The representation theory of $\mathfrak{sl}_2(\mathbb{k})$

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§1. Lie algebras

Lie algebras

1.1. Proposition. *Hello*

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1.2. A ***Lie algebra*** is a vector space $\mathfrak{g}$ endowed with a bilinear function $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the ***bracket*** of $\mathfrak{g}$, which is antisymmetric, so that

$$[y,x] = -[x,y] \tag{1}$$

for all x, $y \in \mathfrak{g}$, and which satisfies *Jacobi's identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{2}$$

for all x, y, $z \in \mathfrak{g}$. In that case, a ***subalgebra*** of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[x,y] \in \mathfrak{h}$ whenever x and y are elements of $\mathfrak{h}$.

1.3. An observation which is useful in checking that concrete examples are Lie algebras is the following:

Proposition. *Let $\mathfrak{g}$ be a vector space and let $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a antisymmetric bilinear function.*

(*i*) *The function $J : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

$$J(x,y,z) = [x,[y,z]] + [y,[z,x]] + [z,[x,y]]$$

for all x, y, $z \in \mathfrak{g}$ is alternating.

(*ii*) *If $\mathfrak{g}$ is finite dimensional and $(x_1, \dots, x_n)$ is a basis of $\mathfrak{g}$, then $[-,-]$ turns $\mathfrak{g}$ into a Lie algebra if and only if we have $J(x_i, x_j, x_k) = 0$ for all i, j, $k \in \{1, \dots, n\}$ such that $i < j < k$.*

Proof. (*i*) We have to check that $J(x,x,y) = J(x,y,y) = 0$ for all x and y in $\mathfrak{g}$, and this follows from a direct calculation. For example, we have $J(x,x,y) = [x,[x,y]] + [x,[y,x]] + [y,[x,x]]$ and the first two terms cancel each other because $[x,y] = -[y,x]$ and the third term is zero.

(*ii*) To see that the bracket turns $\mathfrak{g}$ into a Lie algebra we have to check that the function J of (*i*) vanishes identically and, since that function is trilinear and alternating, that happens if and only if $J(x_i, x_j, x_j) = 0$ when $1 \leq i < x_i < x_j < x_j \leq n$. □

1.4. It follows immediately from this proposition that if $\mathfrak{g}$ is a vector space of dimension at most two any bilinear function $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ turns $\mathfrak{g}$ into a Lie algebra. Indeed, in that case the function J of the proposition vanishes identically simply because it is trilinear and anti-symmetric.

As a less trivial example of how this proposition can be used, let $\mathfrak{g} = \Bbbk^3$ and let $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the cross product. If $\{x, y, z\}$ is the standard basis of $\mathfrak{g}$, we then have

$$[x,y] = z, \qquad [y,z] = x, \qquad [z,x] = y.$$

We claim that we obtain in this way a Lie algebra. As the cross product is bilinear and anti-symmetric, Proposition 1.3 tells us that to check that Jacobi's condition is satisfied we need only compute that

$$J(x,y,z) = [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = [x,x] + [y,y] + [z,z] = 0.$$

More generally, we have the following result:

Proposition. *Let $\mathfrak{g}$ be a vector space of dimension 3 and let $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be an anti-symmetric bilinear map. Let $\{x,y,z\}$ be a basis of $\mathfrak{g}$ and suppose that the scalars a_i, b_i, c_i are such that*

$$[y,z] = a_1 x + b_1 y + c_1 z, \qquad [z,x] = a_2 x + b_2 y + c_2 z, \qquad [x,y] = a_3 x + b_3 y + c_3 z. \tag{3}$$

The bracket $[-,-]$ satisfies the Jacobi identity if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \qquad \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

Proof. The bracket satisfies the Jacobi condition if and only if the function J from Proposition 1.3 vanishes identically. Now $\mathfrak{g}$ is 3-dimensional and J is alternating and bilinear, so this happens if and only if $J(x,y,z) = 0$. Computing the left hand side of this equation using the relations (3) one finds that the condition is equivalent to the three equations given in the lemma. □

1.5. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear function $f : \mathfrak{g} \to \mathfrak{h}$ is a ***morphism of Lie algebras*** if

$$[f(x), f(y)] = f([x,y]) \text{ for all } x, y \in \mathfrak{g}.$$

The brackets appearing here in the left and right hand side are those of of $\mathfrak{h}$ and of $\mathfrak{h}$, respectively.

It is immediate that the identity function of any Lie algebra is a morphism of Lie algebras, and that the composition of morphisms of Lie algebras is again one: it follows from this that there is a category Lie whose objects are Lie algebras and whose arrows are morphisms of Lie algebras.

1.6. An ***ideal*** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{i}$ of $\mathfrak{g}$ such that

$$[x,y] \in \mathfrak{i} \text{ for all } x \in \mathfrak{g} \text{ and all } y \in \mathfrak{i}.$$

In particular, an ideal is a subalgebra.

In every Lie algebra $\mathfrak{g}$ the subspaces 0 and $\mathfrak{g}$ are ideals. We say that $\mathfrak{g}$ is ***simple*** if it has exactly two ideals, which are then of course these two; the zero Lie algebra is thus not simple.

1.7. The ideals of Lie algebras play a role similar to that of ideals in rings:

Proposition. (*i*) *The kernel of a morphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

(*ii*) *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, then there is a unique Lie algebra structure on the quotient space $\mathfrak{g}/\mathfrak{i}$ such that the canonical function $p : \mathfrak{g} \to \mathfrak{g}/i$ is a morphism of Lie algebras. If x and y are elements of $\mathfrak{g}$, then the bracket on $\mathfrak{g}/\mathfrak{i}$ is such that*

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x,y] + \mathfrak{i}.$$

Proof. (*i*) Let $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. If $x \in \mathfrak{g}$ and $y \in \ker f$, then

$$f([x,y]) = [f(x), f(y)] = [f(x), 0] = 0,$$

so that $[x,y] \in \ker f$. This tells us that $\ker f$ is an ideal of $\mathfrak{g}$.

(*ii*) Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$.

Let $x \in \mathfrak{g}$. We consider the function $c_x : y \in \mathfrak{g} \mapsto [x,y] + \mathfrak{i} \in \mathfrak{g}/i$. It is clearly linear and maps $\mathfrak{i}$ to zero: if $y \in \mathfrak{i}$, then $c_x(y) = y + \mathfrak{i} = \mathfrak{i}$. It follows from this that there exists a unique linear map $\bar{c}_x : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}/\mathfrak{i}$ such that $\bar{c}_x(y + \mathfrak{i}) = c_x(y) + \mathfrak{i}$ for all $y \in \mathfrak{g}$.

There is a function $\bar{c} : \mathfrak{g} \to \operatorname{End}_{\Bbbk}(\mathfrak{g}/\mathfrak{i})$ such that $\bar{c}(x) = \bar{c}_x$ for all $x \in \mathfrak{g}$. It is easy to see that it is linear, and it maps $\mathfrak{i}$ to zero: indeed, if $x \in \mathfrak{i}$, then for all $y \in \mathfrak{g}$ we have

$$\bar{c}(x)(y + \mathfrak{i}) = \bar{c}_x(y + \mathfrak{i}) = [x,y] + \mathfrak{i} = \mathfrak{i},$$

since $\mathfrak{i}$ is an ideal. It follows from this that there is a unique linear function $\bar{\bar{c}} : \mathfrak{g}/\mathfrak{i} \to \operatorname{End}(\mathfrak{g}/\mathfrak{i})$ such that

$$\bar{\bar{c}}(x + \mathfrak{i})(y + \mathfrak{i}) = \bar{c}(x)(y) = [x,y] + \mathfrak{i}$$

for all x, $y \in \mathfrak{g}$. We may now define a bracket $[-,-] : \mathfrak{g}/\mathfrak{i} \times \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}/\mathfrak{i}$ so that $[x+\mathfrak{i}, y+\mathfrak{i}] = \bar{\bar{c}}(x)(y)$ for all x, $y \in \mathfrak{g}$ or, in other words,

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x,y] + \mathfrak{i}. \tag{4}$$

This bracket is anti-symmetric and satiafies Jacobi's identity —this follows immediately from the last equation and the fact that $\mathfrak{g}$ is a Lie algebra— so that it turns the quotient $\mathfrak{g}/\mathfrak{i}$ into a Lie algebra. The canonical function $p : x \in \mathfrak{g} \mapsto x + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}$ is a morphism of Lie algebras: this is just a restatement of the equality (4).

To finish the proof of the proposition, we have to check the uniqueness claim. Suppose that $[-,-]' : \mathfrak{g}/\mathfrak{i} \times \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}/\mathfrak{i}$ is a bracket on $\mathfrak{g}/\mathfrak{i}$ such that the canonical function p is a morphism of Lie algebras. If x and y are elements of g, then we have

$$[x,y] + \mathfrak{i} = p([x,y]) = [p(x), p(y)]' = [x + \mathfrak{i}, y + \mathfrak{i}]'$$

and this tells us that in fact the bracket $[-,-]'$ coincides with the one we define before. □

Examples

Abelian Lie algebras

1.8. If $\mathfrak{g}$ is a vector space, the zero bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ turns $\mathfrak{g}$ into a Lie algebra —indeed, the two conditions (1) and (2) are trivially satisfied for this bracket. We say that a Lie algebra is ***abelian*** if its bracket vanishes identically, as in this example. Every subspace of an abelian Lie algebra is a subalgebra and even an ideal. Every linear map between abelian Lie algebras is a morphism of Lie algebras and two abelian Lie algebras are isomorphic if and only if they have the same dimension are vector spaces.

Lie algebras associated to associative algebras

1.9. Let us recall that an ***associative algebra*** is a vector space A endowed with a bilinear multiplication $\cdot : A \times A \to A$ which is associative, in that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in A$; as usual, we will write xy instead of $x \cdot y$ whenever this is convenient and does does not cause any confusion.

From an associative algebra we can construct a Lie algebra, as follows:

Proposition. *Let A be an associative algebra. The map $[-,-] : A \times A \to A$ such that*

$$[x, y] = xy - yx$$

for all $x, y \in A$ turns A into a Lie algebra, which we denote $\mathsf{Lie}(A)$. This Lie algebra is abelian if and only if the associative algebra A is commutative.

Proof. If x and y are elements of A, we have

$$[y, x] = yx - xy = -(xy - yx) = -[x, y]$$

so the bracket is anti-symmetric. On the other hand, let us fix x, y and z in A. We have

$$[x, [y, z]] = [x, yz - zy] = x(yz) - x(zy) - (yz)x + (zy)x$$

and, similarly,

$$[y, [z, x]] = y(zx) - y(xz) - (zx)y + (xz)y$$

and

$$[z, [x, y]] = z(xy) - z(yx) - (xy)z + (yx)z.$$

If follows that the left hand side in Jacobi's condition (2) is

$$x(yz) - x(zy) - (yz)x + (zy)x + y(zx) - y(xz) \\ - (zx)y + (xz)y + z(xy) - z(yx) - (xy)z + (yx)z$$

and since A is an associative algebra the terms appearing in this expression cancel in pairs: $x(yz)$ with $(xy)z$, $x(zy)$ with $(xz)y$, and so on. We see that Jacobi's equation holds and that $\mathsf{Lie}(A)$ is therefore a Lie algebra. The last claim of the proposition is immediate, in view of the definition of the bracket of $\mathsf{Lie}(A)$. □

1.10. The condition that the algebra A be associative was used in the proof the Proposition 1.9 but it is not a necessary condition for the conclusion of that proposition to hold. If A is a possibly non-associative algebra with multiplication $\cdot : A \times A \to A$ we may define a bracket $[-,-] : A \times A \to A$ as before, putting $[x, y] = xy - yx$ for all $x, y \in A$. This bracket is anti-symmetric but Jacobi's condition is no longer automatically satisfied. We say that the possibly non-associative algebra A is ***Lie-admissible*** if $[-,-]$ satisfies Jacobi's condition. If that is the case, we write $\mathsf{Lie}(A)$ the corresponding Lie algebra. In this language, Proposition 1.9 tells us that associative algebras are Lie-admissible. On the other hand, it is immediate to check that a Lie algebra $\mathfrak{g}$ is Lie-admissible.

Let us give a genuinely new example of a class of Lie-admissible algebras. If A is a possibly non-associative algebra, the ***associator*** of A is the function

$$\alpha_A : (x, y, z) \in A \times A \times A \mapsto x(yz) - (xy)z \in A,$$

which is manifestly trilinear. It is clear that the algebra A is associative exactly when its associator α_A is identically zero, so we view α_A as a measure of how badly associativity fails in A. We say that the algebra A is ***alternative*** if its associator is anti-symmetric or, equivalently, if $\alpha_A(x, x, y) = \alpha_A(x, y, y) = 0$ for all $x, y \in A$. We are interested in these notions because of the following result:

Proposition. *An alternative algebra is Lie-admissible.*

Proof. Let A be an alternative algebra and let $\alpha_A : A \times A \times A \to A$ be its associator. We put on A the bracket $[-,-] : (x, y) \in A \times A \mapsto xy - yx \in A$ and consider the function $J : A \times A \times A \to A$ of Proposition 1.3. Since $[-,-]$ is anti-symmetric, to prove that A is Lie-admissible we have to show that the function J is identically zero. As in the proof of Proposition 1.9, we find that if $x, y, z \in A$, then

$$\begin{aligned} J(x, y, z) = x(yz) - x(zy) - (yz)x + (zy)x + y(zx) - y(xz) \\ - (zx)y + (xz)y + z(xy) - z(yx) - (xy)z + (yx)z \end{aligned}$$

and this can be written in terms of the associator as

$$\alpha_A(x, y, z) - \alpha_A(x, z, y) + \alpha_A(z, x, y) - \alpha_A(z, y, x) + \alpha_A(y, z, x) - \alpha_A(y, x, z).$$

As the associator is anti-symmetric, this is easily seen to be equal to zero. □

Of course, this proposition is of interest only if we are able to exhibit Lie-admissible algebras which are not associative. We refer the reader to the book [CS2003] by John H. Conway and Derek Smith and to the survey [Bae2002] for information on the Cayley-Dickson algebra $\mathbb{O}$ of octonions, an important and beautiful example of an alternative algebra which is not associative.

The general and special linear Lie algebra on a vector space

1.11. The most important instance of the construction of Proposition 1.9 is the following. If V is a vector space, then we have the associative algebra $\mathsf{End}_{\Bbbk}(V)$ of all linear functions $V \to V$, whose multiplication is the composition of functions. We write $\mathfrak{gl}(V)$ the Lie algebra $\mathsf{Lie}(\mathsf{End}_{\Bbbk}(V))$ and call it the ***Lie algebra of endomorphisms*** of V or the ***general linear Lie algebra*** on V.

Proposition. *Let V be a vector space.*

(*i*) *The Lie algebra $\mathfrak{gl}(V)$ is finite-dimensional if and only if V is finite-dimensional. If that is the case and $n = \dim V$, then $\dim \mathfrak{gl}(V) = n^2$.*

(*ii*) *The Lie algebra $\mathfrak{gl}(V)$ is abelian if and only if V is of dimension zero or one*

Proof. The first part is clear. The second one follows at once from the fact that the associative algebra $\mathsf{End}_{\Bbbk}(V)$ is commutative if and only if V is of dimension at most one. □

1.12. Let V be now a finite-dimensional vector space, so that we have available ***trace function*** $\mathrm{tr} : \mathrm{End}_{\Bbbk}(V) \to \Bbbk$. This is a linear function such that whenever $f \in \mathrm{End}_{\Bbbk}(V)$ and $\mathscr{B}$ is an ordered basis of V we have

$$\mathrm{tr}\, f = \mathrm{tr}[f]_{\mathscr{B}},$$

where on the right hand side of the equality $[f]_{\mathscr{B}}$ denotes the matrix of the linear map f with respect to the basis $\mathscr{B}$ and $\mathrm{tr}[f]_{\mathscr{B}}$ its trace. This trace function has the property that

$$\mathrm{tr}\,\mathrm{id}_V = \dim V$$

and

$$\mathrm{tr}\, fg = \mathrm{tr}\, gf \tag{5}$$

whenever f and g are elements of $\mathrm{End}_{\Bbbk}(V)$. Moreover, it is easy to see that these two properties uniquely characterize it among linear maps from $\mathrm{End}_{\Bbbk}(V)$ to $\Bbbk$.

1.13. Proposition. *Let V be a finite-dimensional vector space of dimension n. The subspace*

$$\mathfrak{sl}(V) = \{f \in \mathfrak{gl}(V) : \mathrm{tr}(f) = 0\}$$

of $\mathfrak{gl}(V)$ is an ideal —and therefore a subalgebra— of $\mathfrak{gl}(V)$ of dimension $n^2 - 1$.

We call $\mathfrak{sl}(V)$ the ***special linear Lie algebra*** on V.

