composition $h \circ \gamma$ then has $(h \circ \gamma)'(t_0) = 0$ and $(h \circ \gamma)''(t_0) < 0$, and this tells us that $h \circ \gamma$ has a strict local maximum at t_0: there exists a positive number ϵ such that $(t_0 - \epsilon, t_0 + \epsilon)$ is contained in (a, b) and

$$h(\gamma(t)) = (h \circ \gamma)(t) < (h \circ \gamma)(t_0) = 0 \qquad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon) \setminus \{t_0\},$$

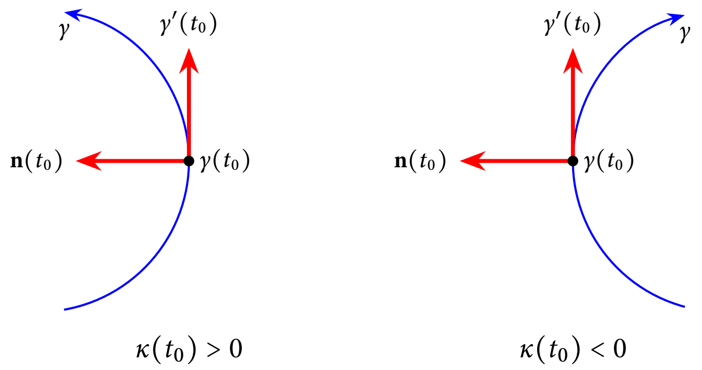
so that $\gamma(t)$ is in the right open half-plane $H^0_+(t_0)$ for all such t. This proves the proposition. □

At a point where the curvature vanishes it can happen that the conclusion of either of the two parts of the proposition holds, or that neither of them holds, so we cannot say anything useful in this respect.

Example 1.7.5. Let us consider the curve

$$\gamma : t \in \mathbb{R} \mapsto (t, t^3) \in \mathbb{R}^2$$

at the point 0. The tangent line L there is clearly the horizontal axis, and there is no open interval in $\mathbb{R}$ containing 0 that is mapped by γ into one of the two half-planes into which L divides the



{fig:sign}

Figure 1.3. A geometric interpretation for the sign of the curvature of a plane curve.

plane. According to the proposition we have just proved, the curvature of γ at 0 must therefore be 0 — notice that this is not a unit-speed curve, so to compute its curvature we need Proposition 1.7.6 below that we are now going to prove.

Our definition of signed curvature applies to unit-speed curves. In principle, if we have any regular curve we can find one of its unit speed reparametrizations and use that to compute its curvature, but this is not really useful in practice: finding that reparametrization can be unfeasible. Having a way to compute curvatures for arbitrary curves is therefore be useful, and that is what the following proposition provides.

Proposition 1.7.6. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular curve in the plane, and let $x, y : (a, b) \to \mathbb{R}^2$ be its components. For all $t \in (a, b)$, the signed curvature of γ at t is* {prop:kappa:nus}

$$\kappa(t) = \frac{\begin{vmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{vmatrix}}{|\gamma'(t)|^3}.$$

The precise meaning of this statement is the following: if $u : (c, d) \to (a, b)$ is a change of parameter such that $\eta := \gamma \circ u$ is a unit speed curve and $s \in (c, d)$ is the point such that $u(s) = t$, then the curvature of η at the point s is given by the formula appearing on the right of the equality in the proposition.

Proof. Let $u : (c, d) \to (a, b)$ be an increasing change of parameter such that the composition $\eta := \gamma \circ u : (c, d) \to \mathbb{R}^2$ is a unit-speed curve, let t_0 be a point in (a, b) and let s_0 be the unique point in (c,d) such that $u(s_0)=t_0$. We have

$$\eta'(s_0)=\gamma'(u(s_0))\cdot u'(s_0)=\gamma'(t_0)\cdot u'(s_0),$$
$$\eta''(s_0)=\gamma''(u(s_0))\cdot u'(s_0)^2+\gamma'(u(s_0))\cdot u''(s_0)=\gamma''(t_0)\cdot u'(s_0)^2+\gamma'(t_0)\cdot u''(s_0)$$

and the unit normal vector to η at s_0 is thus

$$\mathbf{n}(s_0)=(-y'(t_0)u'(s_0),x(t_0)u'(s_0)).$$

The curvature of η at u_0 is

$$\begin{aligned}\langle\eta''(s_0),\mathbf{n}(s_0)\rangle&=\left\langle\gamma''(t_0)\cdot u'(s_0)^2+\gamma'(t_0)\cdot u''(s_0),\mathbf{n}(s_0)\right\rangle\\&=\left\langle\gamma''(t_0)\cdot u'(t_0)^2,\mathbf{n}(u_0)\right\rangle\end{aligned}$$

because the vector $\mathbf{n}(s_0)$ is orthogonal to $\gamma'(t_0)$, and this is

$$\begin{aligned}&=-x''(t_0)y'(t_0)u'(t_0)^3+y''(t_0)x'(t_0)u'(t_0)^3\\&=u'(t_0)^3\cdot\begin{vmatrix}x'(t_0)&x''(t_0)\\y'(t_0)&y''(t_0)\end{vmatrix}.\end{aligned}\tag{1.10}$$ {eq:cnu}

Finally, since η is a unit-speed curve, we have that

$$1=\|\eta'(u_0)\|=\|\gamma'(r(u_0))\cdot r'(u_0)\|=\|\gamma'(t_0)\|\cdot|u'(u_0)|$$

Since the function u is increasing, we have that $u'(s_0)>0$ and this equality tells us that $u'(s_0)=\|\gamma'(t_0)\|^{-1}$. Using this in (1.7) gives us the formula that appears in the proposition. □

Using this proposition we can, for example, compute the curvature of an ellipse.

Example 1.7.7. Let a and b be two positive real numbers. The curve

$$\gamma:t\in\mathbb{R}\mapsto(a\cos t,b\sin t)\in\mathbb{R}^2$$

traces an ellipse centered at the origin whose semi-axes are a and b. For all $t\in\mathbb{R}$ we have that

$$\gamma'(t)=(-a\sin t,b\cos t)$$

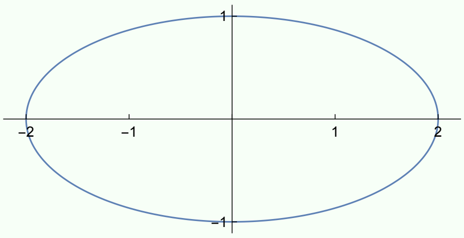
so that $\|\gamma'(t)\|=\sqrt{a^2\sin^2t+b^2\cos^2t}$. We thus see that this is not a unit-speed curve in general. We also have that

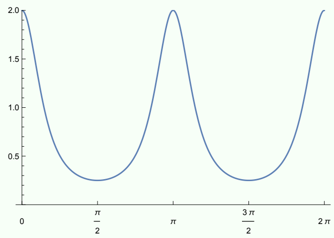
$$\gamma''(t)=(-a\cos t,-b\sin t)$$

so according to the proposition the signed curvature function of γ is

$$\kappa(t) = \frac{\begin{vmatrix} -a\sin t & -a\cos t \\ b\cos t & -b\sin t \end{vmatrix}}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

For example, if we take $a = 2$ and $b = 1$, then the trace of our curve and the graph of its curvature are as follows:





An important observation that we can make about the formula for the curvature of a not necessarily unit-speed curve provided by Proposition 1.7.6 is that it shows that curvature is a *geometric* invariant of curves in the precise sense that it does not depend on the way the curve is parametrizes. The following exercise makes this precise.

Exercise 1.7.8. Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve in the plane, let $x, y : (a,b) \to \mathbb{R}^2$ be its components. Let us consider a change of parameter $u : (c,d) \to (a,b)$, and write $\bar{x} := x \circ u$, $\bar{y} := y \circ u : (c,d) \to \mathbb{R}$ for the components of the reparametrization $\eta := \gamma \circ u : (c,d) \to \mathbb{R}^2$ of γ. Show that if s is an element of (c,d) and $t := u(s)$, then we have that

$$\frac{\begin{vmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{vmatrix}}{|\gamma'(t)|^3} = \frac{\begin{vmatrix} \bar{x}'(s) & \bar{x}''(s) \\ \bar{y}'(s) & \bar{y}''(s) \end{vmatrix}}{|\eta'(s)|^3}.$$

§1.8. Angle functions and the curvature

There is a useful interpretation of the signed curvature of a curve in terms of angles that we will now describe. Let us suppose that $\gamma : (a,b) \to \mathbb{R}^2$ is a regular curve. If t is a point in (a,b), then the tangent vector $\mathbf{t}(t)$ is a unit vector and there exists a real number $\theta(t)$ such that $\mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$. In fact, there are many choices for that number, since for all $k \in \mathbb{Z}$

we have that

$$(\cos\theta(t), \sin\theta(t)) = (\cos(\theta(t) + 2k\pi), \sin(\theta(t) + 2k\pi))$$

In any case, we see that there exist functions $\theta : (a, b) \to \mathbb{R}$ such that $\mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in (a, b)$, and that, in fact, there exist *many* such functions. The following proposition asserts that we can even find such a function that is smooth.

Proposition 1.8.1. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular curve in the plane.* {prop:angle-function}

(*i*) *There is a smooth function $\theta : (a, b) \to \mathbb{R}$ such that*

$$\mathbf{t}(t) = (\cos\theta(t), \sin\theta(t)) \tag{1.11}$$ {eq:theta}

for all $t \in (a, b)$.