Proof. If x and y are elements of $\mathfrak{gl}(V)$, then the identity (5) tells us that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$, so that

$$\mathrm{tr}[f, g] = \mathrm{tr}(fg - gf) = 0$$

and, in particular, $[x, y] \in \mathfrak{sl}(V)$. This implies at once that $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$. The linear function $\mathrm{tr} : \mathrm{End}_{\Bbbk}(V) \to \Bbbk$ is not the zero function, so that its kernel —which is precisely $\mathfrak{sl}(V)$— has codimension 1 in $\mathrm{End}_{\Bbbk}(V)$ and, as $\mathrm{End}_{\Bbbk}(V)$ has dimension n^2, this tells us that we have $\dim \mathfrak{sl}(V) = n^2 - 1$, as the proposition claims. □

Lie algebras associated to bilinear forms

1.14. Let again V be a vector space and let us consider now a bilinear form $\beta : V \times V \to \Bbbk$ on V. We say that a linear map $f : V \to V$ ***preserves*** β if for all $x, y \in V$ we have that

$$\beta\big(f(x), y\big) + \beta\big(x, f(y)\big) = 0.$$

Proposition. *Let V be a vector space and let $\beta : V \times V \to \Bbbk$ be a bilinear form. The subset $\mathfrak{o}(V, \beta)$ of $\mathfrak{gl}(V)$ of all linear maps which preserve β is a Lie subalgebra of $\mathfrak{gl}(V)$.*

Proof. An immediate verification shows that $\mathfrak{o}(V, \beta)$ is a subspace of $\mathfrak{gl}(V)$. On the other hand, if f and g are elements of $\mathfrak{o}(V, \beta)$, since f preserves β we have that

$$\beta\big(f(g(x)), y\big) + \beta\big(g(x), f(y)\big) = 0, \qquad \beta\big(f(x), g(y)\big) + \beta\big(x, f(g(y))\big) = 0$$

and since g preserves β that

$$\beta\big(g(x), f(y)\big) + \beta\big(x, g(f(y))\big) = 0, \qquad \beta\big(g(f(x)), y\big) + \beta\big(f(x), g(y)\big) = 0.$$

The sum of the left hand sides of the first two of these equations minus the sum of the left hand sides of the other two is then

$$\beta\big([f, g](x), y\big) + \beta\big(x, [f, g](y)\big) = 0,$$

so that $[f, g] \in \mathfrak{o}(V, \beta)$. This proves the proposition. □

Lie algebras of derivations

1.15. Let A be a possibly non-associative algebra, that is, a vector space endowed with an arbitrary bilinear map $\cdot : A \times A \to A$ which we view as a multiplication on A. A linear function $f : A \to A$ is a ***derivation*** of A if for all $x, y \in A$ we have that

$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y).$$

Proposition. *Let A be a possibly non-associative algebra. The subset* $\mathrm{Der}(A)$ *of* $\mathfrak{gl}(V)$ *of all derivations of A is a Lie subalgebra of* $\mathfrak{gl}(A)$.

Proof. A straightforward computation proves that $\mathrm{Der}(A)$ is a subspace of $\mathfrak{gl}(V)$. To see that it is a subalgebra, let f and g be two elements of $\mathrm{Der}(A)$ and let us show that $[f, g]$ is also in $\mathrm{Der}(A)$. If x and y are in A, then we have

$$g(x \cdot y) = g(x) \cdot y + x \cdot g(y)$$

because f is a derivation, and then, since g is a derivation,

$$f(g(x \cdot y)) = f(g(x) \cdot y) + f(x \cdot g(y)) = f(g(x)) \cdot y + g(x) \cdot f(y) + f(x) \cdot g(y) + x \cdot f(g(y)),$$

Reversing the roles of f and g, we also have that

$$g(f(x \cdot y)) = g(f(x)) \cdot y + f(x) \cdot g(y) + g(x) \cdot f(y) + x \cdot g(f(y)).$$

and subtracting we find that

$$\begin{aligned}[f, g](x \cdot y) &= f(g(x \cdot y)) - g(f(x \cdot y)) \\ &= f(g(x)) \cdot y + x \cdot f(g(y)) - g(f(x)) \cdot y - x \cdot g(f(y)) \\ &= [f, g](x) \cdot y + x \cdot [f, g](y).\end{aligned}$$

This tells us that $[f, g] \in \mathrm{Der}(A)$, as we wanted. □

1.16. Let us compute as an example of this construction the Lie algebra of derivations of the algebra $\Bbbk[X]$ of polynomials with coefficients in $\Bbbk$ on the variable X. We start by obtaining a description of all derivations of this algebra.

Lemma. *For every $p \in \Bbbk[X]$ the function*

$$d_p : f \in \Bbbk[X] \mapsto pf' \in \Bbbk[X]$$

is a derivation of the algebra $\Bbbk[X]$. Conversely, if $d : \Bbbk[X] \to \Bbbk[X]$ is a derivation, then there is a unique $p \in \Bbbk[X]$ such that $d = d_p$.

Proof. Let $p \in \Bbbk[X]$. If $f, g \in \Bbbk[X]$, then we have

$$d_p(fg) = p(fg)' = pf'g + pfg' = d_p(f)g + fd_p(g)$$

because the derivative satisfies Leibniz's formula, and this tells us that d_p is a derivation of $\Bbbk[X]$. This proves the first part of the lemma.

Suppose now that $d : \Bbbk[X] \to \Bbbk[X]$ is a derivation and let $p = d(X)$. We claim that for all $n \in \mathbb{N}_0$ we have

$$d(X^n) = nX^{n-1}p, \tag{6}$$

and this implies that the function d coincides with the function d_p, for it coincides with it on every element of the basis $\{X^n : n \in \mathbb{N}_0\}$ of $\Bbbk[X]$.

We check (6) by induction on n. We have

$$d(1) = d(1 \cdot 1) = d(1) \cdot 1 + 1 \cdot d(1) = 2d(1)$$

because d is a derivation, and therefore $d(1) = 0$. This means that the equality (6) holds when $n = 0$. On the other hand, if $n \in \mathbb{N}_0$ and we suppose inductively that $d(X^n) = nX^{n-1}p$, then we have that

$$d(X^{n+1}) = d(X^n \cdot X) = d(X^n) \cdot X + X^n \cdot d(X) = nX^{n-1}pX + X^n p = (n+1)X^n p.$$

The induction is thus complete. □

For obvious reasons, if $p \in \Bbbk[X]$ we will write

$$p\frac{\mathrm{d}}{\mathrm{d}X}$$

to denote the derivation $d_p : \Bbbk[X] \to \Bbbk[X]$ described in the lemma. It follows at once from it that the function $p \in \Bbbk[X] \mapsto p\frac{\mathrm{d}}{\mathrm{d}X} \in \Bbbk[X]$ is bijective, and it is easy to check that it is in fact an isomorphism of vector spaces. To complete the description of the Lie algebra $\mathsf{Der}(\Bbbk[X])$ we need to compute its bracket:

Proposition. *If $p, q \in \Bbbk[X]$, the bracket in* $\mathsf{Der}(\Bbbk[X])$ *is such that*

$$\left[p\frac{\mathrm{d}}{\mathrm{d}X}, p\frac{\mathrm{d}}{\mathrm{d}X}\right] = (p'q - pq')\frac{\mathrm{d}}{\mathrm{d}X}.$$

Proof. If $f \in \Bbbk[X]$, then we have

$$d_p(d_q(f)) = d_p(qf') = p(qf')' = pq'f' + pqf''$$

and, similarly,

$$d_p(d_p(f)) = p'qf' + pqf'',$$

so that

$$[d_p, d_q](f) = d_p(d_p(f)) - d_q(d_p(f)) = pq'f' - p'qf' = d_{pq'-p'q}(f).$$

It follows from this that $[d_p, d_q] = d_{pq'-p'q}$, and this is what the proposition claims. □

1.17. HACER: Perfection.

§2. Representations of Lie algebras

Representations, modules and morphisms

2.1. Let $\mathfrak{g}$ be a Lie algebra. A ***representation*** of $\mathfrak{g}$ on a vector space M is a morphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(M)$ from $\mathfrak{g}$ to the Lie algebra of endomorphisms of M. Explicitly, this means that the function ρ is linear, that it maps each $x \in \mathfrak{g}$ to an endomorphism $\rho(x) : M \to M$ of the vector space M, and that whenever $x, y \in \mathfrak{g}$ we have that

$$[\rho(x), \rho(y)] = \rho([x, y]). \tag{7}$$

If $x \in \mathfrak{g}$ we often write x_M instead of $\rho(x)$, and in this notation the condition (7) says that for all $x, y \in \mathfrak{g}$ we have $x_M \circ y_M - y_M \circ x_M = [x, y]_M$.

On the other hand, a $\mathfrak{g}$-***module*** is a pair $(M, \cdot)$ in which M is a vector space and $\cdot : \mathfrak{g} \times M \to M$ is a bilinear function such that

$$x \cdot (y \cdot m) - y \cdot (x \cdot m) = [x, y] \cdot m. \tag{8}$$

whenever $x, y \in \mathfrak{g}$ and $m \in M$. Usually we say that M is itself a $\mathfrak{g}$-module, leaving the action $\cdot$ implicit, and whenever the Lie algebra $\mathfrak{g}$ about which we are talking can be determined from the context, we speak simply of modules instead of $\mathfrak{g}$-modules.

The notion of representations of Lie algebras and that of their modules are related:

- If $\rho : \mathfrak{g} \to \mathfrak{gl}(M)$ is a representation of $\mathfrak{g}$ on a vector space, we can construct an action

  $$\cdot : (x, m) \in \mathfrak{g} \times M \mapsto \rho(x)(m) \in M$$

  of $\mathfrak{g}$ on M. The fact that ρ is a linear map implies at once that this action is a bilinear function, and from the condition that ρ satisfies (7) it follows that we in fact have a $\mathfrak{g}$-module $(M, \cdot)$.
- Conversely, if we are given a $\mathfrak{g}$-module $(M, \cdot)$, then we can construct a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ on M putting, for each $x \in \mathfrak{g}$ and each $m \in M$, $\rho(x)(m) = x \cdot m$. That ρ is a linear function is a consequence of the bilinearity of the action $\cdot$ and, as is to be expected, the condition (7) is a direct consequence of the condition (8).

These two constructions are easily seen to be mutually inverse, and this shows that the two notions presented above are in fact equivalent. We will switch from one point of view to the other whenever we find it convenient.

2.2. If M and N are two $\mathfrak{g}$-modules, then a linear map $f : M \to N$ is a ***morphism of*** $\mathfrak{g}$-***modules*** if for all $x \in \mathfrak{g}$ and all $m \in M$ we have that

$$f(x \cdot m) = m \cdot f(m).$$

We denote $\hom_{\mathfrak{g}}(M, N)$ the set of all morphisms of $\mathfrak{g}$-modules $M \to N$. It is easy to see that it is a subspace of the space $\hom_{\Bbbk}(M, N)$ of all linear functions $M \to N$.

The identity map of a $\mathfrak{g}$-module is a morphism of $\mathfrak{g}$-modules, and the composition of two morphisms of $\mathfrak{g}$-modules is itself a morphism of $\mathfrak{g}$-modules: it follows from this that there is a category whose objects are the $\mathfrak{g}$-modules and whose arrows are the morphisms of $\mathfrak{g}$-modules. We write it ${}_{\mathfrak{g}}\mathsf{Mod}$ and we let ${}_{\mathfrak{g}}\mathsf{mod}$ be its full subcategory spanned by the $\mathfrak{g}$-modules which are finite-dimensional as vector spaces.

2.3. As usual, if M is a $\mathfrak{g}$-module, we say that a subspace N of M is a ***submodule*** of M if $x \cdot n \in N$ for all $x \in \mathfrak{g}$ and all $n \in N$. In that case we can restrict the action $\cdot : \mathfrak{g} \times M \to M$ of $\mathfrak{g}$ on M to an action $\cdot : \mathfrak{g} \times N \to N$ on N, and it is immediate that the latter turns N into a $\mathfrak{g}$-module. We will always view submodules as $\mathfrak{g}$-modules in this way.

Proposition. (*i*) *If $f : M \to N$ is a morphism of $\mathfrak{g}$-modules, then the kernel of f is a submodule of M and the image of f is a submodule of N.*

(*ii*) *If M is a $\mathfrak{g}$-module and $N \subseteq M$ is a submodule of M, there is a unique $\mathfrak{g}$-module structure on the quotient space M/N such that the canonical function $p : M \to M/N$ is a morphism of $\mathfrak{g}$-modules. With respect to that structure, we have*

$$x \cdot (m + N) = x \cdot m + N$$

for all $x \in \mathfrak{g}$ and all $m \in M$.

Proof. (*i*) Let $f : M \to N$ be a morphism of $\mathfrak{g}$-modules. If $x \in \mathfrak{g}$ and $m \in \ker f$, then we have $f(x \cdot m) = x \cdot f(m) = x \cdot 0 = 0$, so that m is in fact in $\ker$. Similarly, if $n \in \operatorname{im} f$ and $m \in M$ is such that $f(m) = n$, we have that $x \cdot n = x \cdot f(m) = f(x \cdot m) \in \operatorname{im} f$. This tells us that $\ker f$ is a submodule of M and that $\operatorname{im} f$ is a submodule of N.

(*ii*) Let M be a $\mathfrak{g}$-module and let N be a submodule of M. Let $\rho_M : \mathfrak{g} \to \mathfrak{gl}(M)$ be the representation of $\mathfrak{g}$ corresponding to M and let $p : M \to M/N$ be the canonical function onto the quotient vector space M/N. If $x \in \mathfrak{g}$, the linear function

$$m \in M \mapsto \rho_M(x)(m) + N \in M/N$$

maps the subspace N to 0, so there exists a unique linear function $\rho_{M/N}(x) : M/N \to M/N$ such that

$$\rho_{M/N}(x)(m + N) = \rho_M(x)(m) + N$$

for all $m \in M$. In this way, we obtain a function $\rho_{M/N} : \mathfrak{g} \to \mathfrak{gl}(M/N)$. It is linear: if $x, y \in \mathfrak{g}$ and $a, b \in \Bbbk$, we have for all $m \in M$ that

$$\begin{aligned}\rho_{M/N}(ax + by)(m + N) &= \rho_M(ax + by)(m) + N \\ &= a\rho_M(x)(m) + b\rho_M(y)(m) + N \\ &= a\rho_{M/N}(x)(m) + b\rho_{M/N}(y)(m),\end{aligned}$$

so that $\rho_{M/N}(ax + by) = a\rho_{M/N}(x) + b\rho_{M/N}(y)$. The action $\cdot : \mathfrak{g} \times M/N \to M/N$ corresponding to the function $\rho_{M/N}$ is such that

$$x \cdot (m + N) = x \cdot m + N \tag{9}$$

for all $x \in \mathfrak{g}$ and all $m \in M$, and then we can compute that for all $x, y \in \mathfrak{g}$ and all $m \in M$ we have

$$x \cdot y \cdot (m+N) - y \cdot x \cdot (m+N) = (x \cdot y \cdot m - y \cdot x \cdot m) + N = [x, y] \cdot m + N = [x, y] \cdot (m+N).$$

it follows that $\rho_{M/N}$ is a representation of $\mathfrak{g}$ on M/N. With respect to this structure, the function $p : M \to M/N$ is a morphism of $\mathfrak{g}$-modules: indeed, the equation (9) means precisely that $x \cdot p(m) = p(x \cdot m)$ for all $x \in \mathfrak{g}$ and all $m \in M$. □

2.4. Submodules of M/N.] [

Examples and constructions

Trivial modules

2.5. If M is a vector space, the zero bilinear function $\cdot : \mathfrak{g} \times M \to M$ turns M into a $\mathfrak{g}$-module, and we say that a $\mathfrak{g}$-module whose action is the zero function is ***trivial***. In particular, if we endow the vector space $\Bbbk$ with its trivial $\mathfrak{g}$-module structure we obtain a $\mathfrak{g}$-module which we call the ***trivial $\mathfrak{g}$-module***. Whenever we view $\Bbbk$ as a $\mathfrak{g}$-module it will be with respect to this trivial structure.

2.6. If M is a module, the ***invariant subspace*** of M is

$$M^{\mathfrak{g}} = \{m \in M : x \cdot m = 0 \text{ for all } x \in \mathfrak{g}\}.$$

The key properties of the invariant subspace are the following:

Proposition. (*i*) *If M is a module, then the invariant subspace $M^{\mathfrak{g}}$ is the unique maximal trivial submodule of M.*

(*ii*) *If M and N are modules and $f : M \to N$ is a morphism of modules, then $f(M^{\mathfrak{g}}) \subseteq N^{\mathfrak{g}}$ and the restriction $f|_{M^{\mathfrak{g}}} : M^{\mathfrak{g}} \to N^{\mathfrak{g}}$ is a morphism of modules.*

(*iii*) *if M is a $\mathfrak{g}$-module, a linear function $f : \Bbbk \to M$ is a morphism of $\mathfrak{g}$-modules if and only if $f(1) \in M^{\mathfrak{g}}$. There is therefore a function*

$$\Phi : f \in \hom_{\mathfrak{g}}(\Bbbk, M) \mapsto f(1) \in M^{\mathfrak{g}}$$

and it is an isomorphism of vector spaces.

Proof. (*i*) Let M be a module. If $m \in M^{\mathfrak{g}}$, then for all $x \in \mathfrak{g}$ we have $x \cdot m = 0 \in M$: this tells us that $M^{\mathfrak{g}}$ is a submodule of M. It is clear from this also that $M^{\mathfrak{g}}$ is a trivial submodule. To show it is the unique maximal one, let us suppose that N is a trivial submodule of M. If $n \in N$, then the triviality of N means that $x \cdot n = 0$ for all $x \in \mathfrak{g}$, and this tells us that $n \in M^{\mathfrak{g}}$: we thus see that $N \subseteq M^{\mathfrak{g}}$, as we wanted.

(*ii*) In the situation of the proposition, if $m \in M$ and $x \in \mathfrak{g}$ we have $x \cdot f(m) = f(x \cdot m) = 0$, so that $f(m) \in N^{\mathfrak{g}}$. This shows that the first claim is true, and the second one is then immediate.