(*ii*) *If $\theta, \bar\theta : (a, b) \to \mathbb{R}$ are two continuous functions such that $\mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$ and $\mathbf{t}(t) = (\cos\bar\theta(t), \sin\bar\theta(t))$ for all $t \in (a, b)$, then there is an integer $k \in \mathbb{Z}$ such that*

$$\theta(t) - \bar\theta(t) = 2k\pi$$

for all $t \in (a, b)$.

We call a function θ as in the part (*i*) of this proposition an ***angle function*** for the curve γ. angle function

Proof. (*i*) Let us suppose first that γ is a unit-speed curve, let t_0 be an arbitrary point in (a, b), and let $\theta_0 \in \mathbb{R}$ be such that

$$\gamma'(t_0) = (\cos\theta_0, \sin\theta_0). \tag{1.12}$$ {eq:theta:a}

Notice that there is such a θ_0 because $\gamma'(t_0)$ is a unit vector. Let x, $y : (a, b) \to \mathbb{R}$ be the components of γ. Since γ is a unit-speed curve, we have that $x'(t)^2 + y'(t)^2 = 1$ for all $t \in (a, b)$, and therefore that

$$x'(t) \cdot x''(t) + y'(t) \cdot y''(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(x'(t)^2 + y'(t)^2\Big) = 0 \tag{1.13}$$ {eq:theta:o}

for all such t. We will use this relation later.

Let us now consider the function $\theta : (a, b) \to \mathbb{R}$ that on each $t \in (a, b)$ takes the value

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \Big(x'(\tau) \cdot y''(\tau) - x''(\tau) \cdot y'(\tau)\Big)\,\mathrm{d}\tau.$$

This makes sense: the integrand

$$\tau \in (a, b) \mapsto x'(\tau) \cdot y''(\tau) - x''(\tau) \cdot y'(\tau) \in \mathbb{R}$$

is a smooth function because x and y are smooth functions on (a, b), so the integral exists for each $t \in (a, b)$.

According to the *Fundamental Theorem of Calculus*, the function θ is differentiable and its derivative at each $t \in (a, b)$ is

$$\theta'(t) = x'(t) \cdot y''(t) - x''(t) \cdot y'(t). \tag{1.14}$$ {eq:theta:1}

In particular, this shows that the function θ is smooth, since x and y are smooth. On the other hand, it is clear that $\theta(t_0) = \theta_0$. We want to check next that the equality ((*i*)) holds.

Let us consider the two functions $X, Y : (a, b) \to \mathbb{R}$ that for all $t \in (a, b)$ have

$$\begin{aligned} X(t) &\coloneqq x'(t) \cdot \cos\theta(t) + y'(t) \cdot \sin\theta(t), \\ Y(t) &\coloneqq x'(t) \cdot \sin\theta(t) - y'(t) \cdot \cos\theta(t). \end{aligned}$$

We have that

$$\begin{aligned} &X'(t) \\ &\quad = x''(t) \cdot \cos\theta(t) - x'(t) \cdot \theta'(t) \cdot \sin\theta(t) + y''(t) \cdot \sin\theta(t) + y'(t) \cdot \theta'(t) \cdot \cos\theta(t) \\ &\quad = \underbrace{\big(x''(t) + y'(t) \cdot \theta'(t)\big)} \cdot \cos\theta(t) + \underbrace{\big(y''(t) - x'(t) \cdot \theta'(t)\big)} \cdot \sin\theta(t) \end{aligned}$$

and that

$$\begin{aligned} &Y'(t) \\ &\quad = x''(t) \cdot \sin\theta(t) + x'(t) \cdot \theta'(t) \cdot \cos\theta(t) - y''(t) \cdot \cos\theta(t) + y'(t) \cdot \theta'(t) \sin\theta(t) \\ &\quad = -\underbrace{\big(y''(t) - x'(t) \cdot \theta'(t)\big)} \cdot \cos\theta(t) + \underbrace{\big(x''(t) + y'(t) \cdot \theta'(t)\big)} \cdot \sin\theta(t). \end{aligned}$$

The expressions we marked with braces vanish. Indeed, let t be an element of (a, b). Using the expression (1.8) for $\theta'(t)$ we see that

$$\begin{aligned} x''(t) + y'(t) \cdot \theta'(t) &= x''(t) + y'(t) \cdot \big(x'(t) \cdot y''(t) - x''(t) \cdot y'(t)\big) \\ &= x''(t) \cdot (1 - y'(t)^2) + x'(t) \cdot y'(t) \cdot y''(t) \end{aligned}$$

and, since $x'(t)^2 + y'(t)^2 = 1$, this is

$$\begin{aligned} &= x''(t) \cdot x'(t)^2 + x'(t) \cdot y'(t) \cdot y''(t) \\ &= \big(x'(t) \cdot x''(t) + y'(t) \cdot y''(t)\big) \cdot x'(t) \\ &= 0, \end{aligned}$$ {eq:theta:2:1}

because of the equation (1.8). Similarly, using the expression (1.8) and later (1.8) again we see that

$$\begin{aligned} y''(t) - x'(t) \cdot \theta'(t) &= y''(t) - x'(t) \cdot \big(x'(t) \cdot y''(t) - x''(t) \cdot y'(t)\big) \\ &= y''(t) \cdot (1 - x'(t)^2) + y'(t) \cdot x'(t) \cdot x''(t) \\ &= y''(t) \cdot y'(t)^2 + y'(t) \cdot x'(t) \cdot x''(t) \\ &= \big(y''(t) \cdot y'(t) + x'(t) \cdot x''(t)\big) \cdot y'(t) \\ &= 0. \end{aligned}$$ {eq:theta:2:2}

It follows from this that $X'(t) = 0$ and $Y'(t) = 0$ for all $t \in (a, b)$, so that the functions X and Y are constant. Moreover, (1.8) implies that

$$X(t_0) = x'(t_0) \cdot \cos\theta(t_0) + y'(t_0) \cdot \sin\theta(t_0) = x'(t_0) \cdot x'(t_0) + y'(t_0) \cdot y'(t_0) = 1$$

and that

$$Y(t_0) = x'(t_0) \cdot \sin\theta(t_0) - y'(t_0) \cdot \cos\theta(t_0) = x'(t_0) \cdot \sin\theta_0 - y'(t_0) \cdot \cos\theta_0 = 0,$$

so that functions X and Y have values 0 and 1, respectively. For all $t \in (a, b)$ we then have that

$$x'(t) \cdot \cos\theta(t) + y'(t) \cdot \sin\theta(t) = 1, \qquad x'(t) \cdot \sin\theta(t) - y'(t) \cdot \cos\theta(t) = 0,$$

and these two equalities imply that for all such t we also have

$$x'(t) = \cos\theta(t), \qquad y'(t) = \sin\theta(t).$$

This proves that θ is a smooth angle function for the curve γ, as we wanted.

All this we did assuming that the curve $\gamma : (a, b) \to \mathbb{R}^2$ is a unit-speed curve. Let us now remove that extra hypothesis. As we know, there is, in any case, a change of parameters $u : (c, d) \to (a, b)$ such that $\eta \coloneqq \gamma \circ u : (c, d) \to \mathbb{R}^2$ is a unit-speed parametrized curve and $u'(t) = 1/\|\gamma'(t)\|$ for all $t \in (a, b)$. What we have already done implies that there is a smooth angle function $\theta_\eta : (c, d) \to \mathbb{R}$ for the curve η, so that

$$\frac{\gamma'(u(s))}{\|\gamma'(u(s))\|} = \gamma'(u(s)) \cdot u'(s) = \eta'(s) = (\cos\theta_\eta(s), \sin\theta(s))$$

for all $s \in (c, d)$. If $\sigma : (a, b) \to (c, d)$ is the function inverse to u, then it follows immediately from this that for all $t \in (a, b)$ we have

$$\mathbf{t}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = (\cos\theta_\eta(\sigma(t)), \sin\theta(\sigma(t))),$$

and thus that the function $\theta \circ \sigma : (a, b) \to \mathbb{R}$ is a smooth angle function for γ.

(*ii*) Let $\theta, \bar\theta : (a, b) \to \mathbb{R}$ be two continuous functions such that $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$ and $\gamma'(t) = (\cos\bar\theta(t), \sin\bar\theta(t))$ for all $t \in (a, b)$. If t is an element of (a, b), we thus have that $\cos\theta(t) = \cos\bar\theta(t)$ and $\sin\theta(t) = \sin\bar\theta(t)$, and this implies, as we know, that there is an integer $k(t) \in \mathbb{Z}$ such that

$$\theta(t) - \bar\theta(t) = 2\pi k(t). \tag{1.15}$$ {eq:thth}

We obtain in this way a function $k : \mathbb{R} \to \mathbb{Z}$, and since θ and $\bar\theta$ are continuous functions, the equality (1.8) implies that the function k is itself continuous. As k only takes integer values, it is in fact constant. The claim (*ii*) of the proposition follows from this at once. □

Once that we have angle functions at hand it is very easy to establish their connection with curvature:

Proposition 1.8.2. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a unit-speed plane curve, let $\theta : (a,b) \to \mathbb{R}$ be an angle function for γ, and let $\kappa : (a,b) \to \mathbb{R}$ be the signed curvature function of γ. For all $t \in (a,b)$ we have that* {prop:angle-kappa}

$$\kappa(t) = \theta'(t).$$

In the proof of Proposition 1.8.1 we explicitly constructed an angle function for a unit-speed curve and showed that its derivative is given by the formula in (1.8): in view of Proposition 1.7.6 we see that this proposition is true for that angle function, and then using the second part of Proposition 1.8.1 it is easy to conclude that in fact the proposition is true for all angle functions. We will prove Proposition 1.8.2 directly, without involving an angle function constructed in any specific way.

Proof. Since θ is an angle function for γ, we have

$$\mathbf{n}(t) = (-\sin\theta(t), \cos\theta(t)), \qquad \gamma''(t) = (-\theta'(t)\cdot\sin\theta(t), \theta'(t)\cdot\cos\theta(t))$$

for all $t \in (a,b)$, and therefore $\kappa(t) = \langle \gamma''(t), \mathbf{n}(t)\rangle = \theta'(t)$. □

Using angle functions we can construct an important invariant of closed curves.

Proposition 1.8.3. *Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed curve, and let $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be the periodic extension of $\tilde{\gamma}$ of period $d \coloneqq b - a$. There is an integer $i(\gamma)$ such that for every smooth angle function $\theta : \mathbb{R} \to \mathbb{R}$ for $\tilde{\gamma}$ and every real number t_0 we have*

$$i(\gamma) = \frac{\theta(t_0 + d) - \theta(t_0)}{2\pi}.$$

We call the integer $i(\gamma)$ described here the ***rotation index*** of the curve γ.

Proof. Let t_0 be a real number, and let $\tilde{\mathbf{t}} : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be the tangent vector field and a smooth angle function for $\tilde{\gamma}$, so that $\tilde{\mathbf{t}}(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in \mathbb{R}$. As $\tilde{\gamma}$ is a periodic function of period d, we have that $\mathbf{t}(t_0 + d) = \mathbf{t}(t_0)$, and therefore we have

$$\cos\theta(t_0 + d) = \cos\theta(t_0), \qquad \sin\theta(t_0 + d) = \sin\theta(t_0),$$

and this implies that $\theta(t_0 + d)$ and $\theta(t_0 0)$ are numbers that differ by an integer multiple of 2π and therefore that the quotient

$$\frac{\theta(t_0 + d) - \theta(t_0)}{2\pi} \tag{1.16}$$ {eq:af:1}

is an integer. We want to show that this integer depends only on the curve γ and not on the way the number t_0 and the smooth angle function θ were chosen.

Let $\theta_1 : \mathbb{R} \to \mathbb{R}$ be another smooth angle function for the parametrized curve $\tilde{\gamma}$. According to the second part of Proposition 1.8.1, there is an integer $k \in \mathbb{Z}$ such that $\theta_1(t) - \theta(t) = 2k\pi$ for all $t \in \mathbb{R}$, and it follows immediately from this that

$$\frac{\theta_1(t_0 + d) - \theta_1(t_0)}{2\pi} = \frac{\theta(t_0 + d) - \theta(t_0)}{2\pi}.$$

This allows us to conclude that the integer (1.8) does not really depend on the way the smooth angle function θ is chosen.

Finally, what we have done implies that the function

$$t \in \mathbb{R} \mapsto \frac{\theta(t + d) - \theta(t)}{2\pi} \in \mathbb{R},$$

which is smooth, takes integer values: as the set $\mathbb{Z}$ is discrete, we can therefore conclude that this function is actually constant, and this means precisely that the value of the quotient (1.8) does not really depend on the way the number t_0 is chosen. This proves the proposition. □

Using the relation between angle functions and curvature that we found earlier, we can prove the following nice result:

Proposition 1.8.4. *Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed curve, and let $\kappa : [a, b] \to \mathbb{R}$ be the corresponding curvature function. The rotation index of γ is*

$$i(\gamma) = \frac{1}{2\pi} \int_a^b \kappa(\tau)\, d\tau.$$

The integral that appears in the right hand side of this equality is usually called the ***total curvature*** of the curve γ.

Proof. Let us suppose first that the curve γ is a unit-speed curve. We know that $\theta'(t) = \kappa(t)$ for all $t \in [a, b]$, so that according to the *Fundamental Theorem of Calculus* we have that

$$\int_a^b \kappa(\tau)\, d\tau = \theta(b) - \theta(a) = 2\pi \cdot i(\gamma),$$

and the equality in the proposition follows immediately from this. □

Let now $\gamma : [a, b] \to \mathbb{R}^2$ be a closed parametrized curve which is not necessarily unit-speed. As we know, if L is the length of γ, there is then a change of parameters $u : [0, L] \to [a, b]$ such that the curve $\eta \coloneqq \gamma \circ u$ is a unit-speed parametrized curve.

§1.9. Euclidean motions

A function $\mathbb{R}^n \to \mathbb{R}^n$ is called an ***Euclidean motion*** or an ***isometry*** if it preserves distances, that is, if whenever p and q are two points in $\mathbb{R}^n$ we have that Euclidean motion

$$d(f(p), f(q)) = d(p, q).$$

The simplest Euclidean motions in the plane $\mathbb{R}^2$ are those described in the following three examples.

{ex:e2:trans}

Example 1.9.1. Let $p = (x_0, y_0)$ be point in $\mathbb{R}^2$. The function

$$\tau_p : (x, y) \in \mathbb{R}^2 \mapsto (x + x_0, y + y_0) \in \mathbb{R}^2$$

is an Euclidean motion that we call the ***translation*** by p. Indeed, if (x, y) and (u, v) are two points in $\mathbb{R}^2$ we have that

$$\begin{aligned} d\big(\tau_p(x,y), \tau_p(u,v)\big)^2 &= d\big((x + x_0, y + y_0), (u + x_0, v + y_0)\big)^2 \\ &= \big((x + x_0) - (u + x_0)\big)^2 + \big((y + y_0) - (v + y_0)\big)^2 \\ &= (x - u)^2 + (y - v)^2 \\ &= d\big((x, y), (u, v)\big)^2. \end{aligned}$$

{ex:e2:rot}

Example 1.9.2. Let θ be a real number. The function

$$\rho_\theta : (x, y) \in \mathbb{R}^2 \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta) \in \mathbb{R}^2$$

is an Euclidean motion that we call the ***rotation*** of angle θ centered at the origin. If (x, y) and (u, v) are two points in $\mathbb{R}^2$, then

$$\begin{aligned} &d\big(\rho_\theta(x,y), \rho_\theta(u,v)\big)^2 \\ &\quad = d\big((x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta), (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)\big)^2 \\ &\quad = \big((x\cos\theta - y\sin\theta) - (u\cos\theta - v\sin\theta)\big)^2 \\ &\qquad\qquad + \big((x\sin\theta + y\cos\theta) - (u\sin\theta + v\cos\theta)\big)^2 \\ &\quad = \big((x-u)\cos\theta - (y-v)\sin\theta\big)^2 + \big((x-u)\sin\theta + (y-v)\cos\theta\big)^2 \\ &\quad = (x-u)^2\cos^2\theta - 2(x-u)(y-v)\cos\theta\sin\theta + (y-v)^2\sin^2\theta \\ &\qquad\qquad (x-u)^2\sin^2\theta + 2(x-u)(y-v)\sin\theta\cos\theta + (y-v)^2\cos^2\theta \\ &\quad = (x-u)^2 + (y-v)^2 \\ &\quad = d\big((x,y),(u,v)\big)^2, \end{aligned}$$

and this is what we are claiming.

{ex:e2:refl}

Example 1.9.3. Finally, the function

$$\sigma : (x, y) \in \mathbb{R}^2 \mapsto (x, -y) \in \mathbb{R}^2$$

is an Euclidean motion that we call the ***reflection*** with respect to the x axis. Indeed, if (x, y) and (u, v) are two points in $\mathbb{R}^2$, then

$$\begin{aligned} d\big(\sigma(x,y), \sigma(u,v)\big)^2 &= d\big((x,-y),(u,-v)\big)^2 \\ &= (x-u)^2 + ((-y)-(-v))^2 \\ &= (x-u)^2 + (y-v)^2 \\ &= d\big((x,y),(u,v)\big). \end{aligned}$$

We will provide below a concrete and useful description of all Euclidean motions. To do that, we need to recall some facts from linear algebra. A matrix A in $\mathrm{M}_n(\mathbb{R})$ is ***orthogonal*** if it satisfies the following equivalent conditions: orthogonal matrix

- $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$.
- $|Ax| = |x|$ for all $x \in \mathbb{R}^n$.
- $AA^t = A^tA = I$, so that A and its transpose A^t are mutually inverse.
- The columns of A form an orthonormal basis for $\mathbb{R}^n$.
- The transposed[3] rows of A form an orthonormal basis for $\mathbb{R}^n$.

We write $\mathrm{O}_n(\mathbb{R})$ for the set of all orthonormal matrices in $\mathrm{M}_n(\mathbb{R})$ and call it the ***orthogonal group***. A few key properties of this set are the following: orthonormal group

- The identity matrix I belongs to $\mathrm{O}_n(\mathbb{R})$.
- If A and B are elements of $\mathrm{O}_n(\mathbb{R})$, then the product AB is also an element of $\mathrm{O}_n(\mathbb{R})$.
- If A is an element of $\mathrm{O}_n(\mathbb{R})$, then A is invertible, its inverse is also an element of $\mathrm{O}_n(\mathbb{R})$ and, in fact, $A^{-1} = A^t$.