(*iii*) Let M be a $\mathfrak{g}$-module and let $f : \Bbbk \to M$ be a linear function. If $f(1)$ is in $M^{\mathfrak{g}}$, then for all $m \in \Bbbk$ we have $f(m) = mf(1) \in M^{\mathfrak{g}})$ and therefore $f(x \cdot m) = 0 = x \cdot f(m)$. This shows that f is a morphism of $\mathfrak{g}$-modules in that case. Conversely, if f is a morphism of $\mathfrak{g}$-modules, for all $x \in \mathfrak{g}$ we have that $x \cdot f(1) = f(x \cdot 1) = f(0) = 0$, so that $f(1) \in M^{\mathfrak{g}}$.

Let us now show that the function Φ defined in the statement of the proposition, which is easily seen to be linear, is an isomorphism. If $f \in \hom_{\Bbbk}(\Bbbk, M)$ is such that $\Phi(f) = f(1) = 0$, then of course f is the function, since 1 generates $\Bbbk$ as a vector space: the function Φ is thus injective. On the other hand, if $m \in M^{\mathfrak{g}}$ then the linear function $f : \lambda \in \Bbbk \mapsto \lambda m \in M$ is a morphism of $\mathfrak{g}$-modules according to what we have already proved, and clearly $\Phi(f) = m$. This proves that Φ is surjective. □

2.7. If M is a $\mathfrak{g}$-module, we denote $[\mathfrak{g}, M]$ the subspace of M generated by the set

$$M' = \{x \cdot m : x \in \mathfrak{g}, m \in M\}.$$

We claim that $[\mathfrak{g}, M]$ is a submodule of M. Indeed, since every element of M' is a linear combination of elements of M' and the action of $\mathfrak{g}$ on M is linear, to see that $[\mathfrak{g}, M]$ is a submodule it is enough to check that for all $x \in \mathfrak{g}$ and all $m \in M'$ we have $x \cdot m \in [\mathfrak{g}, M]$, and this is immediate, since in fact $x \cdot m$ belongs to M'.

It follows from this that the quotient $M/[\mathfrak{g}, M]$ has a canonical $\mathfrak{g}$-module structure. We call it the ***space of coinvariants*** of M and denote it $M_{\mathfrak{g}}$. It has properties dual to the invariant subspace:

Proposition. (*i*) *If M is a $\mathfrak{g}$-module, then the space of coinvariants M_g is a trivial $\mathfrak{g}$-module. The subspace $[\mathfrak{g}, M]$ is the unique minimal $\mathfrak{g}$-submodule N of M such that the quotient M/N is trivial.*

(*ii*) *If $f : M \to N$ is a morphism of $\mathfrak{g}$-modules, then $f([\mathfrak{g}, M]) \subseteq [\mathfrak{g}, N]$ and there is a unique linear function $f_{\mathfrak{g}} : M_{\mathfrak{g}} \to N_{\mathfrak{g}}$ such that $f_{\mathfrak{g}}(m + [\mathfrak{g}, M]) = f(g) + [\mathfrak{g}, N]$ for all $m \in M$.*

Proof. (*i*) If $m \in M$, then in $M_{\mathfrak{g}}$ we have for all $x \in \mathfrak{g}$ that

$$x \cdot (m + [\mathfrak{g}, M]) = x \cdot m + [\mathfrak{g}, M] = [\mathfrak{g}, M],$$

which is the zero element of $M_{\mathfrak{g}}$, because $x \cdot m \in [\mathfrak{g}, M]$. This tells us that $M_{\mathfrak{g}}$ is a trivial module.

Suppose now that N is a submodule of $\mathfrak{g}$ such that the quotient M/M is a trivial $\mathfrak{g}$-module. If $x \in \mathfrak{g}$ and $m \in M$, we then have that

$$x \cdot (m + N) = x \cdot m + N = N,$$

so that in fact $x \cdot m \in N$. This tells us that the set M' defined above is contained in N and, therefore, that the subspace $[\mathfrak{g}, M]$ of M, which is generated by M', is also contained there. This proves what we want.

(*i*) Let $f : M \to N$ be a morphism of $\mathfrak{g}$-modules. To show that $f([\mathfrak{g}, M]) \subseteq [\mathfrak{g}, N]$ it is enough that we show that $f(M') \subseteq [\mathfrak{g}, N]$, for the set M' generates the subspace $[\mathfrak{g}, M]$ and the function f is linear. This is immediate: if $x \in \mathfrak{g}$ and $m \in M$, then the fact that f is a morphism of $\mathfrak{g}$-modules tells us that $f(x \cdot m) = x \cdot f(m) \in N' \subseteq [\mathfrak{g}, N]$. The first claim of (*ii*) is thus proved, and the second one follows immediately from it. □

The adjoint representation

2.8. Let $\mathfrak{g}$ be a Lie algebra. Since the bracket of $\mathfrak{g}$ is a linear function, for each $x \in \mathfrak{g}$ the function

$$\mathsf{ad}(x) : y \in \mathfrak{g} \mapsto [x, y] \in \mathfrak{g}$$

is linear, and there is therefore a function $\mathsf{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $\mathsf{ad}(x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$. It is itself a linear function and it is moreover a representation of $\mathfrak{g}$. To see this, let us observe that the action $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ corresponding to the function ad is simply the bracket of $\mathfrak{g}$, so that $x \cdot y = [x, y]$ whenever $x, y \in \mathfrak{g}$, and that this action satisfies condition (8): if $x, y \in \mathfrak{g}$ and $z \in \mathfrak{g}$, then

$$x \cdot y \cdot z - y \cdot x \cdot z = [x, [y, z]] - [y, [x, z]],$$

and this is equal to

$$[[x, y], z] = [x, y] \cdot z$$

precisely because Jacobi's condition holds in $\mathfrak{g}$. We call the representation ad the ***adjoint representation*** of $\mathfrak{g}$ and the corresponding $\mathfrak{g}$-module $\mathfrak{g}$ the ***adjoint module*** of $\mathfrak{g}$.

Spaces of homomorphisms

2.9. If M and N are $\mathfrak{g}$-modules, there is a $\mathfrak{g}$-module structure on the vector space $\hom_{\mathbb{k}}(M, N)$ of all linear maps $M \to N$ with action $\cdot : \mathfrak{g} \times \hom_{\mathbb{k}}(M, N) \to \hom_{\mathbb{k}}(M, N)$ such that for all $x \in \mathfrak{g}$, $f \in \hom_{\mathbb{k}}(M, N)$ and $m \in M$ we have

$$(x \cdot f)(m) = x \cdot f(m) - f(x \cdot m). \tag{10}$$

To see that this does define a $\mathfrak{g}$-module structure on $\hom_{\mathbb{k}}(M, N)$ we have to do the following calculation: if x and y are elements of $\mathfrak{g}$ and f is element of $\hom_{\mathbb{k}}(M, N)$, then for all $m \in M$ we have that

$$\begin{aligned}(x \cdot y \cdot f)(m) &= x \cdot (y \cdot f)(m) - (y \cdot f)(x \cdot m) \\ &= x \cdot y \cdot f(m) - x \cdot f(y \cdot m) - y \cdot f(x \cdot m) + f(y \cdot x \cdot m)\end{aligned}$$

and, similarly,

$$(y \cdot x \cdot f)(m) = y \cdot x \cdot f(m) - f(x \cdot m) - x \cdot f(y \cdot m) + f(x \cdot y \cdot m),$$

so that

$$\begin{aligned}(x \cdot y \cdot f - y \cdot x \cdot f)(m) &= x \cdot y \cdot f(m) - y \cdot x \cdot f(m) - f(x \cdot y \cdot m - y \cdot x \cdot m) \\ &= [x, y] \cdot f(m) - f([x, y] \cdot m) \\ &= ([x, y] \cdot f)(m).\end{aligned}$$

This tells us that in fact

$$x \cdot y \cdot f - y \cdot x \cdot f = [x, y] \cdot f,$$

which is the required compatibility relation.

2.10. The module structure on hom-spaces described above is compatible with the usual operations on these spaces. For example, maps induced by composition are morphisms of $\mathfrak{g}$-modules:

Proposition. *Let M, N and P be $\mathfrak{g}$-modules. If $f : M \to N$ is a morphism of $\mathfrak{g}$-modules, then the linear maps*

$$f^* : g \in \mathsf{hom}_{\Bbbk}(N,P) \mapsto g \circ f \in \mathsf{hom}_{\Bbbk}(M,P)$$

and

$$f_* : g \in \mathsf{hom}_{\Bbbk}(P,M) \mapsto f \circ f \in \mathsf{hom}_{\Bbbk}(P,N)$$

are morphisms of $\mathfrak{g}$-modules.

Proof. Let $f : M \to N$ be a morphism of $\mathfrak{g}$-modules. The first map f^* described in the proposition is a morphism of $\mathfrak{g}$-modules because for all $x \in \mathfrak{g}$ and all $g \in \mathsf{hom}_{\Bbbk}(N,P)$, we have that

$$\begin{aligned} f^*(x \cdot g)(m) &= (x \cdot g)(f(m)) \\ &= x \cdot g(f(m)) - g(x \cdot f(m)) \\ &= x \cdot g(f(m)) - g(f(x \cdot m)) \\ &= x \cdot f^*(g)(m) - f^*(g)(x \cdot m) \\ &= (x \cdot f^*(g))(m) \end{aligned}$$

for all $m \in M$, so that $f^*(x \cdot g) = x \cdot f^*(g)$.

Similarly, the second map in the proposition is a morphism because for all $x \in \mathfrak{g}$ and all $g \in \mathsf{hom}_{\Bbbk}(P,M)$ we have

$$\begin{aligned} f_*(x \cdot g)(p) &= f((x \cdot g)(p)) \\ &= f(x \cdot g(p)) - f(g(x \cdot p)) \\ &= x \cdot f(g(p)) - f(g(x \cdot p)) \\ &= x \cdot f_*(g)(p) - f_*(g)(x \cdot p) \\ &= (x \cdot f_*(g))(p) \end{aligned}$$

for all $p \in P$, so that $f_*(x \cdot g) = x \cdot f_*(g)$. □

2.11. The following very simple observation is extremely useful:

Proposition. *If M and N are $\mathfrak{g}$-modules, then we have* $\mathsf{hom}_{\Bbbk}(M,N)^{\mathfrak{g}} = \mathsf{hom}_{\mathfrak{g}}(M,N)$.

It is important to notice that the claimed equality makes sense: the space $\mathsf{hom}_{\mathfrak{g}}(M,N)$ of morphisms of $\mathfrak{g}$-modules is by definition a subspace of $\mathsf{hom}_{\Bbbk}(M,N)$.

Proof. A linear map $f : M \to N$ is a morphism of $\mathfrak{g}$-modules if and only if for all $x \in \mathfrak{g}$ and all $m \in M$ we have that $x \cdot f(m) = f(x \cdot m)$, and this condition clearly holds if and only if we have $x \cdot f = 0$ for all $x \in \mathfrak{g}$, that is, if $f \in \mathsf{hom}_{\Bbbk}(M,N)^{\mathfrak{g}}$. □

2.12. If M is a $\mathfrak{g}$-module, then we write M^* its dual space, which is simply $\hom_{\Bbbk}(M, \Bbbk)$. Viewing the vector space $\Bbbk$ that appears here as endowed with its trivial $\mathfrak{g}$-module structure, the construction of 2.9 turns M^* into a $\mathfrak{g}$-module. We call M^* the ***dual $\mathfrak{g}$-module*** of M or the ***contragredient representation***. The action $\cdot : \mathfrak{g} \times M^* \to M^*$ is such that

$$(x \cdot \phi)(m) = -\phi(m)$$

for all $x \in \mathfrak{g}$, all $\phi \in M^*$ and all $m \in M$; this is simply what the formula (10) for the action tells us in this case, since $\Bbbk$ is a trivial module.

As a specific example of this, we may consider the dual representation $\mathfrak{g}^*$ of the adjoint representation of $\mathfrak{g}$ that we described in 2.8. We call $\mathfrak{g}^*$ the ***coadjoint representation*** of $\mathfrak{g}$.

Tensor products

2.13. Let M and N be two $\mathfrak{g}$-modules, and let $\rho_M : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho_N : \mathfrak{g} \to \mathfrak{gl}(N)$ be the associated representations of $\mathfrak{g}$. If $x \in \mathfrak{g}$, then we may consider the linear map

$$\rho(x) = \rho_M(x) \otimes \mathrm{id}_N + \mathrm{id}_M \otimes \rho_N(x) : M \otimes N \to M \otimes N,$$

and in this way we obtain a function $\rho : \mathfrak{g} \to \mathfrak{gl}(M \otimes N)$ which is easily seen to be linear. The corresponding action of $\mathfrak{g}$ on $M \otimes N$ is the unique bilinear map $\cdot : \mathfrak{g} \times (M \otimes N) \to M \otimes N$ such that for all $x \in \mathfrak{g}$, $m \in M$ and $n \in N$ has

$$x \cdot m \otimes n = (x \cdot m) \otimes n + m \otimes (x \cdot n).$$

We claim this turns $M \otimes N$ into a $\mathfrak{g}$-module. Indeed, if $x, y \in \mathfrak{g}$, $m \in M$ and $n \in N$, then we have

$$\begin{aligned} x \cdot y \cdot m \otimes n &= x \cdot \big((y \cdot m) \otimes n + m \otimes (y \cdot n)\big) \\ &= (x \cdot y \cdot m) \otimes n + (y \cdot m) \otimes (x \cdot n) + (x \cdot m) \otimes (y \cdot n) + m \otimes (x \cdot y \cdot n) \end{aligned}$$

and, similarly,

$$y \cdot x \cdot m \otimes n = (y \cdot x \cdot m) \otimes n + (x \cdot m) \otimes (y \cdot n) + (y \cdot m) \otimes (x \cdot n) + m \otimes (y \cdot x \cdot n).$$

Subtracting, we see that

$$\begin{aligned} &x \cdot y \cdot m \otimes n - y \cdot x \cdot m \otimes n \\ &\quad = (x \cdot y \cdot m) \otimes n - (y \cdot x \cdot m) \otimes n + m \otimes (x \cdot y \cdot n) - m \otimes (y \cdot x \cdot n) \\ &\quad = (x \cdot y \cdot m - y \cdot x \cdot m) \otimes n + m \otimes (x \cdot y \cdot n - y \cdot x \cdot n) \\ &\quad = ([x, y] \cdot m) \otimes n + m \otimes ([x, y] \cdot n) \\ &\quad = [x, y] \cdot m \otimes n. \end{aligned}$$

This means that the equality

$$x \cdot y \cdot t - y \cdot x \cdot t = [x, y] \cdot t$$

for all $x, y \in \mathfrak{g}$ and all elementary tensors t of $M \otimes N$. Since the elementary tensors generate $M \otimes N$ as a vector space, this implies at once that the equation holds in fact for all t in $M \otimes N$, an this proves that we do have a $\mathfrak{g}$-module structure on $M \otimes N$, as we claimed.

2.14. The usual properties of the tensor product of vector spaces hold in the context of modules over a Lie algebra:

Proposition. (*i*) *If M, N and P are $\mathfrak{g}$-modules, then the linear map*

$$\alpha : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$$

such that $\alpha(m \otimes (n \otimes p)) = \alpha((m \otimes n) \otimes p)$ for all $m \in M$, $n \in N$ and $p \in P$ is an isomorphism of $\mathfrak{g}$-modules.

(*ii*) *If M and N are $\mathfrak{g}$-modules, then the linear map*

$$\beta : M \otimes N \to N \otimes M$$

such that $\beta(m \otimes n) = n \otimes m$ for all $m \in M$ and all $n \in N$ is an isomorphism of $\mathfrak{g}$-modules.

(*iii*) *If M is a $\mathfrak{g}$-module, then the linear maps*

$$\lambda : m \in M \mapsto 1 \otimes m \in \Bbbk \otimes M, \qquad \rho : m \in M \mapsto m \otimes 1 \in M \otimes \Bbbk$$

are isomorphisms of $\mathfrak{g}$-modules.

Proof. We know from linear algebra that there are maps α, β, λ and ρ as described in these three statements and that they are isomorphisms of vector spaces. In order to prove the proposition, we need only show that they are morphisms of $\mathfrak{g}$-modules and this follows from a direct computation in each case. □

2.15. Similarly, the well-known adjoint relation between $\hom_\Bbbk$ and $\otimes$ is compatible with module structures:

Proposition. *Let M, N and P be $\mathfrak{g}$-modules. The linear map*

$$\Phi : \hom_\Bbbk(M \otimes N, P) \to \hom_\Bbbk(N, \hom_\Bbbk(M, P))$$

such that $\Phi(f)(n)(m) = f(m \otimes n)$ for all $f \in \hom_\Bbbk(M \otimes N, P)$, $n \in N$ and $m \in M$ is an isomorphism of $\mathfrak{g}$-modules.