In fact, it is because of these properties that we call the set $\mathrm{O}_n(\mathbb{R})$ a group.

If A is an orthogonal matrix, then $1 = \det I = \det AA^t = \det A \cdot \det A^t = (\det A)^2$, and therefore the determinant of A is either 1 or -1. If the determinant of A is 1 we say that the matrix A is ***special orthogonal***. We write $\mathrm{SO}_n(\mathbb{R})$ for the set of all special orthogonal matrices in $\mathrm{M}_n(\mathbb{R})$ and call it the ***special orthogonal group***. This set has properties similar to those of $\mathrm{O}_n(\mathbb{R})$:

- The identity matrix I belongs to $\mathrm{SO}_n(\mathbb{R})$.
- If A and B are elements of $\mathrm{SO}_n(\mathbb{R})$, then the product AB is also an element of $\mathrm{SO}_n(\mathbb{R})$.
- If A is an element of $\mathrm{SO}_n(\mathbb{R})$, then A is invertible, its inverse is also an element of $\mathrm{O}_n(\mathbb{R})$ and, in fact, $A^{-1} = A^t$.

[3] Remember that the elements of $\mathbb{R}^n$ are *column* vectors, so we have to transpose the rows of A to obtain elements of that vector space.

The following two examples describe the orthogonal matrices in dimensions 2 and 3. Similar analyses can be carried out in all dimensions, in fact.

{ex:O2}

Example 1.9.4. Suppose that A is an element of $O_2(\mathbb{R})$. The first column of A is a unit vector in $\mathbb{R}^2$, so we know that there exists a number $\theta \in \mathbb{R}$ such that column is equal to $\left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$. The second column of the matrix is orthogonal to the first column, so is a scalar multiple of $\left(\begin{smallmatrix}-\sin\theta\\ \cos\theta\end{smallmatrix}\right)$ and has norm 1, so it is either this vector or its opposite. We thus see that A is one of the two matrices

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \qquad \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}.$$

The first one has determinant 1 while the second one has determinant −1. In this way we obtain an enumeration of all the element so $O_2(\mathbb{R})$ and of $SO_2(\mathbb{R})$.

Example 1.9.5. Let now A be an element of $SO_3(\mathbb{R})$. We have that

$$\det(A-I) = \det A^t \cdot \det(A-I) = \det A^t(A-I) = \det(A^tA - A^t) = \det(I-A^t) = -\det(A-I),$$

and this implies that $\det(A-I) = 0$, so that 1 is an eigenvalue of A and there exists a unit vector u in $\mathbb{R}^3$ such that $Au = u$. Let (v, w) be an ordered orthonormal basis for the orthogonal complement of the subspace $\operatorname{span}(u)$. Let C be the matrix that has as columns the three vectors u, v and w: since these three vectors are orthonormal, we have that $C^tC = I_3$. and therefore the matrix C is orthogonal and its inverse is its transpose C^t.

We have that

$$\langle Av, u\rangle = \langle Av, Au\rangle = \langle v, u\rangle = 0$$

and similarly $\langle Aw, u\rangle = 0$, to that the vectors Av and Aw are orthogonal to u and therefore the two belong to the subspace $\operatorname{span}(v, w)$. This implies that there are four numbers a, b, c and d such that

$$C^tAC = \begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & c & d\end{pmatrix}.$$

If we let B be the matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, we can write this in the form

$$C^tAC = \begin{pmatrix}1 & 0\\ 0 & B\end{pmatrix}. \qquad (1.17)$$

{eq:so3}

From this it follows that

$$C^tA^tC = (C^tAC)^t \begin{pmatrix}1 & 0\\ 0 & B^t\end{pmatrix}$$

and thus, since $CC^t = AA^t = I_3$, that

$$I_3 = C^tC = C^tA^tAC = C^tA^tC \cdot C^tAC = \begin{pmatrix} 1 & 0 \\ 0 & B^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B^tB \end{pmatrix}.$$

It follows from this that $B^tB = I_2$, so that the matrix B is orthogonal. From (1.9.5) we see immediately that $\det B = 1$, so that in fact B is *special* orthogonal and we know that there is a number θ such that $B = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. The conclusion of all this is that

> *if A is an element of $\mathrm{SO}_3(\mathbb{R})$, then there exists an orthogonal matrix C of determinant 1 and a real number θ such that*
>
> $$C^tAC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$
>
> *If u, v and w are the elements of $\mathbb{R}^3$ given by the columns of the matrix C, then the function $x \in \mathbb{R}^3 \mapsto Ax \in \mathbb{R}^3$ is then a rotation of angle θ around the line of direction u through the origin.*

It is very easy to see that, conversely, for every orthogonal matrix C of determinant 1 and every $\theta \in \mathbb{R}$ the matrix

$$C \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} C^t$$

is an element of $\mathrm{SO}_3(\mathbb{R})$, so this provides a precise description of the elements of that group.

Exercise 1.9.6. Obtain a similar description for the elements of $\mathrm{O}_3(\mathbb{R})$ of determinant -1.

The following simple lemma that allows us to compare elements of $\mathrm{SO}_2(\mathbb{R})$ will be useful later.

Lemma 1.9.7. *Let A and B be two elements of $\mathrm{SO}_2(\mathbb{R})$, and let v be a non-zero element of $\mathbb{R}^2$. If $A \cdot v = B \cdot v$, then in fact $A = B$.* □ {lemma:so2:unique}

Exercise 1.9.8. Prove the lemma.

With the information we have about orthogonal matrices at hand we can now state and prove the following result that completely describes Euclidean motions.

{prop:motions}

Proposition 1.9.9.
(*i*) *If $A \in \mathrm{O}_n(\mathbb{R})$ is an orthogonal matrix and $b \in \mathbb{R}^n$ is a vector, then the function*

$$f : x \in \mathbb{R} \mapsto Ax + b \in \mathbb{R}^n$$

is an Euclidean motion.
(*ii*) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an Euclidean motion. There exist an orthogonal matrix $A \in \mathrm{O}_n(\mathbb{R})$ and a vector $v \in \mathbb{R}^n$, both uniquely determined by f, such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^n$.*

Proof. (*i*) Let $A \in \mathrm{O}_n(\mathbb{R})$ be an orthogonal matrix, let $b \in \mathbb{R}^n$ be a vector, and let us consider the function $f : x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^n$. If x and y are any two elements of $\mathbb{R}^n$, then we have that

$$d(f(x), f(y)) = \|f(x) - f(y)\| = \|(Ax + b) - (Ay + b)\| = \|A(x - y)\| = \|x - y\| = d(x, y),$$

and this tells us that the function f is an Euclidean motion.

(*ii*) Let us consider the function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\bar{f}(x) = f(x) - f(0)$ for all $x \in \mathbb{R}^n$. If x and y are two elements of $\mathbb{R}^n$, then

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2,$$

and of course also

$$\|f(x - y)\|^2 = \|f(x)\|^2 - 2\langle f(x), f(y)\rangle + \|f(y)\|^2.$$

We can use this and the fact that f preserves distances to see that

$$\begin{aligned}
2\langle \bar{f}(x), \bar{f}(y)\rangle &= \|\bar{f}(x)\|^2 + \|\bar{f}(y)\|^2 - \|\bar{f}(x - y)\|^2 \\
&= \|f(x) - f(0)\|^2 + \|f(y) - f(0)\|^2 - \|f(x) - f(y)\|^2 \\
&= d(f(x), f(0))^2 + d(f(y), f(0))^2 - d(f(x), f(y))^2 \\
&= d(x, 0)^2 + d(y, 0)^2 - d(x, y)^2 \\
&= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \\
&= 2\langle x, y\rangle.
\end{aligned}$$

This tells us that the function $\bar{f}$ preserves inner products:

for all x, $y \in \mathbb{R}$ we have $\langle \bar{f}(x), \bar{f}(y)\rangle = \langle x, y\rangle$.

Let now $(e_1, \dots, e_n)$ be a orthonormal basis for $\mathbb{R}^n$, so that for all i, $j \in \{1, \dots, n\}$ be have that

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{in any other case.} \end{cases}$$

Since the function $\bar{f}$ preserves inner products, we also have that $\bar{(}f(e_1),\ldots,\bar{f}(e_n))$ is an orthonormal basis for $\mathbb{R}^n$, since $\langle\bar{f}(e_i),\bar{f}(e_j)\rangle = \langle e_i, e_j\rangle$ for all $i, j \in \{1,\ldots,n\}$. As a consequence of this, we have that when x is an element of $\mathbb{R}^n$,

if $\langle x, \bar{f}(e_i)\rangle = 0$ *for all* $i \in \{1,\ldots,n\}$, *then* $x = 0$.

Let x and y be two elements of $\mathbb{R}^n$. If $i \in \{1,\ldots,n\}$, then

$$\begin{aligned}
&\langle\bar{f}(x)+\bar{f}(y)-\bar{f}(x+y),\bar{f}(e_i)\rangle \\
&\qquad = \langle\bar{f}(x),\bar{f}(e_i)\rangle + \langle\bar{f}(y),\bar{f}(e_i)\rangle - \langle\bar{f}(x+y),\bar{f}(e_i)\rangle \\
&\qquad = \langle x, e_i\rangle + \langle y, e_i\rangle - \langle x+y, e_i\rangle \\
&\qquad = \langle x+y-(x-y), e_1\rangle \\
&\qquad = 0,
\end{aligned}$$

and we can therefore conclude that $\bar{f}(x)+\bar{f}(y)-\bar{f}(x+y) = 0$, so that $\bar{f}(x+y) = \bar{f}(x)+\bar{f}(y)$.

Similarly, if x is an element of $\mathbb{R}^n$ and λ one of $\mathbb{R}$, we have that for each $i \in \{1,\ldots,n\}$

$$\begin{aligned}
&\langle\bar{f}(\lambda x)-\lambda\bar{f}(x),\bar{f}(e_i)\rangle \\
&\qquad = \langle\bar{f}(\lambda x),\bar{f}(e_i)\rangle - \lambda\langle\bar{f}(x),\bar{f}(e_i)\rangle \\
&\qquad = \langle\lambda x, e_i\rangle - \lambda\langle x, e_i\rangle \\
&\qquad = \langle\lambda x - \lambda x, e_i\rangle \\
&\qquad = 0,
\end{aligned}$$

so again we see that $\bar{f}(\lambda x)-\lambda\bar{f}(x)$, so that in fact $\bar{f}(\lambda x) = \lambda\bar{f}(x)$. Putting everything together, we have proved that the function $\bar{f}:\mathbb{R}^n\to\mathbb{R}^n$ is linear. It follows from this, as usual, that there is a matrix $A \in \mathrm{M}_n(\mathbb{R})$ such that $\bar{f}(x) = Ax$ for all $x \in \mathbb{R}^n$. The columns of A are precisely the vectors $\bar{f}(e_1),\ldots,\bar{f}(e_n)$, and these form an orthonormal basis for $\mathbb{R}^n$, so the matrix A is orthogonal. If we put $b \coloneqq f(0)$, then $f(x) = \bar{f}(x)+f(0) = Ax+b$ for all $x \in \mathbb{R}^n$, and we see that the existence statement of the proposition is true. Let us prove the uniqueness statement next.

Suppose that A_1 and b_1 are an orthogonal matrix in $\mathrm{O}(n)$ and a vector in $\mathbb{R}^n$ such that $f(x) = A_1x+b_1$ for all $x \in \mathbb{R}^n$. We then have that $b = f(0) = b_1$, and for all $x \in \mathbb{R}^n$ that

$$0 = f(x)-f(x) = (Ax+b)-(A_1x+b_1) = (A-A_1)x.$$

This clearly implies that the matrix $A-A_1$ is zero, so that also $A = A_1$, and this completes the proof of the proposition. □

Suppose that $f:\mathbb{R}^n\to\mathbb{R}^n$ is an Euclidean motion, so that, according to the proposition, there exist an orthogonal matrix $A \in \mathrm{O}_n(\mathbb{R})$ and a vector $b \in \mathbb{R}^n$, both uniquely determined by f, such that $f(x) = Ax+b$ for all $x \in \mathbb{R}^n$. We say that the Euclidean motion f is ***proper***, ***direct*** or ***positive*** if the determinant of A is 1, and that it is ***improper***, ***inverse*** or ***negative*** it that determinant is -1. We write $\mathrm{E}(n)$ for the set of all Euclidean motions $\mathbb{R}^n \to \mathbb{R}^n$, and $\mathrm{SE}(n)$ for the set of those Euclidean motions that are proper. The set $\mathrm{E}(n)$ is the ***Euclidean group***, while $\mathrm{SE}(n)$ is the ***special Euclidean group***.

Exercise 1.9.10. Let G be one of the sets $\mathrm{E}(n)$ or $\mathrm{SE}(n)$.

- The identity function $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ belongs to G.
- If f and g are elements of G, then the composition $f \circ g$ is also an element of G.
- If f is an element of G, then f is a bijective and its inverse is also an element of G.

Using the description that we gave in Example 1.9.4 of the orthogonal matrices in $M_2(\mathbb{R})$ and Proposition 1.9.9 we can enumerate all Euclidean motions in the plane.

- If $\theta \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, then the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

  $$f(x, y) = (x\cos\theta - y\sin\theta + a, x\sin\theta + y\cos\theta + b)$$

  for all $(x, y) \in \mathbb{R}^2$ is a proper Euclidean motion, and every direct Euclidean motion is obtained in this way.
- Similarly, if $\theta \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, then the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

  $$f(x, y) = (x\cos\theta + y\sin\theta + a, x\sin\theta - y\cos\theta + b)$$

  for all $(x, y) \in \mathbb{R}^2$ is a improper Euclidean motion, and every improper Euclidean motion is obtained in this way.

In particular, it is immediate to check that the translations of Example 1.9.1, the rotations of Example 1.9.2 and the reflection of Example 1.9.3 are all accounted for in this enumeration.

Exercise 1.9.11. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a proper Euclidean motion, then there exist a translation $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ and a rotation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f = \tau \circ \rho$. Are τ and ρ uniquely determined by f? Similarly, show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an improper Euclidean motion, then there exist a translation $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ and a rotation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f = \tau \circ \sigma \circ \rho$, with σ the reflection of Example 1.9.3.

Exercise 1.9.12. A ***reflection*** in the plane is an isometry $r : \mathbb{R}^2 \to \mathbb{R}^2$ that is not the identity and that leaves all the points in a line L fixed — in that case, the line L is completely determined by r and is the ***axis*** of the reflection. For example, the map S from Example 1.9.3 is a reflection, as it leaves every point of the horizontal axis fixed, so that axis is the axis of the reflection. reflection

(1) Show that the composition of two reflections with parallel axes is a translation, and that every translation is the composition of two reflections.
(2) Similarly, show that that the composition of two reflections with non-parallel axes is a

rotation with center at the point of intersection of the two axes, and that every rotation is a composition of two reflections.

One of the reasons for which reflections are important is that they are simultaneously very simple and enough to generate all Euclidean motions efficiently, in the sense that the following result holds:

(3) Show that every Euclidean motion of the plane is the composition of at most three reflections, and that every positive Euclidean motion is the composition of at most two reflections.

§1.10. The fundamental theorem of plane curves

If $\gamma : (a,b) \to \mathbb{R}^2$ is a unit speed curve and $x, y : (a,b) \to \mathbb{R}$ are its components, the normal field $\mathbf{n} : (a,b) \to \mathbb{R}^2$ of γ has $\mathbf{n}(t) = (-y'(t), x'(t))$ for all $t \in (a,b)$. We will use below several times the fact that we have the equality

$$\mathbf{n}(t) = R_{\pi/2}\gamma'(t),$$

with $R_{\pi/2} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the orthogonal matrix that corresponds to the rotation of angle $\pi/2$ described in Example 1.9.2. We will also use below the following property of the matrix $R_{\pi/2}$.

Lemma 1.10.1. *If $A \in O_2(\mathbb{R})$ be an orthogonal matrix and $d \coloneqq \det A$ is its determinant, then $R_{\pi/2}A = d \cdot AR_{\pi/2}$.*

Proof. The determinant of A is either $+1$ or -1. In the first case, there is a real number θ such that $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, and in the second case there is a real number θ such that $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$. In each of these two cases we can compute directly that the equality $R_{\pi/2}A = d \cdot AR_{\pi/2}$ holds. □

The following proposition describes what happens to the curvature of a curve when we «move» the curve using an Euclidean motion: it almost does not change.

Proposition 1.10.2. *Let $\gamma : (a,b) \to \mathbb{R}^2$ be a unit speed plane curve and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an Euclidean motion. The composition $\eta \coloneqq f \circ \gamma : (a,b) \to \mathbb{R}^2$ is also a unit speed plane curve and if $\kappa_\gamma, \kappa_\eta : (a,b) \to \mathbb{R}$ are the signed curvature functions of γ and of η, respectively, then $\kappa_\gamma = \kappa_\eta$ if f is proper, and $\kappa_\eta = -\kappa_\gamma$ if f is improper.*

Proof. We know from Proposition 1.9.9 that there are an orthogonal matrix $A \in O_2(\mathbb{R})$ and a vector $b \in \mathbb{R}^2$ such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^2$. We then have that $\eta(t) = A\gamma(t) + b$ for all

$t \in (a, b)$, and therefore we see immediately that η is a smooth curve and that

$$\eta'(t) = A\gamma'(t)$$

for all $t \in (a, b)$. In particular,for any such t we have

$$\|\eta'(t)\| = \|A\gamma'(t)\| = \|\gamma'(t)\| = 1,$$

since A is orthogonal and γ is a unit speed curve, and this tells us that η is a unit speed curve. As $\eta''(t) = A\gamma''(t)$ for all $t \in (a, b)$, the signed curvature of η is

$$\kappa_\eta(t) = \langle \eta''(t), \mathbf{n}_\eta(t) \rangle = \langle A\gamma''(t), R_{\pi/2}\eta'(t) \rangle = \langle A\gamma''(t), R_{\pi/2}A\gamma'(t) \rangle.$$

If f is a proper Euclidean motion, then the matrix A is special orthogonal and, as we noted above, $R_{\pi/2}A = AR_{\pi/2}$. It follows from this that

$$\kappa_\eta(t) = \langle A\gamma''(t), AR_{\pi/2}\gamma'(t) \rangle = \langle \gamma''(t), R_{\pi/2}\gamma'(t) \rangle = \langle \gamma''(t), \mathbf{n}'_\gamma(t) \rangle = \kappa_\gamma(t).$$