Proof. We know from linear algebra that there is such a linear map Φ and that it is an isomorphism of vector spaces, so we need only show that that map is a morphism of $\mathfrak{g}$-modules. Let $x \in \mathfrak{g}$ and $f \in \hom_\Bbbk(M \otimes N, P)$. If $m \in M$ and $n \in N$, we have that

$$\begin{aligned} \Phi(x \cdot f)(n)(m) &= (x \cdot f)(m \otimes n) \\ &= x \cdot f(m \otimes n) - f(x \cdot m \otimes n) \\ &= x \cdot f(m \otimes n) - f((x \cdot m) \otimes n) - f(m \otimes (x \cdot n)). \end{aligned} \tag{11}$$

On the other hand,

$$(x \cdot \Phi(f))(n) = x \cdot \Phi(f)(n) - \Phi(f)(x \cdot n),$$

so that

$$\begin{aligned}(x\cdot\Phi(f))(n)(m) &= (x\cdot\Phi(f)(n))(m) - \Phi(f)(x\cdot n)(m)\\ &= x\cdot\Phi(f)(n)(m) - \Phi(f)(n)(x\cdot m) - \Phi(f)(x\cdot n)(m)\\ &= x\cdot f(m\otimes n) - f((x\cdot m)\otimes n) - f(m\otimes(x\cdot n)).\end{aligned}$$

Comparing this with (11) we see that

$$\Phi(x\cdot f)(n)(m) = (x\cdot\Phi(f))(n)(m),$$

and this equality, which holds for all $n\in N$ and all $m\in M$, implies at once that $\Phi(x\cdot f) = x\cdot\Phi(f)$. This shows that Φ is a morphism of $\mathfrak{g}$-modules, as we wanted. □

§3. The Lie algebra $\mathfrak{sl}_2(\Bbbk)$ and its finite-dimensional representations

3.1. We fix an algebraically closed field $\Bbbk$ of characteristic zero and let $\mathfrak{sl}_2(\Bbbk)$ be the Lie subalgebra of $\mathfrak{gl}_2(\Bbbk)$ consisting of those matrices with trace equal to zero. We put

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The set $\{E, H, F\}$ is a basis for $\mathfrak{sl}_2(\Bbbk)$ and a computation show that we have

$$[H, E] = 2E, \qquad [E, F] = H, \qquad [H, F] = -2E.$$

In this chapter we will work almost exclusively with this Lie algebra, which we will write simply as $\mathfrak{g}$ and to whom we will omit omit explicit references unless they are needed. In particular, we will speak of modules instead of $\mathfrak{sl}_2(\Bbbk)$-modules, and so on.

Simple modules

3.2. A module is ***simple*** if it is nonzero and it does not have any non-zero proper submodules. Our first result is that somewhat miraculously we have a complete description of all finite-dimensional simple modules over $\mathfrak{sl}_2(\Bbbk)$. The basic observation that allows us to do this is the following:

Lemma. *Let M be a module. If $m \in M$ is an eigenvector for H_M of eigenvalue $\lambda \in \Bbbk$ then*

$$H \cdot E \cdot m = (\lambda + 2)E \cdot m, \qquad H \cdot F \cdot m = (\lambda - 2)F \cdot m.$$

In particular, if the vector $E \cdot m$ is non-zero, then it is an eigenvector of H_M of eigenvalue $\lambda - 2$ and if the vector $F \cdot m$ is non-zero, then it is an eigenvector of H_M of eigenvalue $\lambda - 2$.

Proof. Let $m \in M$ be a non-zero vector and let $\lambda \in \Bbbk$ be such that $H \cdot m = \lambda m$. We have

$$H \cdot E \cdot m = E \cdot H \cdot m + [H, E] \cdot m = \lambda E \cdot m + 2E \cdot m = (\lambda + 2)E \cdot m$$

and, similarly, $H \cdot F \cdot m = (\lambda - 2)F \cdot m$, as the lemma claims. □

3.3. The lemma tells us that if M is a module, then the maps E_M and F_M interact in a very special way with the eigenspaces of H_M. Building up on this, we can completely describe simple modules:

Proposition. *Let M be a finite-dimensional simple module of dimension $r + 1$. There exists a basis $\{m_0, m_1, \dots, m_r\}$ of M such that for each $i \in \{0, \dots, r\}$ we have*

$$E \cdot m_i = \begin{cases} 0, & \text{if } i = 0; \\ (r - i + 1)m_{i-1}, & \text{if } 0 < i \le r; \end{cases}$$

$$H \cdot m_i = (r - 2i)m_i;$$

$$F \cdot m_i = \begin{cases} (i + 1)m_{i+1}, & \text{if } 0 \le i < r; \\ 0, & \text{if } i = r. \end{cases}$$

Proof. Let M be a simple finite-dimensional module. We consider on our ground field $\Bbbk$ the partial order $\leq$ such that whenever λ and μ are elements of $\Bbbk$ we have

$$\lambda \leq \mu \iff \text{there exists an } i \in \mathbb{N}_0 \text{ such that } \mu - \lambda = 2i.$$

Since $\Bbbk$ is algebraically closed, we know that the map $H_M : M \to M$ has eigenvalues and, since it has finitely many, it is clear that there exists an eigenvalue λ of H_M which is maximal with respect to the order of $\Bbbk$. Let $m \in M$ be an eigenvector for H_M corresponding to λ. As

$$H \cdot E \cdot m = E \cdot H \cdot M + 2E \cdot m = (\lambda + 2)E \cdot m,$$

we must have that

$$E \cdot m = 0, \tag{12}$$

for otherwise $E \cdot m$ would be an eigenvector de H_M corresponding to the eigenvalue $\lambda + 2$ and $\lambda < \lambda + 2$ in $\Bbbk$, contradicting the choice of λ.

If $j \in \mathbb{N}_0$ is such that $F^j \cdot m \neq 0$, then the $j + 1$ vectors

$$m, \quad F \cdot m, \quad F^2 \cdot m, \quad \dots, \quad F^j \cdot m \tag{13}$$

are all non-zero. Since m is an eigenvector of H_M of eigenvalue λ, a straightforward induction using Lemma 3.2 shows that we more generally have that

$$H \cdot F^i \cdot m = (\lambda - 2i)F^i \cdot m \qquad \text{for all } i \in \{0, \dots, j\}. \tag{14}$$

It follows that the $j + 1$ vectors listed in (13) are eigenvectors of H_M corresponding to distinct eigenvalues and are therefore linearly independent. Of course, this implies that $j + 1$ is at most equal to $\dim M$. We thus see that we may consider the number

$$\ell = \max\{j \in \mathbb{N}_0 : F^j \cdot m \neq 0\},$$

for the set whose maximum we are taking is non-empty and bounded.

Let us write $m_i = \frac{1}{i!}F^i \cdot m$ for each $i \in \{0, \dots, \ell\}$. We want to see how the elements of $\mathfrak{g}$ act on these vectors.

- From their very definition and the choice of the number ℓ, it is clear that we have

$$F \cdot m_i = \begin{cases} (i + 1)m_{i+1}, & \text{if } 0 \leq i < \ell; \\ 0, & \text{if } i = \ell. \end{cases} \tag{15}$$

- On the other hand, the equalities (14) tell us that

$$H \cdot m_i = (\lambda - 2i)m_i \qquad \text{for all } i \in \{0, \dots, \ell\}. \tag{16}$$

- Finally, we claim that

$$E \cdot m_i = \begin{cases} 0, & \text{if } i = 0; \\ (\lambda - i + 1)m_{i-1}, & \text{if } 0 < i \leq \ell. \end{cases} \tag{17}$$

  That this holds when $i = 0$ is precisely the content of the equality (12). If instead $i = 1$, we have

$$E \cdot m_1 = E \cdot F \cdot m = F \cdot E \cdot m + H \cdot m = \lambda m = 1(\lambda - 1 + 1)m,$$

  so that (17) also holds in this case. Finally, if we suppose that $1 \leq i < \ell$ and that (17) holds for i, we have

$$\begin{aligned} E \cdot m_{i+1} &= \frac{1}{i+1} E \cdot F \cdot m_i = \frac{1}{i+1} F \cdot E \cdot m_i + \frac{1}{i+1} H \cdot m_i \\ &= \frac{\lambda - i + 1}{i+1} F \cdot m_{i-1} + \frac{\lambda - 2i}{i+1} m_i \\ &= (\lambda - i)m_i. \end{aligned}$$

We thus see that the subspace $\langle m_0, m_1, \ldots, m_\ell \rangle$ is a submodule of M: indeed, the equalities (15), (16) and (17) tell us that this subspace is preserved under the action of F, H and E, respectively. Now, M does not have any proper non-zero submodules, so that subspace must coincide with M itself. It follows that $\dim M = \ell + 1$ and, since

$$\begin{aligned} 0 &= E \cdot F^{\ell+1} \cdot m = \ell!\, E \cdot F \cdot m_\ell = \ell!\, F \cdot E \cdot m_\ell + \ell!\, H \cdot m_\ell = \ell!\big(\ell(\lambda - \ell + 1) + \lambda - 2\ell\big)m_\ell \\ &= (\ell + 1)!(\lambda - \ell)m_\ell \end{aligned}$$

and $m_\ell \neq 0$, that $\lambda = \ell$. Using this last equality, we see at once that the identities that the proposition claims are precisely those in (15), (16) and (17). □

3.4. We can restate the description of simple modules provided by Proposition 3.3 in terms of matrices: the proposition tells us that if M is a finite-dimensional simple module of dimension $r + 1$, then there exists an ordered basis $\mathscr{B}$ of M such that the matrices of the linear maps $E_M, H_M, F_M : M \to M$ with respect to $\mathscr{B}$ are

$$\|E_M\|_{\mathscr{B}} = \begin{pmatrix} 0 & r & 0 & 0 & & & & \\ 0 & 0 & r-1 & 0 & & & & \\ 0 & 0 & 0 & r-2 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 2 & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & & 0 & 0 & 0 \end{pmatrix},$$

$$\|H_M\|_{\mathscr{B}} = \begin{pmatrix} r & 0 & 0 & 0 & & & & \\ 0 & r-2 & 0 & 0 & & & & \\ 0 & 0 & r-4 & 0 & & & & \\ 0 & 0 & 0 & r-6 & & & & \\ & & & & \ddots & & & \\ & & & & & -r+4 & 0 & 0 \\ & & & & & 0 & -r+2 & 0 \\ & & & & & 0 & 0 & -r \end{pmatrix}$$

and

$$\|F_M\|_{\mathscr{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 2 & 0 & 0 & & & & \\ 0 & 0 & 3 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & r-1 & 0 & 0 \\ & & & & & 0 & r & 0 \end{pmatrix}.$$

3.5. A direct consequence of Proposition 3.3 is the following:

Proposition. *Two finite-dimensional simple modules of the same dimension are isomorphic.*

Proof. Let us suppose that $r \in \mathbb{N}_0$ and that M and M' are two simple modules of dimension $r+1$. According to Proposition 3.3, there exist bases $\mathscr{B} = \{m_0, \dots, m_r\}$ and $\mathscr{B}' = \{m'_0, \dots, m'_r\}$ of M and M' such that the action of E, H and F on the elements of $\mathscr{B}$ and of $\mathscr{B}'$ are given by the formulas given in that proposition. This implies at once that the linear map $f : M \to M'$ such that $f(m_i) = m'_i$ for each $i \in \{0, \dots, r\}$ is an isomorphism of modules. □

3.6. On the other hand, the description of finite-dimensional simple modules provided by Proposition 3.3 suggests a way of proving the existence of such modules:

Proposition. *For each $r \in \mathbb{N}_0$ there exist simple modules of dimension $r+1$.*

Proof. Let $r \in \mathbb{N}_0$ and consider a vector space M of dimension $r+1$ with an ordered basis $\mathscr{B} = (m_0, \dots, m_r)$. We define linear maps $e, h, f : M \to M$ mimicking the formulas of Proposition 3.3, so that for each $i \in \{0, \dots, r\}$ we have

$$e(m_i) = \begin{cases} 0, & \text{if } i = 0; \\ (r-i+1)m_{i-1}, & \text{if } 0 < i \le r; \end{cases}$$

$$h(m_i) = (r-2i)m_i;$$

$$f(m_i) = \begin{cases} (i+1)m_{i+1}, & \text{if } 0 \le i < r; \\ 0, & \text{if } i = r. \end{cases}$$

We claim that these linear maps satisfy the relations

$$[h,e] = 2e, \qquad [e,f] = h, \qquad [h,f] = -2f \tag{18}$$

in the Lie algebra $\mathfrak{gl}(M)$. To check this it is enough, in each case, to compute the result of applying both sides of the equalities to each element of the basis $\mathscr{B}$, and this can be done by direct computation. For example, let us do this for the first of the three equalities. Let $i \in \{0,\dots,r\}$. We have

$$\begin{aligned} m_i &\overset{e}{\longmapsto} \begin{cases} 0, & \text{if } i = 0; \\ (r-i+1)m_{i-1}, & \text{if } 0 < i \le r; \end{cases} \\ &\overset{h}{\longmapsto} \begin{cases} 0, & \text{if } i = 0; \\ (r-i+1)(r-2i+2)m_{i-1}, & \text{if } 0 < i \le r; \end{cases} \end{aligned}$$

and

$$\begin{aligned} m_i &\overset{h}{\longmapsto} (r-2i)m_i \\ &\overset{e}{\longmapsto} \begin{cases} 0, & \text{if } i = 0; \\ (r-2i)(r-i+1)m_{i-1}, & \text{if } 0 < i \le r; \end{cases} \end{aligned}$$

and then $[h,e](m_0) = 0$ and if $0 < i \le r$

$$\begin{aligned} [h,e](m_i) &= \big((r-i+1)(r-2i+2) - (r-2i)(r-i+1)\big)m_{i-1} = 2(r-i+1)m_{i-1} \\ &= 2e(m_{i-1}). \end{aligned}$$

It follows from this that $[h,e] = 2e$, as we wanted.

The fact that the equalities (18) are satisfied implies at once that the linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(M)$ such that $\rho(E) = e$, $\rho(H) = h$ and $\rho(F) = f$ is a morphism of Lie algebras, and this map ρ turns M into a module over $\mathfrak{g}$. As $\dim M = r+1$, to prove the proposition it is enough that we show that M is a simple module.

In order to do that, let us suppose that N is a non-zero submodule of M and let n be a non-zero element of N. As $\mathscr{B}$ is a basis of M, there exist scalars $a_0,\dots,a_r \in \Bbbk$ such that $n = a_0 m_0 + \cdots + a_r m_r$, and since N is a submodule of M for each $i \in \mathbb{N}_0$ we have that

$$a_0 r^i m_0 + a_1(r-2)^i m_1 + \cdots + a_r(-r)^i m_r = a_0 H^i \cdot m_0 + \cdots + a_r H^i \cdot m_r = H^i \cdot n \in N.$$

Since the $r+1$ scalars $r, r-2, \dots, -r$ are distinct, this implies that for each $i \in \{0,\dots,r\}$ we have $a_i m_i \in N$. Now n is not the zero element, so there exists a $j \in \{0,\dots,r\}$ such that $a_j \neq 0$ and we then have $m_j \in N$. It follows from this that

$$m_0 = \frac{1}{r!} E^j \cdot m_j \in N$$

and, using again that N is a submodule, we see that for each $i \in \{0,\dots,r\}$ we have

$$m_i = i!\, F^i \cdot m_0 \in N.$$

The whole basis $\mathscr{B}$ is thus contained in N, so that, of course, $N = M$. This proves that M is simple, as we wanted. □

3.7. It is useful to observe that using the same ideas as in the proof of Proposition 3.3 we can obtain the following slightly stronger result:

Proposition. *Let M be a finite-dimensional module. If $m \in M$ is a singular weight vector of weight λ, then λ is a non-negative integer and there is a simple module S of M of dimension $\lambda+1$ containing m. If we put $m_i = \frac{1}{i!}F^i \cdot m$ for each $i \in \{0,\dots,\lambda\}$, the set $\{m_0,\dots,m_\lambda\}$ is a basis for S whose elements are weight vectors, with*

$$H \cdot m_i = (\lambda - 2i)m_i$$

for each $i \in \{0,\dots,\lambda\}$.

Proof. Using Lemma 3.2 and induction, we see at once that

$$H \cdot F^i \cdot m = (\lambda - 2i)F^i \cdot m \text{ for all } i \in \mathbb{N}_0.$$

As in the proof of Proposition 3.3, this implies that if $j \in \mathbb{N}_0$ is such that $F^j \cdot m \neq 0$ then the vectors m, $F \cdot M$, …, $F^i \cdot m$ are linearly independent. This, together with the fact that M is finite-dimensional implies that we may consider the number

$$\ell = \max\{j \in \mathbb{N}_0 : F^j \cdot m \neq 0\}.$$

For each $i \in \{0,\dots,\ell\}$ we put $m_i = \frac{1}{i!}F^i \cdot m$ and let S be the subspace of M spanned by $\mathscr{B} = \{m_0,\dots,m_\ell\}$. For each $i \in \{0,\dots,\ell\}$ we have

$$H \cdot m_i = (\lambda - 2i)m_i, \qquad F \cdot m_i = \begin{cases} (i+1)m_{i+1}, & \text{if } 0 \le i < \ell; \\ 0, & \text{if } i = \ell. \end{cases}$$

One the other hand, an induction just like the one we did in the proof of Proposition 3.3 shows that

$$E \cdot m_i = \begin{cases} 0, & \text{if } i = 0; \\ (r+1-i)m_{i-1}, & \text{if } 0 < i \le \ell. \end{cases}$$

It follows from this formulas that S is a submodule of M, and it is obvious that S is isomorphic to the simple module constructed in the proof of Proposition 3.6. This proves the proposition, as $m = m_0 \in S$. □

Schur's Lemma and the Casimir operator

3.8. If M is a module, the ***Casimir operator*** of M is the linear map

$$\Omega_M : m \in M \mapsto E \cdot F \cdot m + F \cdot E \cdot m + \tfrac{1}{2}H \cdot H \cdot m \in M.$$

This map is a natural endomorphism of M, in the following sense:

Proposition. (*i*) *If M is a module, then $\Omega_M : M \to M$ is a morphism of modules.*

(*ii*) *If $f : M \to N$ is a morphism of modules, then the square*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \Omega_M}\downarrow & & \downarrow{\scriptstyle \Omega_N} \\ M & \xrightarrow{f} & N \end{array}$$

commutes.

Proof. (*i*) Let M be a module. That the map $\Omega_M : M \to M$ is an endomorphism of modules follows from a direct computation. We have

$$\begin{aligned} \Omega_M(E \cdot m) &= E \cdot F \cdot E \cdot m + F \cdot E \cdot E \cdot m + \tfrac{1}{2} H \cdot H \cdot E \cdot m \\ &= E \cdot F \cdot E \cdot m + (E \cdot F \cdot E \cdot m - H \cdot E \cdot m) + \tfrac{1}{2}(H \cdot E \cdot H \cdot m + 2H \cdot E \cdot m) \\ &= E \cdot F \cdot E \cdot m + E \cdot F \cdot E \cdot m + \tfrac{1}{2} H \cdot E \cdot H \cdot m \\ &= (E \cdot E \cdot F \cdot m - E \cdot H \cdot m) + E \cdot F \cdot E \cdot m + \tfrac{1}{2}(E \cdot H \cdot H \cdot m + 2E \cdot H \cdot m) \\ &= E \cdot E \cdot F \cdot m + E \cdot F \cdot E \cdot m + \tfrac{1}{2} E \cdot H \cdot H \cdot m \\ &= E \cdot (E \cdot F \cdot m + F \cdot E \cdot m + \tfrac{1}{2} H \cdot H \cdot m) \\ &= E \cdot \Omega_M(m) \end{aligned}$$

and similar computations show that $\Omega_M(F \cdot m) = F \cdot \Omega_M(m)$ and $\Omega_M(H \cdot m) = H \cdot \Omega_M(m)$. These equalities tell us that Ω_M is a morphism of modules.

(*ii*) Let now $f : M \to N$ be a morphism of modules. If $m \in M$, we have for all x and y in $\mathfrak{g}$ that $f(x \cdot y \cdot m) = x \cdot y \cdot f(m)$, and using this we see that

$$\begin{aligned} f(\Omega_M(m)) &= f(E \cdot F \cdot m + F \cdot E \cdot m + \tfrac{1}{2} H \cdot H \cdot m) \\ &= E \cdot F \cdot f(m) + F \cdot E \cdot f(m) + \tfrac{1}{2} H \cdot H \cdot f(m) \\ &= \Omega_N(f(m)), \end{aligned}$$

and this equality es precisely what the proposition claims. □

3.9. Since we have at this point a complete description of the finite-dimensional simple modules, we are able to compute their Casimir operators. We start with the following famous observation due to Issai Schur [Sch1904].

Lemma. *If $f : M \to M$ is an endomorphism of a finite dimensional simple module, then there exists a scalar $\lambda \in \Bbbk$ such that $f = \lambda \mathrm{id}_M$.*

Proof. Since our ground field is algebraically closed, we know that the linear map f admits an eigenvalue $\lambda \in \Bbbk$; in particular, there exists a non-zero vector $m \in M$ such that $f(m) = \lambda m$. The linear function $h = f - \lambda \mathrm{id}_M : M \to M$ is a morphism of modules, so its kernel $\ker h$ is a submodule of M. As $h(m) = 0$, this submodule is non-zero, and since M is simple it must coincide with M. Of course, this means that the map $h = f - \lambda \mathrm{id}_M$ is the zero map, so that $f = \lambda \mathrm{id}_M$. □

3.10. Proposition. *If M is a finite-dimensional simple module of dimension $r+1$, then the Casimir operator of M is*

$$\Omega_M = \tfrac{1}{2}r(r+2)\mathsf{id}_M.$$

Proof. Let M be a finite-dimension finite module of dimension $r+1$. The linear map $\Omega_M : M \to M$ is an endomorphism of M, so Schur's Lemma 3.9 tells us that there exists a scalar $\lambda \in \Bbbk$ such that $\Omega_M = \lambda \mathsf{id}_M$. Now, in view of Proposition 3.3 we know that there exists an ordered basis $\{m_0, \dots, m_r\}$ of M such that the action of the generators E, H and F of $\mathfrak{g}$ is as described there. In particular, we have that

$$\begin{aligned}
\lambda m &= \Omega_M(m_0) \\
&= E \cdot F \cdot m_0 + F \cdot E \cdot m_0 + \tfrac{1}{2} H \cdot H \cdot m_0 \\
&= E \cdot m_1 + \tfrac{1}{2} r^2 m_0 \\
&= r m_0 + \tfrac{1}{2} r^2 m_0 \\
&= \tfrac{1}{2} r(r+2) m_0
\end{aligned}$$

and this implies that we must have $\lambda = \frac{1}{2}r(r+2)$, because the vector m_0 is not zero. □

Semisimplicity

3.11. We want to analyze now the structure of an arbitrary finite-dimensional module and we will do this by reducing the problem to the description we already have of simple modules. The key tool in that reduction is the following definition.

If M is a module, a ***composition series*** for M is a finite increasing sequence of submodules

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t$$

of M such that $M_0 = 0$, $M_t = M$ and for each $i \in \{1, \dots, t\}$ the quotient module M_i/M_{i-1} is simple. We call the quotients $M_1/M_0, \dots, M_n/M_{n-1}$ the ***factors*** of the composition series.

An important fact is that composition series exist in the situation which interests us:

Proposition. *Every non-zero finite-dimensional module admits a composition series.*

Proof. If M is simple, then

$$M_0 \subseteq M_1$$

is a composition series for M. If not, then among all the proper submodules of M we may pick one of maximal dimension. Call it N. As $\dim N < \dim M$, we may suppose inductively that there exists a composition series

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t$$

of N. Since N is a proper submodule of M of maximal dimension, the quotient M/N is a simple module, and therefore

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t \subseteq M$$

is a composition series for M. □

3.12. The next step is to establish a key property of the trivial module:

Proposition. *The trivial module* $\Bbbk$ *is projective relative to the class* ${}_{\mathfrak{g}}\mathsf{mod}$ *of all finite-dimensional modules.*

Proof. According to Proposition 3.11, every module in ${}_{\mathfrak{g}}\mathsf{mod}$ has a finite filtration whose subquotients are finite-dimensional simple modules. According to Proposition 5.12, then, to show that the trivial module $\Bbbk$ is projective relative to ${}_{\mathfrak{g}}\mathsf{mod}$ it is enough that we show that it is projective relative to the class of all finite-dimensional simple modules. Let then

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} \Bbbk \longrightarrow 0 \tag{19}$$

be an extension of $\Bbbk$ by a finite-dimensional simple module M. We consider two cases now:

- First, let us suppose that M is not the trivial module, and let us write ℓ its dimension. According to Proposition 3.8, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & E & \xrightarrow{g} & \Bbbk & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Omega_M} & \swarrow{\scriptstyle \tilde r} & \downarrow{\scriptstyle \Omega_E} & & \downarrow{\scriptstyle \Omega_{\Bbbk}=0} & & \\ 0 & \longrightarrow & M & \xrightarrow[f]{} & E & \xrightarrow[g]{} & \Bbbk & \longrightarrow & 0 \end{array}$$

  in which the vertical arrows are Casimir operators. Since $\Omega_{\Bbbk} = 0$, this tells us that $g \circ \Omega_E = 0$ and the exactness of the bottom row implies then that there exists a morphism of modules $\tilde r : E \to M$ such that $f \circ \tilde r = \Omega_E$. As

$$f \circ \tilde r \circ f = \Omega_E \circ f = f \circ \Omega_M$$

  and the morphism f is injective, we see that

$$\tilde r \circ f = \Omega_M = \tfrac{1}{2}\ell(\ell+2)\mathrm{id}_M.$$

  It follows at once that the map $r = \frac{2}{\ell(\ell+2)}\tilde r$ is a retraction of f and, as a consequence of this, that the extension (19) is split.
- Let us next suppose that M is a trivial module. Let $e \in E$ and let $x, y \in \mathfrak{g}$. We have $g(y \cdot e) = y \cdot g(e) = 0$, because $g(e)$ is an element of the trivial module $\Bbbk$. This implies that there exists an $m \in M$ such that $y \cdot e = f(m)$. Now, as M is also a trivial module, we have

$$x \cdot y \cdot e = x \cdot f(m) = f(x \cdot m) = 0.$$

Of course, we also have $y \cdot x \cdot e = 0$ and then, in fact, we see that

$$[x, y] \cdot e = 0$$

for all $x, y \in \mathfrak{g}$. Now the algebra $\mathfrak{g}$ is perfect, so that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and the last equality then tells us that E is itself also a trivial module. In particular, if e is any element of E such that $g(e) = 1$, then the linear map $s : \lambda \in \Bbbk \mapsto \lambda eE$ is a morphism of modules which is a section of g. The extension (19) is therefore split also in this case.

In this way we conclude that every extension of $\Bbbk$ by a finite-dimensional simple module is split, as we wanted. □

3.13. Proposition. *Every extension of finite-dimensional modules is split.*

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \tag{20}$$

be a short exact sequence of finite-dimensional modules. Applying the functor $\mathsf{hom}_{\Bbbk}(P, -)$ we obtain another short exact sequence of modules,

$$0 \longrightarrow \mathsf{hom}_{\Bbbk}(P, M) \xrightarrow{f_*} \mathsf{hom}_{\Bbbk}(P, N) \xrightarrow{g_*} \mathsf{hom}_{\Bbbk}(P, P) \longrightarrow 0 \tag{21}$$

Let us now consider the linear map

$$\phi : \lambda \Bbbk \mapsto \lambda \mathsf{id}_P \, \mathsf{hom}_{\Bbbk}(P, P),$$

which is easily seen to be a morphism of modules. Since the three modules appearing in (21) are finite-dimensional, it follows from Proposition 5.10 that there exists a morphism of modules $\bar{\phi} : \Bbbk \to \mathsf{hom}_{\Bbbk}(P, N)$ such that $g_* \circ \bar{\phi} = \phi$. In particular, the linear map $s = \bar{\phi}(1) : P \to N$ is such that

$$x \cdot s = x \cdot \bar{\phi}(1) = \bar{\phi}(x \cdot 1) = 0$$

for all $x \in \mathfrak{g}$, so that s is in fact a morphism of modules, and

$$g \circ s = g_*(s) = g_*(\bar{\phi}(1)) = \phi(1) = \mathsf{id}_P.$$

We see in this way that s is a section of the morphism g appearing in the short exact sequence (20) and therefore that that short exact sequence is split. □

3.14. We can now state the main result of this section:

Theorem. *Every finite-dimensional module is isomorphic to a direct sum of simple modules. In fact, if M is a module and*

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$$

is a composition series for M, then there exists an isomorphism

$$M \cong \bigoplus_{i=1}^{n} M_i / M_{i-1}.$$

Proof. Let M be a finite-dimensional module and let us consider a composition series like in the statement of the theorem. We proceed by induction on its length n. We may suppose that M is not the zero module, as otherwise there is nothing to prove, and then we have $n \geq 1$.

If $n = 1$, then M is simple, and the result is clear. If $n \geq 2$, then

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1}$$

is a composition series for M_{n-1}, and therefore the inductive hypothesis tells us that

$$M_{n-1} \cong \bigoplus_{i=1}^{n-1} M_i/M_{i-1}, \tag{22}$$

a direct sum of simple modules. On the other hand, we have a short exact sequence

$$0 \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0$$

and it is, according to Proposition 3.13, split, so that

$$M = M_n \cong M_{n-1} \oplus M_n/M_{n-1}.$$

Putting together this isomorphism with (22), we see that there is an isomorphism as the one whose existence the theorem claims. □

Multiplicities

3.15. We have shown that every finite-dimensional module is isomorphic to a direct sum of simple ones. We now propose to prove that this isomorphism is essentially unique. We start with a simple result, which is really a continuation of Schur's Lemma 3.9.

Lemma. *If S and T are finite-dimensional simple modules, then we have*

$$\dim \hom_{\mathfrak{g}}(S,T) = \begin{cases} 1, & \text{if } S \text{ and } T \text{ are isomorphic modules;} \\ 0, & \text{if not.} \end{cases}$$

Proof. Let S and T be finite-dimensional simple modules. If there is an isomorphism of modules $f : S \to T$, then the function

$$f^* : g \in \hom_{\mathfrak{g}}(S,S) \mapsto f \circ g \in \hom \mathfrak{g}(S,T)$$

is linear and an isomorphism, with inverse the function

$$(f^{-1})^* : g \in \hom_{\mathfrak{g}}(S,T) \mapsto f^{-1} \circ g \in \hom_{\mathfrak{g}}(S,S).$$

It follows from this that the vector spaces $\hom_{\mathfrak{g}}(S,T)$ and $\hom_{\mathfrak{g}}(S,S)$ have the same dimension, and we know from Lemma 3.9 that $\hom_{\mathfrak{g}}(S,S)$ is one dimensional, since it is generated by its non-zero element $\mathrm{id}_S : S \to S$. This proves the first claim.

Suppose next that S and T are not isomorphic and, to reach a contradiction and prove the second claim, that there is a non-zero morphism of modules $f : S \to T$. Since f is not the zero map, its kernel is a proper submodule of S: as S is simple, it follows then that $\ker f = 0$, so that f is injective. Similarly, the image of f is a non-zero submodule of T, which is also simple, so that $\operatorname{im} f = T$. We see in this way that f is in fact a isomorphism, and this is impossible in view of our hypothesis. □

3.16. Our first uniqueness result is that the number of times a simple module appears —up to isomorphism— in a direct sum decomposition of a finite-dimensional modules is independent of the particular decomposition under consideration:

Proposition. *Let M be a module and let S be a simple module. If $n \in \mathbb{N}_0$ and $S_1, \dots, S_n$ are simple modules such that $M \cong \bigoplus_{i=1}^n S_i$, then*

$$\#\{i \in \{1, \dots, n\} : S_i \cong S\} = \dim \hom_{\mathfrak{g}}(S, M), \tag{23}$$

so that the number appearing in the left hand side of the equality depends only on M and S, and not on the choice of n and the simple modules $S_1, \dots, S_n$.

In view of this, we may denote that number $[M : S]$. We call it the ***multiplicity*** of S in M.

Proof. Let $n \in \mathbb{N}$ and let $S_1, \dots, S_n$ be simple modules as in the statement. We have

$$\hom_{\mathfrak{g}}(S, M) \cong \hom_{\mathfrak{g}}\Big(S, \bigoplus_{i=1}^n S_i\Big) \cong \bigoplus_{i=1}^n \hom_{\mathfrak{g}}(S, S_i),$$

so that

$$\dim \hom_g(S, M) = \sum_{i=1}^n \dim \hom_{\mathfrak{g}}(S, S_i).$$

Using Lemma 3.15 we see immediately that this sum is equal to the number of elements of the set $\{i \in \{1, \dots, n\} : S_i \cong S\}$, and this proves the (23) of the proposition. □

3.17. Using the well-definedness of the multiplicity of direct summands, we can prove the following precise form of the uniqueness of direct sum decompositions:

Proposition. *Let M be a finite-dimensional module, and suppose that $m, n \in \mathbb{N}_0$ and that $S_1, \dots, S_m$, $T_1, \dots, T_n$ are simple modules. If $M \cong \bigoplus_{i=1}^m S_i$ and $M \cong \bigoplus_{j=1}^n T_j$, then $m = n$ and there is a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that for all $i \in \{1, \dots, n\}$ we have $S_i \cong T_{\sigma(i)}$.*

Proof. Let $m, n \in \mathbb{N}_0$ and let $S_1, \dots, S_m$ and $T_1, \dots, T_n$ be simple modules such that $M \cong \bigoplus_{i=1}^m S_i$ and $M \cong \bigoplus_{j=1}^n T_j$. The set

$$I = \{i \in \{1, \dots, m\} : \ S_i \neq S_j \text{ for all } j \in \{1, \dots, i-1\}\}$$

contains 1, so that it is not empty: let $k \in \mathbb{N}$ be its cardinal and let $j_1, \dots, j_k$ be its elements.

We know from Proposition 3.16 that for each $l \in \{1, \dots, k\}$ there exists a bijective function

$$\pi_l : \{i \in \{1, \dots, m\} : S_i \cong S_{j_l}\} \to \{i \in \{1, \dots, n\} : T_i \cong S_{j_l}\}$$

and we use this to define a function $\pi : \{1,\ldots,m\} \to \{1,\ldots,n\}$ as follows: if $i \in \{1,\ldots,m\}$, it is clear from the definition of the set I that there exists a unique $l \in \{1,\ldots,k\}$ such that $S_i \cong S_{j_l}$, and we may therefore set $\pi(i) = \pi_l(i)$. We have that $S_i \cong T_{\pi(i)}$ for all $i \in \{1,\ldots,m\}$.