If instead the map f is an improper Euclidean motion, then the orthogonal matrix A has determinant -1 and we know that $R_{\pi/2}A = -AR_{\pi/2}$, so what we have is that

$$\kappa_\eta(t) = \langle A\gamma''(t), -AR_{\pi/2}\gamma'(t) \rangle = -\langle \gamma''(t), R_{\pi/2}\gamma'(t) \rangle = -\langle \gamma''(t), \mathbf{n}'_\gamma(t) \rangle = -\kappa_\gamma(t).$$

This proves the proposition. □

The second important observation that we have to make is that two curves that have the same curvature functions are in fact related by an Euclidean motion:

Proposition 1.10.3. *Let γ, $\eta : (a, b) \to \mathbb{R}^2$ be two unit-speed curves defined on the same interval. If the corresponding curvature functions κ_γ, $\kappa_\eta : (a, b) \to \mathbb{R}$ are equal, then there exists a unique proper Euclidean motion $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\eta = f \circ \gamma$.*

Proof. Let us suppose that $\kappa_\gamma(t) = \kappa_\eta(t)$ for all $t \in (a, b)$, let t_0 be any element of (a, b), and let θ_γ, $\theta_\eta : (a, b) \to \mathbb{R}$ be smooth angle functions for γ and θ, respectively. For each $t \in (a, b)$ we then have that

$$\theta'_\eta(t) = \kappa_\eta(t) = \kappa_\gamma(t) = \theta'_\gamma(t),$$

so that there exists a real number α such that $\theta_\eta(t) = \theta_\gamma(t) + \alpha$ for all $t \in (a, b)$. This implies that

$$\begin{aligned}\eta'(t) &= (\cos\theta_\eta(t), \sin\theta_\eta(t)) = (\cos(\theta_\gamma(t) + \alpha), \sin(\theta_\gamma(t) + \alpha)) \\ &= R_\alpha \cdot (\cos\theta_\gamma(t), \sin\theta_\gamma(t)) = R_\alpha \cdot \gamma'(t)\end{aligned}$$

for all $t \in (a,b)$. Let us now put $b \coloneqq \eta(t_0) - R_\alpha \cdot \gamma(t_0)$ and consider the proper Euclidean motion

$$f : x \in \mathbb{R}^2 \mapsto R_\alpha \cdot x + b \in \mathbb{R}^2.$$

The function $h \coloneqq \eta - f \circ \gamma : (a,b) \to \mathbb{R}^2$ is differentiable and has $h(t_0) = \eta(t_0) - (R_\alpha \cdot \gamma(t_0) + b) = 0$ and $h'(t) = \eta'(t) - R_\alpha \cdot \gamma'(t) = 0$ for each $t \in (a,b)$, so that in fact it is constant of value 0. We can therefore conclude that $\eta = f \circ \gamma$, and this proves the existence claim of the proposition.

To prove the uniqueness claim, let us suppose that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is another proper Euclidean motion such that $\eta = g \circ \gamma$, and let A and c be the special orthogonal matrix and the element of $\mathbb{R}^2$ such that $g(x) = A \cdot x + c$ for all $x \in \mathbb{R}^2$. We then have that $\eta'(t_0) = A \cdot \gamma'(t_0)$, and we can conclude from this and Lemma 1.9.7 that $A = R_\alpha$ because we also have that $\eta'(t_0) = R_\alpha \cdot \gamma'(t_0)$. As also $\eta(t_0) = g(\gamma(t_0)) = R_\alpha \cdot \gamma(t_0) + c$, we have that $c = \eta(t_0) - R_\alpha \cdot \gamma(t_0) = b$, and therefore $g = f$. This completes the proof of the proposition. □

We say that two units-speed parametrized curves $\gamma, \eta : (a,b) \to \mathbb{R}^2$ are ***equivalent under proper Euclidean motions*** if there is a proper Euclidean motion $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\eta = f \circ \gamma$.

Exercise 1.10.4. Let (a,b) be an open interval. Show that the relation of equivalence under proper Euclidean motions is an equivalence relation on the set of all unit-speed parametrized curves defined on (a,b).

The combination of the last two propositions allows us to obtain the following key property of the curvature:

Corollary 1.10.5. *Two unit-speed parametrized curves defined on the same open interval are equivalent under proper Euclidean motions if and only if they have the same curvature function.*

In other words, an equivalence class of unit-speed parametrized curves under proper Euclidean motions is completely determined by the curvature function of any one of its elements.

In view of this, it is interesting ask *what* possible curvature functions can a unit-speed parametrized curve have. Our following proposition answers this question.

Proposition 1.10.6. *If $\kappa : (a,b) \to \mathbb{R}$ is a continuous function defined on an open interval, then there exists a unit-speed parametrized curve $\gamma : (a,b) \to \mathbb{R}^2$ whose curvature function is precisely κ.*

Proof. Let $\kappa : (a,b) \to \mathbb{R}$ be a continuous function, and let t_0 be an element of (a,b). Since κ is continuous, we can define a function $\theta : (a,b) \to \mathbb{R}$ putting

$$\theta(t) = \int_{t_0}^{t} \kappa(\tau)\,\mathrm{d}\tau,$$

and it follows from the *Fundamental Theorem of Calculus* that θ is a differentiable function such whose derivative is $\theta' = \kappa$; in particular, θ is actually of class C^1.

Let us consider the function $T : (a,b) \to \mathbb{R}^2$ such that

$$T(t) = (\cos\theta(t), \sin\theta(t))$$

for all $t \in (a,b)$. This is also a function of class C^1, and this allows us to define a final function $\gamma : (a,b) \to \mathbb{R}^2$ that on each $t \in (a,b)$ takes the value

$$\gamma(t) = \int_{t_0}^{t} T(\tau)\,\mathrm{d}\tau.$$

As before, the *Fundamental Theorem of Calculus* tells us that γ is a differentiable function whose derivative is $\gamma' = T$, so that in fact γ is of class C^2. Moreover, using we see that

$$\|\gamma'(t)\| = \|T(t)\| = 1,$$

so that γ is a unit-speed parametrized curve in $\mathbb{R}^2$. As $\gamma'(t) = T(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in (a,b)$ and the function θ is continuous, we see that θ is an angle function for γ, and therefore the curvature function $\kappa_\gamma : (a,b) \to \mathbb{R}$ of γ is such that $\kappa_\gamma(t) = \theta'(t) = \kappa(t)$ for all $t \in (a,b)$. We have thus proved the proposition. □

The proof that we have given for this proposition is of a constructive nature: given a function $\kappa : (a,b) \to \mathbb{R}$ we can explicitly construct, following the steps of the proof, a unit-speed parametrized curve $\gamma : (a,b) \to \mathbb{R}^2$ whose curvature function is κ.

Example 1.10.7. Let R be a positive number, and let $\kappa : \mathbb{R} \to \mathbb{R}$ be the constant function of value $1/R$. Following the proof of the proposition, we define $\theta : \mathbb{R} \to \mathbb{R}$ putting, for each $t \in \mathbb{R}$,

$$\theta(t) = \int_0^t \kappa(\tau)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\tau}{R} = \frac{t}{R}.$$

We then consider the function $T : \mathbb{R} \to \mathbb{R}^2$ with

$$T(t) = (\cos\theta(t), \sin\theta(t)) = \left(\cos\frac{t}{R}, \sin\frac{t}{R}\right),$$

and its integral $\gamma : t \in \mathbb{R} \mapsto \int_0^t T(\tau)\,\mathrm{d}\tau$. As

$$\int_0^t \cos\frac{\tau}{R}\,\mathrm{d}\tau = R\sin\frac{t}{R}, \qquad \int_0^t \sin\frac{\tau}{R}\,\mathrm{d}\tau = -R\cos\frac{t}{R},$$

we see that

$$\gamma(t) = \left(R\sin\frac{t}{R}, -R\cos\frac{t}{R}\right).$$

Clearly γ is the standard unit-speed parametrization if the circle of radius R centered at the origin, with initial point $\gamma(0)$ equal to $(0, -R)$.

Example 1.10.8. Let us now take $\kappa : \mathbb{R} \to \mathbb{R}$ be the function such that

$$\kappa(t) = \frac{1}{1+t^2}$$

for all $t \in \mathbb{R}$. Following the steps of the proof, we find that

$$\theta(t) = \arctan t$$

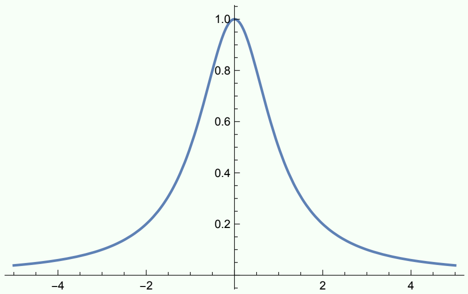
and, since

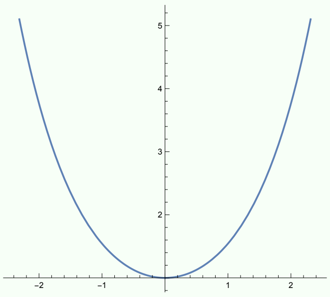
$$\int_0^t \cos \arctan \tau \, \mathrm{d}\tau = \operatorname{arcsinh} t, \qquad \int_0^t \sin \arctan \tau \, \mathrm{d}\tau = \sqrt{1+t^2},$$

that

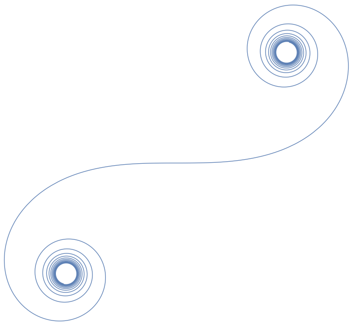
$$\gamma(t) = (\arcsin t, \sqrt{1+t^2})$$

for all $t \in \mathbb{R}$. The following two graphs show the function κ and the trace of the parametrized curve γ.

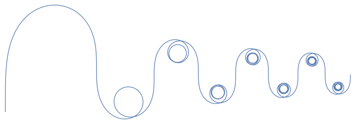




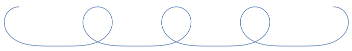
In most situations, if we start with a curvature function we cannot actually compute analytically the curve whose existence is claimed in the proposition, but in that case we can use numerical methods to obtain approximations. In Figure 1.4 on page 48 we show the result of doing this in some examples.



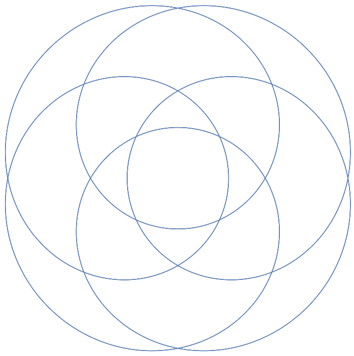
$\kappa(t) = t$



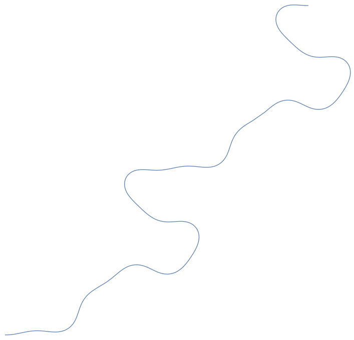
$\kappa(t) = -t \sin t$



$\kappa(t) = 1 + \cos^2 t$



$\kappa(t) = 2 + \cos^2 t$



$\kappa(t) = 2\sin 2t + \cos t - 2\cos 6t$

{fig:ftc}

Figure 1.4. Some examples of curves determined by their curvatures.

§1.11. Exercises

Exercise 1.11.1. Determine conditions on a function $r : (a, b) \to \mathbb{R}$ that ensure that the parametrized curve $\gamma : (a, b) \to \mathbb{R}^2$ with

$$\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$$

for all $\theta \in (a, b)$ is a unit-speed curve.

Solution. As

$$\gamma'(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta),$$

we have that $\|\gamma'(\theta)\| = \sqrt{r'(\theta)^2 + r(\theta)^2}$. The curve will have unit-speed, then, exactly when $r'(\theta)^2 + r(\theta)^2 = 1$. □

Exercise 1.11.2. Find all curves with the property that the angle formed by the segment from the origin to one of its points and the tangent line at that point does not depend on the point. We call this curves ***logarithmic spirals*** — in Figure 1.5 the reader can find a drawing of one. {exer:logarithmic-spiral}

Solution. Let $\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ be a parametrized curve with that property, determined by a positive function r. The angle α formed by the segment from the origin to $\gamma(\theta)$ and the tangent line there is such that

$$\langle\gamma(\theta), \gamma'(\theta)\rangle = \|\gamma(\theta)\| \cdot \|\gamma'(\theta)\| \cdot \cos\alpha$$

As $\|\gamma(\theta)\| = r(\theta)$,

$$\gamma'(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta)$$

and thus $\|\gamma'(\theta)\| = \sqrt{r'(\theta)^2 + r(\theta)^2}$, this tells us that

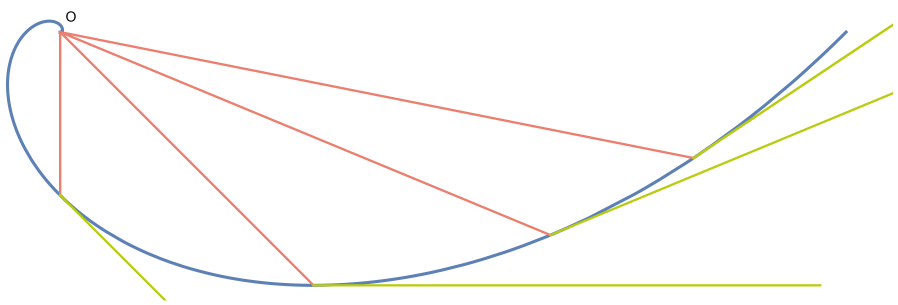
$$\cos\alpha = \frac{r(\theta)r'(\theta)}{r(\theta)\sqrt{r'(\theta)^2 + r(\theta)^2}} = \frac{r'(\theta)}{\sqrt{r'(\theta)^2 + r(\theta)^2}}.$$

The function r therefore is a solution to the differential equation

$$(r'(\theta)^2 + r(\theta)^2)\cos^2\alpha = r'(\theta)^2,$$

which we can rewrite in the form

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log r(\theta) = \frac{r'(\theta)}{r(\theta)} = \frac{1}{\sqrt{\sec^2\alpha - 1}}.$$



{fig:logarithmic-spiral}

Figure 1.5. The logarithmic spiral with polar equation $r = e^\theta$ from Exercise 1.11.2.

We can immediately solve this: if θ_0 is an element of the domain of r and $r_0 \coloneqq r(\theta_0)$, then

$$r(\theta) = r_0 \exp \frac{\theta - \theta_0}{\sqrt{\sec^2 \alpha - 1}}.$$

Conversely, using this formula we can clearly define a smooth positive function $t : \mathbb{R} \to \mathbb{R}$ that gives rise to a parametrized curve with the property we want. □

Exercise 1.11.3. Find all regular curves all of whose normal lines pass through the origin.

Solution. Let $\gamma : (a, b) \to \mathbb{R}^2$ be a unit-speed parametrized curve all of whose normal lines pass through the origin. If t is an element of (a, b), then a point p of $\mathbb{R}^2$ is on the normal line to γ at t exactly when $\langle p - \gamma(t), \gamma'(t)\rangle = 0$. It follows from this that if the curve has the property described in the statement of the exercise then $\langle \gamma(t), \gamma'(t)\rangle = 0$ for all $t \in (a, b)$.

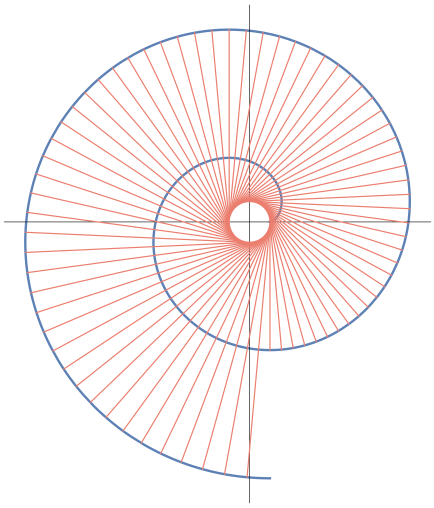
It follows from this that $\gamma(t)$ is orthogonal to $\gamma'(t)$, so it is a multiple of the normal vector: there exists a function $\lambda : (a, b) \to \mathbb{R}$ such that $\gamma(t) = \lambda(t) \cdot n(t)$, and this function is smooth because $\lambda(t) = \langle \gamma(t), n(t)\rangle$ for all $t \in (a, b)$. Since

$$\lambda'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), n(t)\rangle = \langle \gamma'(t), n(t)\rangle + \langle \gamma(t), n'(t)\rangle = \langle \gamma(t), \kappa(t) \cdot \gamma'(t)\rangle = 0,$$

the function λ is actually constant. This tells us that for all $t \in (a, b)$ the point $\gamma(t)$ has $\|\gamma(t)\| = \|\lambda \cdot n(t)\| = |\lambda|$, so that the trace of γ is contained in the circle of radius $|\lambda|$ centered at the origin. Let us notice that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \gamma'(t)\rangle = \langle \gamma(t), \gamma'(t)\rangle + \langle \gamma(t), \gamma''(t)\rangle = 1 + \langle \lambda \cdot n(t), \kappa(t) \cdot n(t)\rangle \\ &= 1 + \lambda \cdot \kappa(t), \end{aligned}$$

so $\kappa(t) = -1/\lambda$ is constant, as it should be. □



{fig:involute-circle}

Figure 1.6. The curve of Exercise 1.11.4, the involute of the unit circle.

{exer:involute-circle}

Exercise 1.11.4.

(1) Show that the parametrized curve $\gamma : (0, +\infty) \to \mathbb{R}^2$ with

$$\gamma(\theta) = (\cos\theta + \theta \sin\theta, \sin\theta - \theta\cos\theta)$$

for all $\theta \in (0, +\infty)$ has the property that all of its normal lines are at distance 1 from the origin. We have drawn this curve and its normal lines in Figure 1.6.