This function is surjective. Indeed, let $t \in \{1,\ldots,n\}$. Proposition 3.16 implies that the set $\{i \in \{1,\ldots,m\} : S_i \cong T_t\}$ is not empty and it is easy to see that its minimum element is equal to j_l for some $l \in \{1,\ldots,k\}$. It follows immediately from this that $\pi(\pi_l^{-1}(t)) = t$. It follows, of course, that $m \geq n$. Now, reversing the roles of the two direct sum decompositions in all that we have done, can can obtain the reverse inequality in the same way, and therefore we actually have that $n = m$ and, in particular, that the function π is a bijection, and as it has the required property, this proves the proposition. □

3.18. If $n \in \mathbb{N}_0$ and S is a module, we write nS the module

$$\underbrace{S \oplus \cdots \oplus S}_{n \text{ summands}}.$$

Proposition. *If M is a finite-dimensional module M, then there exist $k \geq 0$ and simple modules $S_1, \ldots, S_k$ such that for all $i, j \in \{1,\ldots,k\}$ we have $S_i \cong S_j$ exactly when $i = j$, and*

$$M \cong \bigoplus_{i=1}^{k} [M : S_i] S_i.$$

Proof. This follows immediately from the results above. □

3.19. A consequence of the caracterization of multiplicity given in Proposition 3.16 is that it is monotone for inclusions:

Proposition. *If M is a finite-dimensional module and N is a submodule of M, then $[N : S] \leq [M : S]$ for every simple module S.*

Proof. Let M be a finite-dimensional module, let N be a submodule of M and let S be a finite-dimensional simple module. If $\iota : N \to M$ is the inclusion, we have a linear map

$$\iota_* : f \in \hom_{\mathfrak{g}}(S, N) \mapsto \iota \circ f \in \hom_{\mathfrak{g}}(S, M)$$

and it is injective, so that in particular $\dim \hom_{\mathfrak{g}}(S, N) \leq \dim \hom_{\mathfrak{g}}(S, M)$. The desired conclusion follows from this using Proposition 3.16. □

3.20. Proposition. *Let M be a finite-dimensional module and let $n \in \mathbb{N}_0$ and $S_1, \ldots, S_n$ be simple modules such that $M \cong \bigoplus_{i=1}^{n} S_i$. If N is a submodule of N, then there exists a subset $I \subseteq \{1,\ldots,n\}$ such that $N \cong \bigoplus_{i \in I} S_i$.*

Proof. HACER □

Isotypic components

3.21. If M is a finite-dimensional module, we know at this point that there exist $n \in \mathbb{N}_0$ and simple modules $S_1, \ldots, S_n$ such that there is an isomorphism $\phi : \bigoplus_{i=1}^n S_i \to M$. As a consequence of this, if for each $i \in \{1, \ldots, n\}$ we let $T_i = \phi(S_i)$ then we have an *internal* decomposition

$$M = \bigoplus_{i=1}^n T_i$$

as a direct sum of simple submodules. It is important to remark that, in contrast to the uniqueness results that we have obtained so far, it is not true that the submodules $T_1, \ldots, T_n$ are well-determined by the module M: it is only on their isomorphism classes that we have information. This is in the nature of things, and we cannot do better. For example, if M is a 2-dimensional trivial module and T_1 and T_2 are any two 1-dimensional subspaces such that $T_1 \cap T_2 = 0$ then we have an internal direct sum decomposition $M = T_1 \oplus T_2$ of M as a *module* — as there are infinitely many choices for the pair (T_1, T_2), uniqueness obviously fails.

There is a way to partially fix this lack of uniqueness, at the cost of considering coarser decompositions. Doing this will be our next task.

3.22. We start with the following auxiliary result, which is a generalization of Schur's Lemma that tells us that a non-zero morphism with simple domain is injective.

Lemma. *Let M and S be finite-dimensional modules and suppose that S is simple. If $n \in \mathbb{N}$ and $f_1, \ldots, f_n : S \to M$ are morphisms of modules which are linearly independent elements of the vector space $\hom_{\mathfrak{g}}(S, M)$, then the submodules $f_1(S), \ldots, f_n(S)$ of M, all of which are isomorphic to S, are independent and we therefore have that $\bigoplus_{i=1}^n f_i(S) \subseteq M$.*

Proof. We proceed by induction on the number n of morphisms. Since the morphisms are linearly independent, they are non-zero and, as S is simple, they are injective: this implies that for each $i \in \{1, \ldots, n\}$ we have $f_i(S) \cong S$. In particular, the lemma holds if $n = 1$.

Let us now suppose that $n > 1$ and show that the submodules $f_1(S), \ldots, f_n(S)$ are independent: this will prove the lemma. In fact, we will only verify that

$$f_1(S) \cap (f_2(S) + \cdots + f_n(S)) = 0,$$

as the rest of what there is to be done is similar. We assume, in order to reach a contradiction, that this intersection is not zero: as it is then a non-zero submodule of the simple submodule $f_1(S)$, it cannot be a proper one and it follows from this that we in fact have that

$$f_1(S) \subseteq f_2(S) + \cdots + f_n(S).$$

Of course, the morphisms $f_2, \ldots, f_n$ are linearly independent, so we inductively know that the submodules $f_2(S), \ldots, f_n(S)$ of M are independent and that their sum is direct. In particular, for each $j \in \{2, \ldots, n\}$ there is a morphism of modules $p_j : f_2(S) + \cdots + f_n(S) \to f_j(S)$ whose restriction to $f_j(S)$ is the identity of $f_j(S)$ and which vanishes on $f_k(S)$ for all $k \in \{2, \dots, k\} \setminus \{j\}$, and we have

$$\sum_{j=2}^{n} p_j(t) = t \text{ for all } t \in f_2(S) + \cdots + f_n(S). \tag{24}$$

If now $j \in \{2, \dots, n\}$, using the fact that the functions f_1 and f_j are injective, it is easy to see that there is unique endomorphism $g_j : S \to S$ such that the diagram

$$\begin{array}{ccccc} S & & \xrightarrow{\quad g_j \quad} & & S \\ \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_j} \\ f_1(S) & \hookrightarrow & f_2(S) + \cdots + f_n(S) & \xrightarrow{p_j} & f_j(S) \end{array}$$

commutes and, since S is simple, Schur's Lemma 3.9 tells us that there exists a scalar $\lambda_j \in \Bbbk$ such that $g_j = \lambda \mathrm{id}_S$. We have thus found scalars $\lambda_2, \dots, \lambda_n$ such that for each $j \in \{2, \dots, n\}$ we have

$$p_j(f_1(s)) = \lambda_j f_j(s)$$

for all $s \in S$. In view of (24), we then have that

$$f_1(s) = \sum_{j=2}^{n} p_j(f_1(s)) = \sum_{j=2}^{n} \lambda_j f_j(s)$$

for all $s \in S$, that is, that $f_1 = \sum_{j=2}^{n} \lambda_i f_i$: this is absurd, because the morphisms $f_1, \dots, f_n$ are linearly independent. This contradition is the one we wanted. □

3.23. If M is a finite-dimensional module and S is a simple module, then the ***isotypic component*** of M of ***type*** S is the submodule M_S obtained as the sum of all submodules of M which are isomorphic to S:

$$M_S = \sum_{\substack{N \subseteq M \\ N \cong S}} N.$$

It is obvious, in view of the form of this definition, that $M_S = M_{S'}$ whenever S and S' are isomorphic simple modules: this means that the component M_S depends only on the isomorphism class of S.

3.24. Isotypic components generalize a construction we have already considered in 2.6:

Proposition. *If M is a finite-dimensional module, the isotypic component of trivial type $M_\Bbbk$ coincides with the invariant subspace $M^{\mathfrak{g}}$.*

Proof. **HACER** □

3.25. In Proposition 2.6*(iii)* we described the invariant subspace of a module M in terms of morphisms $\Bbbk \to M$. This generalizes to the other isotypic components as follows:

Proposition. *Let M be a finite-dimensional module and let S be a finite-dimensional simple module.*

(*i*) *There is a linear function $\phi_S : S \otimes \hom_{\mathfrak{g}}(S, M) \to M$ such that $\phi_S(s \otimes f) = f(s)$ for all $s \in S$ and all $f \in \hom_{\mathfrak{g}}(S, M)$. This function is an injective homomorphism of modules, provided that we view the vector space $\hom_{\mathfrak{g}}(S, M)$ as a trivial module, and its image is exactly the isotypic component M_S. In particular, it corestricts to an isomorphism $S \otimes \hom_{\mathfrak{g}}(S, M) \to M_S$. In particular, we have that $\dim M_S = [M : S] \cdot \dim S$.*

(*ii*) *If $n = [M : S]$ and $\{f_1, \ldots, f_n\}$ is a basis of $\hom_{\mathfrak{g}}(S, M)$, then for each $i \in \{1, \ldots, n\}$ the submodule $f_i(S)$ of M is isomorphic to S and there is an internal direct sum decomposition*

$$M_S = \bigoplus_{i=1}^{n} f_i(S).$$

Proof. It is easy to check that there is a linear map as in (*i*), and that it is a morphism of modules is a consequence of the fact that for each $x \in \mathfrak{g}$, $s \in S$ and $f \in \hom_{\mathfrak{g}}(S, M)$ we have

$$\phi_S(x \cdot s \otimes f) = \phi_S((x \cdot s) \otimes f) = f(x \cdot s) = x \cdot f(s) = x \cdot \phi_S(s \otimes f).$$

The second equality here is due to the fact that we are viewing $\hom_{\mathfrak{g}}(S, M)$ as a trivial module.

Let $n = [M : S]$ and let $\mathscr{B} = \{f_1, \ldots, f_n\}$ be a basis of $\hom_{\mathfrak{g}}(S, M)$. We know from Lemma 3.22 that the submodules $f_1(S), \ldots, f_n(S)$ are all isomorphic to S and independent, so that their sum, which we will denote M'_S, is direct and contained in M_S. We claim that in fact $M_S = M'_S$ and that therefore (*ii*) holds. To see this it is enough that we show that if N is a submodule of M which is isomorphic to S, we then have $N \subseteq M'_S$, as M_S is the sum of all such submodules.

Let N be a submodule of M isomorphic to S. There is then an injective morphism $f : S \to M$ whose image is N and, since $\mathscr{B}$ is a basis of $\hom_{\mathfrak{g}}(S, M)$, scalars $\lambda_1, \ldots, \lambda_n \in \Bbbk$ such that $f = \sum_{i=1}^{n} \lambda_i f_i$. For each $s \in S$ we have that

$$f(s) = \sum_{i=1}^{n} \lambda_i f_i(s) \in \sum_{i=1}^{n} \lambda_i f_i(S) = M'_S,$$

and this tells us that N is contained in M'_S, as we wanted.

Now that we know that (*ii*) holds, in particular we have that

$$\dim M_S = [M : S] \cdot \dim S = \dim S \otimes \hom_{\mathfrak{g}}(S, M).$$

The domain and codomain of the map ϕ thus have the same finite dimension: as it is a surjection, it is necessarily an isomorphism. This completes the proof of the proposition. □

3.26. We can now describe the direct sum decomposition in which we are interested:

Proposition. *Let M be a finite-dimensional module. There are finitely many non-zero isotypic components in M and M is their direct sum.*

This means that if M is a finite-dimensional module, then there exist $n \in \mathbb{N}_0$ and simple modules $S_1, \dots, S_n$, pairwise non-isomorphic, such that $M_{S_i} \neq 0$ for all $i \in \{1, \dots, n\}$ and $M = \bigoplus_{i=1}^n M_{S_i}$. This internal direct sum decomposition of M is canonical, in that up to permutation it is well-determined by M. We call it the ***isotypic decomposition*** of M.

Proof. Let $\mathscr{S}$ be a set of representatives for the isomorphism classes of finite-dimensional simple modules. We want to prove first that whenever $n \in \mathbb{N}$ and $S_1, \dots, S_n$ are pairwise distinct elements of $\mathscr{S}$, then the isotypic components $M_{S_1}, \dots, M_{S_n}$ are independent, and we do it by induction on n. Of course, if $n = 1$ there is nothing to do.

Let then $n > 1$ and let $S_1, \dots, S_n$ be pairwise distinct elements of $\mathscr{S}$. To show that the corresponding isotypic components are independent it will be enough to show that

$$M_{S_1} \cap (M_{S_2} + \cdots + M_{S_n}) = 0.$$

Let I be the intersection in question. As I is a submodule of M_{S_1} and $M_{S_1} \cong [M : S_1]S_1$, it follows from Proposition 3.20 that $I \cong k_1 S_1$ for some integer k_1 such that $0 \leq k_1 \leq [M : S_1]$. On the other hand, we know inductively that the direct sum $M_{S_2} + \cdots + M_{S_n}$ is direct and isomorphic to

$$[M : S_2]S_2 \oplus \cdots \oplus [M : S_n]S_n,$$

and that same proposition tells us that there are integers $k_2, \dots, k_n$ such that $0 \leq k_i \leq [M : S_i]$ for each $i \in \{2, \dots, n\}$ and $I \cong \bigoplus_{i=2}^n k_i S_i$. We have thus proved that there is an isomorphism

$$k_1 S_1 \cong \bigoplus_{i=2}^{n} k_i S_i.$$

Since $S_1 \ncong S_j$ for all $j \in \{2, \dots, n\}$, computing the multiplicity of S_1 on both sides of this isomorphism we find that $k_1 = 0$, so that in fact $I = 0$.

As every finite set of isotypic components is independent, it follows that the set of all isotypic components is independent, and their sum $M' = \bigoplus_{S \in \mathscr{S}} M_S$ is direct. This is a submodule of M, so its dimension is finite: this implies that the set $\mathscr{S}_M = \{S \in \mathscr{S} : M_S \neq 0\}$ is finite. To finish the proof, we have to show that $M = M'$.

There is a short exact sequence

$$0 \longrightarrow M' \hookrightarrow M \xrightarrow{g} M/M' \longrightarrow 0$$

and, as all exact sequences of finite-dimensional modules, it is split: there is then a section $\sigma : M/M' \to M$ of g. Suppose that $M' \subsetneq M$, so that the quotient M/M' is non-zero and has a simple submodule T. The intersection $M' \cap \sigma(T)$ is a submodule of $\sigma(T)$ contained in the kernel of g, and since the restriction of g to the image of σ is injective, this implies that $M' \cap \sigma(T) = 0$. This is absurd: we have $0 \neq \sigma(T) \subseteq M_T \subseteq M'$ We must therefore that $M = M'$, as we wanted. ☐

3.27. The isotypic decomposition of a finite-dimensional module is canonical and natural, in that it is preserved by all homomorphisms. This is the generalization of Proposition 2.6*(ii)* to isotypic components of non-trivial type.

Proposition. *Let S be a finite-dimensional simple module. If $g : M \to N$ is a morphism of finite-dimensional modules, then we have $g(M_S) \subseteq N_S$, so that g restricts to a morphism of modules $g_S : M_S \to N_S$.*

Proof. Let $g : M \to N$ be morphism of finite-dimensional modules. In order to prove that $g(M_S) \subseteq N_S$ it is enough that we show that if T is a submodule of M which is isomorphic to S, then we have $g(T) \subseteq N_S$. But this is clear: the restriction $g|_T : T \to N$ is either zero or injective, because T is simple: in the first case we have $g(T) = 0$ and in the second one $g(T)$ is a submodule of N isomorphic to S, so that it is contained in N_S. □

Characters

3.28. Let M be a finite-dimensional module. A consequence of the information we have thus far is that

$$\text{the linear map } H_M : M \to M \text{ is diagonalizable and that it has integer eigenvalues.} \tag{25}$$

Indeed, we know that there exist $n \in \mathbb{N}_0$ and simple submodules $S_1, \ldots, S_n$ of M such that $M = \bigoplus_{i=1}^n S_i$, and this implies in particular that for each $i \in \{1, \ldots, n\}$ the subspace S_i of M is H_M-invariant and $H_M|_{S_i} = H_{S_i}$. It follows from this that H_M is diagonalizable if for all $i \in \{1, \ldots, n\}$ the map H_{S_i} is diagonalizable, and we know from Proposition 3.3 that this holds. Moreover, it is clear from this that a scalar is an eigenvalue of H_M if and only if it is an eigenvalue of one of the maps H_{S_i}, and the latter have integer eigenvalues.

For each $\lambda \in \Bbbk$, we denote M^λ the eigenspace of the linear map $H_M : M \to M$ corresponding to the eigenvalue λ and call this subspace the ***weight subspace*** of M of ***weight*** λ. The non-zero elements of M^λ are the ***weight vectors*** of M of that weight. In view of our observation (25) above, we have that $M = \bigoplus_{\lambda \in \mathbb{Z}} M^\lambda$ and, since M is finite-dimensional, that $M^\lambda = 0$ for all $\lambda \in \Bbbk$ except finitely many,

We denote $\mathbb{Z}[q^{\pm 1}]$ the ring of Laurent polynomials with integer coefficients and we refer to its elements simply as polynomials —this should not be cause for any confusion. The ***character*** of the module M is the polynomial

$$\chi_M = \sum_{\lambda \in \mathbb{Z}} \dim M^\lambda \, q^\lambda.$$

This makes sense: as we have observed, we have $M^\lambda = 0$ for almost all $\lambda \in \Bbbk$, so the sum is finite.