(2) Let ℓ be a positive number. Show that if $\gamma : (a, b) \to \mathbb{R}^2$ us a unit-speed curve all of whose normal lines are at distance ℓ from the origin, then there are numbers $\alpha, \beta \in \mathbb{R}$ and $\epsilon \in \{\pm 1\}$ such that the curvature function of γ has

$$\kappa(t) = \frac{\epsilon}{\sqrt{\alpha t + \beta}}$$

for all $t \in (a, b)$.

(3) Find *all* curves with that property.

Solution. Let us suppose that $\gamma : (a, b) \to \mathbb{R}^2$ is a unit-speed parametrized curve with the property described in the statement of the exercise. The line normal to the curve at $\gamma(t)$ can be parametrized by the function

$$\alpha : s \in \mathbb{R} \mapsto \gamma(t) + s \cdot n(t) \in \mathbb{R}.$$

In particular, the square of the distance from $\alpha(s)$ to the origin is

$$\|\alpha(s)\|^2 = \|\gamma(t) + s \cdot n(t)\|^2 = \|\gamma(t)\|^2 + 2s \cdot \langle \gamma(t), n(t)\rangle + s^2$$

and the critical points of this occur at values of s such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\alpha(s)\|^2 = 2\langle \gamma(t), n(t)\rangle + 2s = 0.$$

There is therefore a unique critical point at $s_0 = -\langle \gamma(t), n(t)\rangle$, and it has to be a minimum, for geometric reasons. The square of the distance from the line to the origin is therefore

$$\ell^2 = \|\alpha(s_0)\|^2 = \|\gamma(t)\|^2 - \langle \gamma(t), n(t)\rangle^2. \tag{1.18}$$ {eq:qq}

This is true for all $t \in (a, b)$, so differentiating we see that

$$0 = 2\langle \gamma(t), \gamma'(t)\rangle - 2\langle \gamma(t), n(t)\rangle\big(\langle \gamma'(t), n(t)\rangle + \langle \gamma(t), n'(t)\rangle\big).$$

As $\gamma'(t) \perp n(t)$, $\gamma''(t) = \kappa(t)n(t)$, and $n'(t) = -\kappa(t)\gamma'(t)$, this tells us that

$$0 = \langle \gamma(t), \gamma'(t)\rangle \cdot \big(1 + \langle \gamma(t), \gamma''(t)\rangle\big). \tag{1.19}$$ {eq:qq:2}

Let us suppose for a moment that there is an element t_0 of (a, b) such that $\langle \gamma(t_0), \gamma''(t_0)\rangle \neq -1$. Since γ is smooth, there is then a positive number ϵ such that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq (a, b)$ and $\langle \gamma(t), \gamma''(t)\rangle \neq -1$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. It follows then from (1.11), that

$$\langle \gamma(t), \gamma'(t)\rangle = 0$$

for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$, and this implies that the normal line to γ at $\gamma(t)$ passes through the origin. As the number ℓ is positive, this is absurd.

We thus see that

$$\kappa(t) \cdot \langle \gamma(t), n(t)\rangle = \langle \gamma(t), \gamma''(t)\rangle = -1$$

for all $t \in (a, b)$. Going back to (1.11) we see that for such t we have

$$\ell^2 = \|\gamma(t)\|^2 + \frac{1}{\kappa(t)^2}. \tag{1.20}$$ {eq:qqq}

On the other hand, for such t we also have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|\gamma(t)\|^2 = 2\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \gamma'(t)\rangle = 2\langle \gamma'(t), \gamma'(t)\rangle + 2\langle \gamma(t), \gamma''(t)\rangle = 2 - 2 = 0,$$

so there are numbers α and ν such that

$$\|\gamma(t)\|^2 = -\alpha t + \nu$$

for all $t \in (a, b)$. Using this and (1.11), and putting $\beta \coloneqq \ell^2 - \nu$ we can therefore conclude that

$$\kappa(t)^2 = \frac{1}{\alpha t + \beta} \qquad \text{for all } t \in (a, b).$$

The claim of the exercise follows immediately from this. □

Exercise 1.11.5. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a *closed* unit-speed curve. If the image of γ is contained in a circle of radius r, then there exists a point t_0 in $\mathbb{R}$ such that the curvature of γ at t_0 is $\kappa(t_0) \geq 1/r$.

This tells us that if a closed curve is contained in a small circle then it must have a point where its curvature is large, a fact that should be intuitively clear.

Solution. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a closed unit-speed curve whose image is contained in the circle centered at a point $p \in \mathbb{R}^2$ with radius r, so that the smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = \langle \gamma(t) - p, \gamma(t) - p \rangle$ is bounded by r^2. Since the curve is closed, the function f is periodic, and since it is continuous it attains its maximum: there is a point t_0 in $\mathbb{R}$ such that $f(t) \leq f(t_0)$ for all $t \in \mathbb{R}$. As f is smooth and has a maximum at t_0, we have that

$$0 = f'(t_0) = 2\langle \mathbf{t}(t_0), \gamma(t_0) - p \rangle,$$

and this implies that there is a number ρ such that $\gamma(t_0) - p = \rho \cdot \mathbf{n}(t_0)$, and since

$$r^2 \geq f(t_0) = \langle \gamma(t_0) - p, \gamma(t_0) - p \rangle$$

we see that $|\rho| \leq r$. Moreover, the fact that f has a maximum at t_0 implies that

$$\begin{aligned} 0 &\geq \tfrac{1}{2} f''(t_0) = \langle \mathbf{t}'(t_0), \gamma(t_0) - p \rangle + \langle \mathbf{t}(t_0), \mathbf{t}(t_0) \rangle = \kappa(t_0) \cdot \langle \mathbf{n}(t_0), \gamma(t_0) - p \rangle + 1 \\ &= \rho \cdot \kappa(t_0) + 1. \end{aligned}$$

We thus see that $|\rho| \cdot |\kappa(t_0)| \geq 1$, so $|\kappa(t_0)| \geq 1/|\rho| \geq 1/r$. □

Exercise 1.11.6. Let $\gamma : [0, L] \to \mathbb{R}^2$ be a closed unit-speed curve in the plane with strictly positive curvature, let $\mathbf{n} : [0, L] \to \mathbb{R}^2$ be its normal field, and let r be a positive number.

We consider the curve $\delta : [0, L] \to \mathbb{R}^2$ such that

$$\delta(t) = \gamma(t) - r \cdot \mathbf{n}(t)$$

for each $t \in \mathbb{R}$. We call it a ***parallel curve*** of γ. Make pictures of these curves in some easy examples to see why it is called like that.

(1) Prove that the length of δ is

$$\operatorname{len}(\delta) = \operatorname{len}(\gamma) + 2\pi r.$$

(2) Prove that the signed curvature κ_δ of δ is given by in terms of that of κ by the formula

$$\kappa_\delta = \frac{\kappa_\gamma}{1 + r \cdot \kappa_\gamma}.$$

Solution. We have that

$$\delta'(t) = \gamma'(t) - r \cdot \mathbf{n}'(t) = \mathbf{t}(t) - r \cdot \mathbf{n}'(t) = \mathbf{t}(t) + r\kappa(t) \cdot \mathbf{t}(t) = (1 + r\kappa(t)) \cdot \mathbf{t}(t),$$

so that the length of δ is

$$\operatorname{len}(\delta) = \int_0^L \|\delta'(t)\| \, \mathrm{d}t = \int_0^L (1 + r\kappa(t)) \, \mathrm{d}t = \int_0^L \mathrm{d}t + r \int_0^L \kappa(t) \, \mathrm{d}t.$$

Of course, $\int_0^L \mathrm{d}t = L = \operatorname{len}(\gamma)$. On the other hand, if $\theta : [0, L] \to \mathbb{R}$ is a smooth angle function for the curve γ, so that $\mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in [0, L]$, then we know that $\theta'(t) = \kappa(t)$ for all $t \in [0, L]$, so that $\int_0^L \kappa(t) \, \mathrm{d}t = \theta(L) - \theta(0)$. This difference is 2π times the rotation index $i(\gamma)$ of the curve γ: since γ has strictly positive curvature, it is a simple convex closed curve, and we know that its rotation index is +1 or −1, and since the curvature is positive, it is in fact +1. Putting everything together, we see that $\operatorname{len}(\delta) = \operatorname{len}(\gamma) + 2\pi r$. This proves the first part of the exercise.

As we noted above, $\delta' = (1 + r\kappa) \cdot \mathbf{t}$, so that $\|\delta'\| = (1 + r \cdot \kappa)$, and

$$\delta'' = (1 + r\kappa)' \cdot \mathbf{t} + (1 + r\kappa) \cdot \mathbf{t}' = (1 + r\kappa)' \cdot \mathbf{t} + (1 + r\kappa)\kappa \cdot \mathbf{n},$$

so that

$$\det(\delta', \delta'') = \det\big((1 + r\kappa) \cdot \mathbf{t}, (1 + r\kappa)' \cdot \mathbf{t} + (1 + r\kappa)\kappa \cdot \mathbf{n}\big) = (1 + r\kappa)^2\kappa.$$

The possibly non-unit-speed curve δ has therefore signed curvature given by

$$\kappa_\delta = \frac{\det(\delta', \delta'')}{|\delta'(t)|^3}. = \frac{(1 + r\kappa)^2\kappa}{(1 + r\kappa)^3} = \frac{\kappa}{1 + r\kappa},$$

as the second part of the exercise claims. □

Bibliography

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