3.29. The following is often a useful rephrasing of this definition:

Lemma. *Let M be a finite-dimensional module of dimension r. If $\mathscr{B} = \{m_1, \ldots, m_r\}$ is a basis of M whose elements are eigenvectors of H_M, so that for each $i \in \{1, \ldots, r\}$ there is a scalar $\lambda_i \in \Bbbk$ (which* *is necessarily an integer, as we know) such that* $H \cdot m_i = \lambda_i m_i$, *then we have*

$$\chi_M = \sum_{i=1}^{r} q^{\lambda_i}.$$

Proof. Grouping terms, we see that

$$\sum_{i=1}^{r} q^{\lambda_i} = \sum_{\lambda\in\mathbb{Z}} \left(\sum_{\substack{i\in\{1,\dots,r\}\\ \lambda_i=\lambda}} q^{\lambda_i} \right) = \sum_{\lambda\in\mathbb{Z}} \#\{i \in \{1,\dots,r\} : \lambda_i = \lambda_i\}\, q^{\lambda}$$

and, since the set $\{m_i : i \in \{1,\dots,r\} : \lambda_i = \lambda\}$ is a basis of the weight space M_i^{λ}, this is

$$= \sum_{\lambda\in\mathbb{Z}} \dim M^{\lambda}\, q^{\lambda} = \chi_M. \qquad \square$$

3.30. As we will amply demonstrate in what follows, the characters of modules are actually quite amenable to computation. The key properties that enable that are codified in the following result:

Proposition. (*i*) *If*

$$0 \longrightarrow M \xrightarrow{\ f\ } E \xrightarrow{\ g\ } N \longrightarrow 0$$

is a short exact sequence of finite-dimensional modules, then $\chi_E = \chi_M + \chi_N$. *In particular, if* M *and* N *are finite-dimensional modules, then* $\chi_{M\oplus N} = \chi_M + \chi_M$.

(*ii*) *If* M *and* N *are finite-dimensional modules, then* $\chi_{M\otimes N} = \chi_M \cdot \chi_N$.

(*iii*) *If* M *is a finite-dimensional simple module of dimension* $r+1$, *then*

$$\chi_M = \frac{q^{r+1} - q^{-r-1}}{q - q^{-1}}.$$

Proof. (*i*) Let $r = \dim M$ and $s = \dim N$. As H_M and H_N are diagonalizable maps, there exist ordered bases $\mathscr{B}' = (m_1,\dots,m_r)$ and $\mathscr{B}'' = (n_1,\dots,n_s)$ of M and of N, respectively, whose elements are eigenvectors of H_M and of H_N. There exist $\tilde{n}_1$, …, $\tilde{n}_r \in E$ such that $g(\tilde{n}_i) = n_i$ for each $i \in \{1,\dots,s\}$, and it is easy to see that $\mathscr{B} = (m_1,\dots,m_r,\tilde{n}_1,\dots,\tilde{n}_s)$ is an ordered basis of E and that the matrix of H_E with respect to $\mathscr{B}$ is a upper triangular block matrix of the form

$$\|H_E\|_{\mathscr{B}} = \begin{pmatrix} \|H_M\|_{\mathscr{B}'} & * \\ 0 & \|H_N\|_{\mathscr{B}''} \end{pmatrix}.$$

It follows immediately from this that the multiplicity of a scalar as an eigenvalue of H_E is the sum of its multiplicity as an eigenvalue of H_M and its multiplicity as an eigenvalue of H_N, that is, that for all $\lambda \in \Bbbk$ we have

$$\dim E^{\lambda} = \dim M^{\lambda} + \dim N^{\lambda}.$$

That $\chi_E = \chi_M + \chi_N$ follows immediately from this.

To prove the second claim of (i) we need only observe that if M and N are finite-dimensional modules, then there is a short exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix}\mathrm{id}_M\\0\end{pmatrix}} M \oplus N \xrightarrow{(\,0\ \mathrm{id}_N\,)} N \longrightarrow 0$$

and that what we have already proved therefore tells us that $\chi_{M\oplus N} = \chi_M + \chi_N$.

(ii) Let M and N be finite-dimensional modules of dimensions r and s, respectively, and let $\mathscr{B}' = (m_1, \dots, m_r)$ and $\mathscr{B}'' = (n_1, \dots, n_s)$ be ordered bases of M and of N whose elements are eigenvectors of H_M and of H_N. There are then integers $\lambda_1, \dots, \lambda_r$ and $\mu_1, \dots, \mu_s$ such that $H \cdot m_i = \lambda_i m_i$ for all $i \in \{1, \dots, r\}$ and $H \cdot n_j = \mu_j n_j$ for all $j \in \{1, \dots, s\}$.

We know from linear algebra that the set $\mathscr{B} = \{m_i \otimes n_j : i \in \{1, \dots, r\}, j \in \{1, \dots, s\}\}$ is basis of the vector space $M \otimes N$. We claim that its elements are weight vectors. Indeed, if $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, we have

$$\begin{aligned} H \cdot m_i \otimes n_j &= (H \cdot m_i) \otimes n_j + m_i \otimes (H \cdot n_j) = \lambda_i m_i \otimes n_j + m_i \otimes \mu_j n_j \\ &= (\lambda_i + \mu_j) m_i \otimes n_j. \end{aligned}$$

so that the elementary tensor $m_i \otimes n_j$ is an weight vector of $M \otimes N$ of weight $\lambda_i + \mu_j$. It follows from this and from the lemma above that

$$\chi_{M\otimes N} = \sum_{\substack{1\le i\le r\\ 1\le j\le s}} q^{\lambda_i+\mu_j} = \sum_{i=1}^{r} q^{\lambda_i} \cdot \sum_{j=1}^{s} q^{\mu_j} = \chi_M \cdot \chi_N.$$

(iii) If M is a finite-dimensional simple module of dimension $r+1$, then Proposition 3.3 tells us that the eigenvalues of H_M are precisely the numbers of the form $r - 2i$ with $i \in \{1, \dots, r\}$, so that

$$\chi_M = \sum_{i=0}^{r} q^{r-2i} = \frac{q^{r+1} - q^{-r-1}}{q - q^{-1}}.$$

This completes the proof of the proposition. □

3.31. The usefulness of the character of a module is that while it is an object of a much simpler nature than a representation it allows us to recover information about it:

Proposition. *If M be a finite-dimensional module, then $\chi_M(1) = \dim M$ and for all $r \ge 0$ we have*

$$[M : V_r] = \operatorname*{Res}_{q=0} (q^{-1} - q) q^r \chi_M(q). \tag{26}$$

Proof. Let M be a finite-dimensional module. It follows immediately from the definition that

$$\chi_M(1) = \sum_{\lambda\in\mathbb{Z}} \dim M^\lambda,$$

and this is equal to $\dim M$ because the linear map $H_M : M \to M$ is diagonalizable. Let, on the other hand, $r \in \mathbb{N}_0$. We know that there exists a $d \geq 0$ such that $M = \bigoplus_{i=0}^{d}[M : V_i]V_i$, so that

$$(q^{-1} - q)q^r \chi_M(q) = \sum_{i=0}^{d}[M : V_i](q^{-1} - q)q^r \chi_{V_i}(q) = \sum_{i=0}^{d}[M : V_i]q^r(q^{i+1} - q^{-i-1}),$$

and the coefficient of q^{-1} in this polynomial is precisely $[M : V_r]$. This is the meaning of the equality (26) that appears in the proposition. □

3.32. An immediate corollary of Proposition 3.31 is the following fundamental observation:

Proposition. *Two finite-dimensional modules with the same character are isomorphic.*

In other words, we do not lose any information about the isomorphism class of a module if you pass to its character.

Proof. If M and N are two finite-dimensional modules which have the same character, then Proposition 3.31 tells us that for all simple modules S we have $[M : S] = [N : S]$, so that, according to Proposition 3.18, $M \cong N$. □

§4. Some applications

Tensor products and the Clebsch–Gordan formula

4.1. The following result is due to Alfred Clebsch (1833–1872, Germany) and Paul Gordan (1837–1912, Germany):

Proposition. *If $r, s \in \mathbb{N}_0$ are such that $0 \leq s \leq r$, then there is an isomorphism*

$$V_r \otimes V_s \cong V_{r+s} \oplus V_{r+s-2} \oplus V_{r+s-3} \oplus \cdots \oplus V_{r-s}. \tag{27}$$

Since the tensor product of modules is commutative —this is what Proposition 2.14*(ii)* states— and distributes over direct sums, the above result allows us to describe the tensor product of two arbitrary finite-dimensional modules, at least in principle.

Proof. Let r and s be as in the statement of the proposition. In order to prove that there is an isomorphism (27) it is enough, in view of Proposition 3.32, to show that the two modules appearing there have the same character, and this just a matter of a simple computation in $\mathbb{Z}[q^{\pm 1}]$:

$$\begin{aligned}
\chi_{V_r \otimes V_s} &= \chi_{V_r} \cdot \chi_{V_s} = \frac{q^{r+1} - q^{-r-1}}{q - q^{-1}} \cdot \frac{q^{s+1} - q^{-s-1}}{q - q^{-1}} \\
&= \frac{1}{q - q^{-1}} \left(\frac{q^{r+s+2} - q^{r-s}}{q - q^{-1}} - \frac{q^{-r+s} - q^{-r-s-2}}{q - q^{-1}} \right) \\
&= \frac{1}{q - q^{-1}} \left(\left(q^{r+s+1} + q^{r+s-1} + \cdots + q^{r-s+1} \right) - \left(q^{-r+s-1} + \cdots + q^{-r-s+1} + q^{-r-s-1} \right) \right) \\
&= \frac{q^{r+s+1} - q^{-r-s-1}}{q - q^{-1}} + \frac{s^{r+s-1} - q^{-r-s+1}}{q - q^{-1}} + \cdots + \frac{q^{r-s+1} - q^{-r+s-1}}{q - q^{-1}} \\
&= \chi_{V_{r+s}} + \chi_{V_{r+s-2}} + \cdots + \chi_{V_{r-s}} \\
&= \chi_{V_{r+s} \oplus V_{r+s-2} \oplus \cdots \oplus V_{r-s}}.
\end{aligned}$$

□

4.2. The proposition we have just proved describes the structure of tensor products of simple modules as a direct sum but does not tell us what the actual submodules that appear in that decomposition are. In many situations we need this finer information, and that is provided by the following result:

Proposition. *Let M and N be two finite dimensional modules, and let m and n be singular weight vectors of weights λ and μ in M and in N, respectively, so that in particular λ and μ are non-negative integers. If p is a integer such that $0 \leq p \leq \min\{\lambda, \mu\}$, then*

$$\sum_{i=0}^{p} (-1)^i \frac{(\lambda - i)!(\mu - p + i)!}{(\lambda - p)!\mu!} \frac{F^i \cdot m}{i!} \otimes \frac{F^{p-i} \cdot n}{(p-i)!}$$

is a singular weight vector of weight $\lambda + \mu - 2p$ in $M \otimes N$.

Proof. Let p be a non-negative integer such that $0 \le p \le \min\{\lambda, \mu\}$. For each $i \in \{0, \dots, p\}$ let us put $m_i = \frac{1}{i!}F^i \cdot m$ and $n_i = \frac{1}{i!}F^i \cdot n$. We know from Proposition 3.7 that m_i and n_i are weight vectors of weights $\lambda - 2i$ and $\mu - 2i$, respectively, and that the vectors $m_0, \dots, m_p$ and the vectors $n_0, \dots, n_p$ are linearly independent. It follows from this, in particular, that the vector —let us denote it w— that appears in the statement of the proposition is non-zero and, since each summand $m_i \otimes n_{p-i}$ is a weight vector of weight $(\lambda - 2i) + (\mu - 2(p-i)) = \lambda + \mu - 2p$ in $M \otimes N$, that so is w.

To complete the proof, then, we need only show that $E \cdot w = 0$. For this, we write

$$a_i = \frac{(\lambda - i)!(\mu - p + i)!}{(\lambda - p)!\mu!}$$

and compute that

$$\begin{aligned}
E \cdot w &= \sum_{i=0}^{p} (-1)^i a_i \left((E \cdot m_i) \otimes n_{p-i} + m_i \otimes (E \cdot n_{p-i})\right) \\
&= \sum_{i=1}^{p} (-1)^i a_i (\lambda + 1 - i) m_{i-1} \otimes n_{p-i} + \sum_{i=0}^{p-1} (-1)^i a_i (\mu + 1 - p + i) m_i \otimes n_{p-i-1} \\
&= \sum_{i=0}^{p-1} (-1)^i \big(-a_{i+1}(\lambda - i) + a_i(\mu + 1 - p + i)\big) m_i \otimes n_{p-i-1} \\
&= 0,
\end{aligned}$$

since for each $i \in \{0, \dots, p-1\}$ we have that $a_{i+1}(\lambda - i) = a_i(\mu + 1 - p + i)$. □

Invariant bilinear forms

4.3. Let M be a vector space. A ***bilinear form*** on M is a bilinear map $\beta : M \times M \to \Bbbk$. Such a thing, as we know, can be identified to a linear map $M \otimes M \to k$, and we will usually switch from one point of view to the other without mention. In terms of this identification, the set of all bilinear forms on M is simply $\hom_{\Bbbk}(M \otimes M, \Bbbk)$, the dual space of $M \otimes M$.

We say that a bilinear form $\beta : M \times M \to \Bbbk$ is ***non-degenerate*** if

- for all $m \in M$ there exists an $n \in M$ such that $\beta(m, n) \ne 0$, and
- for all $n \in M$ there exists an $m \in M$ such that $\beta(n, m) \ne 0$.

On the other hand, we say that β is ***symmetric*** if $\beta(m, n) = \beta(n, m)$ for all $m, n \in M$, and that it is ***anti-symmetric*** if $\beta(m, n) = -\beta(n, m)$ for all $m, n \in M$.

4.4. If M is a module, then we say that a bilinear form $\beta : M \times M \to \Bbbk$ is ***invariant*** if for all $x \in \mathfrak{g}$ and all $m, n \in M$ we have that

$$\beta(x \cdot m, n) + \beta(m, x \cdot n) = 0.$$

If we view β as a linear map $M \otimes M \to \Bbbk$, this means precisely that it is a morphism of modules —provided that we put on $\Bbbk$ the trivial module structure— and therefore the set of invariant bilinear forms on M is the subspace $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$ of $\hom_{\Bbbk}(M \otimes M, k)$. We have shown that $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$ coincides with the subspace of invariants $\hom \Bbbk(M \otimes M, \Bbbk)$: a bilinear form is invariant if and only if it is an invariant element of the space of bilinear forms.

4.5. An easy consequence of the Clebsch–Gordan Formula is that we can describe the invariant bilinear forms on finite-dimensional modules:

Proposition. *Let M be a finite-dimensional simple module. The vector space $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$ of invariant bilinear forms on M is 1-dimensional. If $\beta : M \times M \to \Bbbk$ is a non-zero invariant bilinear form on M, then β is non-degenerate, and it is symmetric or anti-symmetric if $\dim M$ is odd or even, respectively.*

Proof. Let $r \in \mathbb{N}_0$ be such that $\dim M = r+1$. As we know, this implies that there is an isomorphism of modules $M \cong V_r$, and we may just as well suppose that M is V_r for the purpose of this proof. From the Clebsch–Gordan Formula 4.1 we know that there is an isomorphism of modules

$$V_r \otimes V_r \cong \bigoplus_{i=0}^{r} V_{2i}$$

and then we have an isomorphism of vector spaces

$$\hom_{\mathfrak{g}}(V_r \otimes V_r, \Bbbk) \cong \bigoplus_{i=0}^{r} \hom_{\mathfrak{g}}(V_{2i}, \Bbbk).$$

As $k \cong V_0$, we know from Lemma 3.15 that $\hom_{\mathfrak{g}}(V_{2i}, \Bbbk) = 0$ if $i \neq 0$ and that $\dim \hom_{\mathfrak{g}}(V_0, \Bbbk) = 1$. This proves the first part of the proposition.

Let us now fix an non-zero invariant bilinear form $\beta : M \times M \to \Bbbk$ on our finite-dimensional simple module M, and let us consider the set

$$M' = \{m \in M : \beta(m, n) = 0 \text{ for all } n \in M\}.$$

It is easy to see that M' is a subspace of M and it is a submodule, because if $x \in \mathfrak{g}$ and $m \in M'$ we have for all $n \in M$ that $\beta(x \cdot m, n) = -\beta(m, x \cdot n) = 0$, so that $x \cdot m \in M'$. As M is simple, it follows from this that M' is either the zero subspace of M or equal to M itself, and the second possibility cannot occur, since β is not the zero bilinear form. A similar argument shows that the set $M'' = \{n \in M : \beta(m, n) = 0 \text{ for all } m \in M\}$ is also the zero subspace of M and, in conclusion, that the form β is non-degenerate.

Let us denote S and A the sets of symmetric and anti-symmetric invariant bilinear forms on M, respectively. These are subspaces of $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$, as one can readily check, and in fact we have a direct sum decomposition

$$\hom_{\mathfrak{g}}(M \otimes M, \Bbbk) = S \oplus A. \tag{28}$$

Indeed, a bilinear form $\beta : M \otimes M \to \Bbbk$ which is in $S \cap A$ is necessarily zero, as for all $m, n \in M$ we have that

$$\beta(m \otimes n) = -\beta(n \otimes m) = -\beta(m \otimes n),$$

with the first equality coming from the anti-symmetry and the second one from symmetry. On the other hand, if $\beta : M \otimes M \to \Bbbk$ is an invariant bilinear form on M, then there are invariant bilinear forms $\beta_s, \beta_a : M \otimes M \to \Bbbk$ such that for all $m, n \in M$ we have

$$\beta_s(m \otimes n) = \tfrac{1}{2}(\beta(m \otimes n) + \beta(n \otimes m))$$

and

$$\beta_a(m \otimes n) = \tfrac{1}{2}(\beta(m \otimes n) - \beta(n \otimes m)),$$

they are symmetric and anti-symmetric, respectively, and $\beta = \beta_s + \beta_a$.

In view of the decomposition (28) and the fact that the vector space $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$ is 1-dimensional, we see immediately that we have in fact that $\hom_{\mathfrak{g}}(M \otimes M, \Bbbk)$ is equal to one of A or S. This is what the proposition claims. □

4.6. The proposition we have just proved tells us that each finite-dimensional module can be canonically endowed with a non-degenerate bilinear form, uniquely determined up to a scalar, and that it is either symmetric or anti-symmetric. It does not tell us *what* this form is, nor does the argument we have used allow us to decide if it is symmetric of anti-symmetric. Let us show how we can use the precise form of the Clebsch–Gordan Formula provided by Proposition 4.2 to actually construct the form.

Let us fix $r \geq 0$ and let V be a simple module of dimension $r+1$. We want to exhibit a non-zero vector in $\hom_{\mathfrak{g}}(V \otimes V, \Bbbk)$, which, as we know, is the invariant subspace of $\hom(V \otimes V, \Bbbk)$, the dual space $(V \otimes V)^*$ of $V \otimes V$. There is a linear function $\Phi : V^* \otimes V^* \to (V \otimes V^*)$ such that for each $\phi, \psi \in V^*$ and each $v, w \in V$ we have $\Phi(\phi \otimes \psi)(v \otimes w) = \phi(v)\psi(w)$, and this map Φ is an isomorphism of modules. We can therefore look for an non-zero invariant element of $V^* \otimes V^*$.

There is a basis $\mathscr{B} = \{m_0, \dots, m_r\}$ of V whose elements are weight vectors, with

$$H \cdot m_i = (r - 2i)m_i$$

for each $i \in \{0, \dots, r\}$ and

$$E \cdot m_i = \begin{cases} 0, & \text{if } i = 0; \\ (r - i + 1)m_{i-1}, & \text{if } 0 < i \leq r; \end{cases} \qquad F \cdot m_i = \begin{cases} (i+1)m_{i+1}, & \text{if } 0 \leq i < r; \\ 0, & \text{if } i = r. \end{cases}$$

Let $\mathscr{B}^* = \{\phi_0, \dots, \phi_r\}$ be the basis of V^* dual to $\mathscr{B}$, so that $\phi_i(m_j) = \delta_{i,j}$ for all $i, j \in \{0, \dots, n\}$. One sees at once, using the definition of the action $\mathfrak{g}$ on V^*, that

$$H \cdot \phi_i = (2i - r)\phi_i$$

for each $i \in \{0, \dots, n\}$, and that

$$E \cdot \phi_i = \begin{cases} -(r - i + 1)\phi_{i+1}, & \text{if } 0 \leq i < r; \\ 0, & \text{if } i = r; \end{cases} \qquad F \cdot \phi_i = \begin{cases} 0, & \text{if } i = 0; \\ -(i+1)\phi_{i-1}, & \text{if } 0 < i \leq r. \end{cases}$$

In particular, the vector ϕ_r is a singular weight vector of weight r and for each $i \in \{0, \dots, r\}$ we have,

$$F^i \cdot \phi_r = (-1)^i \frac{(r+1)!}{(r+1-i)!}\phi_{r-i}. \tag{29}$$

It follows from Proposition 4.2 that

$$\frac{1}{r!}\sum_{k=0}^{r}(-1)^k\,(F^k\cdot\phi_r)\otimes(F^{r-k}\cdot\phi_r)$$

is a weight vector of $V^*\otimes V^*$ of weight 0, and using (29) this is easily seen to be a scalar multiple of

$$\omega=\sum_{k=0}^{r}(-1)^k\binom{r+2}{k+1}\phi_{r-k}\otimes\phi_k.$$

Proposition. *Let V be a finite-dimensional simple module of dimension $r+1$ and let $\{m_0,\dots,m_r\}$ be a basis of V as in Proposition 3.3. There is a non-degenerate invariant bilinear form $\beta : V\otimes V\to\Bbbk$ such that*

$$\beta(m_i,m_j)=(-1)^j\binom{r+2}{j+1}\delta_{i+j,r}$$

for all $i, j\in\{0,\dots,r\}$, and it is symmetric if r is even, and anti-symmetric if r is odd.

Proof. Our observations above imply that $\Phi(\omega)$ is an invariant bilinear form on V, and it is manifestly non-zero. If $i, j\in\{0,\dots,r\}$, then

$$\Phi(\omega)(m_i\otimes m_j)=\sum_{k=0}^{r}(-1)^k\binom{r+2}{k+1}\phi_{r-k}(m_i)\phi_k(m_j)=(-1)^j\binom{r+2}{j+1}\delta_{r-j,i},$$

and this has the same evalue as the expression given in the proposition. Using that formula, it is immediate to see that $\beta(m_i,m_j)=(-1)^r\beta(m_j,m_i)$ for all $i, j\in\{0,\dots,r\}$, and the last claim of the statement follows at once from this. □

Tensor powers

4.7. In many contexts it is useful to understand the structure of the tensor powers of representations and this can be done using the Clebsch–Gordan Formula. The simplest non-trivial example of this is that of the powers of the 2-dimensional simple representation:

Proposition. *If $d, r\geq 0$, then the multiplicity of V_r as a direct summand of the dth tensor power $V_1^{\otimes r}$ is*

$$[V_1^{\otimes d}:V_r]=\begin{cases}\displaystyle\binom{d}{\frac{d+r}{2}}\frac{2(r+1)}{d+r+2}, & \text{if } d+r \text{ is even;}\\ 0, & \text{it it is not.}\end{cases}\tag{30}$$

We have computed some of these multiplicities in Table 1 on the following page.

	r												
d	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1		1											
2	1		1										
3		2		1									
4	2		3		1								
5		5		4		1							
6	5		9		5		1						
7		14		14		6		1					
8	14		28		20		7		1				
9		42		48		27		8		1			
10	42		90		75		35		9		1		
11		132		165		110		44		10		1	
12	132		297		275		154		54		11		1

Table 1. The multiplicites $[V_1^{\otimes d} : V_r]$, with the zeroes omitted. These numbers can be computed very efficiently using the recurrence relation found in the proof of Proposition 4.7.

Proof. Let us write $a_r^d = [V_1^{\otimes d} : V_r]$ for each $d, r \geq 0$. If $d \geq 0$, then we have $V_1^{\otimes d} = \bigoplus_{r \geq 0} a_r^d V_r$ and using the Clebsch–Gordan Formula we see that

$$\begin{aligned} V_1^{\otimes(d+1)} &= V_1^{\otimes d} \otimes V_1 \cong \bigoplus_{r\geq 0} a_r^d V_r \otimes V_1 \cong \bigoplus_{r \geq 1} a_r^d (V_{r+1} \oplus V_{r-1}) \oplus a_0^d V_1 \\ &\cong \bigoplus_{r\geq 1} a_r^d V_{r+1} \oplus \bigoplus_{r\geq 1} a_r^d V_{r-1} \oplus a_0^d V_1 \cong \bigoplus_{r\geq 2} a_{r-1}^d V_r \oplus \bigoplus_{r\geq 0} a_{r+1}^d V_r \oplus a_0^d V_1 \\ &\cong \bigoplus_{r\geq 1} (a_{r-1}^d + a_{r+1}^d) V_r \oplus a_1^d V_0. \end{aligned}$$

This tells us that for all $r, d \geq 0$ we have

$$a_r^{d+1} = \begin{cases} a_1^d, & \text{if } r = 0; \\ a_{r-1}^d + a_{r+1}^d, & \text{if } r \geq 1. \end{cases} \tag{31}$$

It is clear that $V_1^{\otimes 0} \cong V_0$, so that $a_0^0 = 1$ and $a_r^0 = 0$ for all $r > 0$: this means that the equation (30) holds if $d = 0$. Let us now fix $d \in \mathbb{N}_0$, suppose that for all $r \geq 0$ we have

$$a_r^d = \begin{cases} \binom{d}{\frac{d+r}{2}} \frac{2(r+1)}{d+r+2}, & \text{if } d + r \text{ is even;} \\ 0, & \text{if it is not.} \end{cases} \tag{32}$$

and show that

$$a_r^{d+1} = \begin{cases} \binom{d+1}{\frac{d+1+r}{2}} \frac{2(r+1)}{d+r+3}, & \text{if } d+r+1 \text{ is even;} \\ 0, & \text{if it is not.} \end{cases} \tag{33}$$

for all $r \geq 0$. To do this we consider two cases:

- Suppose first that the number $d+r$ is even. According to the recurrence relations (31) we have that $a_0^{d+1} = a_1^d$ and $a_r^{d+1} = a_{r-1}^d + a_{r+1}^d$ for all $r > 0$, and the right hand sides of these equalities vanish in view of our hypothesis (32): this proves (33) in this case.
- Suppose next that $d+r$ is an odd number. As $d+r-1$ and $d+r+1$ are even, from the recurrence (31) and the inductive hypothesis (32) we see that

$$\begin{aligned} a_r^{d+1} &= a_{r-1}^d + a_{r+1}^d = \binom{d}{\frac{d+r-1}{2}} \frac{2r}{d+r+1} + \binom{d}{\frac{d+r+1}{2}} \frac{2(r+2)}{d+r+3} \\ &= \frac{d!}{\frac{d+r-1}{2}!\,\frac{d-r+1}{2}!} \frac{2r}{d+r+1} + \frac{d!}{\frac{d+r+1}{2}!\,\frac{d-r-1}{2}!} \frac{2(r+2)}{d+r+3} \\ &= \frac{d!}{\frac{d+r-1}{2}!\,\frac{d-r-1}{2}!} \left(\frac{2r}{\frac{d-r+1}{2}(d+r+1)} + \frac{2(r+2)}{\frac{d+r+1}{2}(d+r+3)} \right) \\ &= \frac{d!}{\frac{d+r+1}{2}!\,\frac{d-r-1}{2}!} \left(\frac{2r}{d-r+1} + \frac{2(r+2)}{d+r+3} \right) = \frac{d!}{\frac{d+r+1}{2}!\,\frac{d-r-1}{2}!} \frac{4(d+1)(r+1)}{(d-r+1)(d+r+3)} \\ &= \frac{(d+1)!}{\frac{d+r+1}{2}!\,\frac{d-r+1}{2}!} \frac{2(r+1)}{(d+r+3)} = \binom{d+1}{\frac{d+r+1}{2}} \frac{2(r+1)}{d+r+3}, \end{aligned}$$

  This is exactly what (33) claims in this case.

The proposition is thus proved. □

4.8. An interesting observation that one can make is that Proposition 4.7 implies that the whole finite-dimensional representation theory of $\mathfrak{sl}_2(\Bbbk)$ can be "reconstructed" from its 2-dimensional simple module V_1, in the sense that all simple modules appear as direct summands of its tensor powers. In fact, an immediate consequence of that proposition is that for all $r \geq 0$ the sum of the isotypic components of $V_1^{\otimes r}$ with composition factors in the set $\{V_0, \dots, V_{r-1}\}$ is a submodule R which allows us to find the simple module V_r, up to isomorphism, as the quotient $V_1^{\otimes r}/R$: this provides a recursive construction of all simple modules starting from V_1.

4.9. As a special case of Proposition 4.7, we have:

Corollary. *For all $d \geq 0$,*

$$\dim(V_1^{\otimes d})^{\mathfrak{g}} = [V_1^{\otimes d} : V_0] = \begin{cases} 0, & \text{if } d \text{ is odd;} \\ C_{\frac{d}{2}}, & \text{if } d \text{ is even;} \end{cases} \tag{34}$$

where for each $n \in \mathbb{N}_0$ we are writing C_n the nth **Catalan number**,

$$C_n = \binom{2n}{n} \frac{1}{n+1}.$$

□

These numbers, named after the mathematician Eugène Charles Catalan (1814–1894, France and Belgium), are very well-known in combinatorics and famously count many different types of objects. Richard Stanley's beautiful monograph [Sta2015] provides a wealth of information about these numbers and describes, in particular, over 200 types of combinatorial objects which are enumerated by them.

4.10. Let us show another argument which also proves the formula (34) of Corollary **4.9** and which is of a rather different nature. Since $\chi_{V_1}(q) = q + q^{-1}$, we know from Proposition 3.30*(ii)* that for each $d \geq 0$ we have

$$\chi_{V_1^{\otimes d}}(q) = (q + q^{-1})^d.$$

Let us consider the series

$$f(q,t) = \sum_{d\geq 0} \chi_{V_1^{\otimes d}}(q) t^d = \sum_{d \geq 0} (q+q^{-1})^d t^d = \frac{1}{1-(q+q^{-1})t},$$

with converges absolutely for all pairs (q,t) in the open set

$$\Omega = \{(q,t) \in \mathbb{C}^2 : q \neq 0, |(q+q^{-1})t| < 1\}$$

of $\mathbb{C}^2$ and uniformly on compact sets contained there. Let $\rho \in (0, \frac{1}{4})$, let $Q = \overline{B}(0,1)$ and $T = \overline{B}(0,\rho)$ be the closed discs in $\mathbb{C}$ centered at the origin and of radii 1 and ρ, respectively, and let $S^1 \subseteq \mathbb{C}$ be the unit circle. We have for all $t \in T$ that $S^1 \times \{t\}$ is contained in Ω, so that it makes sense to consider the integral

$$\frac{1}{2\pi i} \int_{S^1} (q^{-1} - q) f(q,t) \, \mathrm{d}q = \frac{1}{2\pi i} \int_{S^1} (q^{-1} - q) \sum_{d \geq 0} \chi_{V_1^{\otimes d}}(q) t^d \, \mathrm{d}q.$$

Since the series converges uniformly on $S^1 \times T$, this is

$$\begin{aligned} &= \sum_{d\geq 0} \frac{1}{2\pi i} \int_{S^1} (q^{-1} - q) \chi_{V_1^{\otimes d}}(q) t^d \, \mathrm{d}q \\ &= \sum_{d \geq 0} \operatorname{Res}_{q=0} (q^{-1} - q) \chi_{V_1^{\otimes d}}(q) \cdot t^d \\ &= \sum_{d\geq 0} [V_1^{\otimes d} : V_0] \cdot t^d. \end{aligned}$$

Let us now fix $t \in T \setminus \{0\}$. We have

$$\frac{1}{2\pi i} \int_{S^1} (q^{-1} - q) f(q,t) \, \mathrm{d}q = \frac{1}{2\pi i} \int_{S^1} \frac{q^{-1} - q}{1 - (q + q^{-1})t} \, \mathrm{d}q = \frac{1}{2\pi i} \int_{S^1} \frac{q^2 - 1}{q^2 t - q + t} \, \mathrm{d}q.$$

The integrand in this last integral is a meromorphic function on $\mathbb{C}$ whose only poles are at the points

$$A(t) = \frac{1 + \sqrt{1 - 4t^2}}{2t}, \qquad B(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t}$$

and are both simple; indeed, we have

$$q^2 t - q + t = t(q - A(t))(q - B(t))$$

and $A(t) - B(t) = \sqrt{1-4t^2}/2t \neq 0$. As $\lim_{t\to 0}|A(t)| = \infty$ and $\lim_{t\to 0}|B(t)| = 0$, we can choose ρ small enough in $(0, \frac{1}{4})$ so that the only pole in Q is $B(t)$ and belongs to the interior. Choosing ρ in that way, we have that for each $t \in T \smallsetminus \{0\}$

$$\begin{aligned}\frac{1}{2\pi i}\int_{S^1} \frac{q^2-1}{q^2t-q+t}\,\mathrm{d}q &= \operatorname*{Res}_{q=B(t)} \frac{q^2-1}{q^2t-q+t} = \lim_{q\to B(t)} \frac{q^2-1}{q^2t-q+t}(q-B(t)) \\ &= \lim_{q\to B(t)} \frac{q^2-1}{t(q-A(t))} = \frac{B(t)^2-1}{t(B(t)-A(t))} = \frac{2}{1+\sqrt{1-4t^2}}.\end{aligned}$$

We can therefore conclude that for all $t \in T$ we have

$$\sum_{d\geq 0}[V_1^{\otimes d} : V_0]\cdot t^d = \frac{2}{1+\sqrt{1-4t^2}} = \frac{1-\sqrt{1-4t^2}}{2t^2}$$

which, according to Newton's generalized binomial formula, is

$$= \frac{1}{2t^2}\left(1 - \sum_{k\geq 0}\binom{\frac{1}{2}}{k}4^k t^{2k}\right) = \sum_{k\geq 0}\binom{2k}{k}\frac{t^{2k}}{k+1}.$$

This recovers the result of Corollary 4.9.

A nice observation to make at this point is that if we put

$$h_{TV_1}(t) = \sum_{d\geq 0}[V_1^{\otimes d} : V_0]\cdot t^d = \frac{1-\sqrt{1-4t^2}}{2t^2},$$

then we have that

$$t^2 h_{TV_1}(t)^2 - h_{TV_1}(t) + 1 = 0. \tag{35}$$

As $Th_{V_1}(t) = \sum_{d\geq 0} C_d t^{2d}$, replacing in this equation we find that

$$1 - C_0 \sum_{d\geq 2} + \left(\sum_{i+j=d-1} C_i C_j - C_d\right) t^{2d} = 0,$$

so that $C_0 = 1$ and

$$C_d = \sum_{i+j=d-1} C_i C_j$$

for all $d \geq 1$. This recurrence relation for the Catalan numbers is often used as their definition and is a key fact in many of their combinatorial interpretations.

On the other hand, we can rewrite the equation (35) in the form

$$h_{TV_1}(t) = \frac{1}{1 - t^2 h_{TV_1}(t)},$$

and iterating this formula —after making sure the right hand side makes sense— we can obtain the following expression in terms of a continued fraction:

$$h_{TV_1}(t) = \sum_{d\geq 0} [V_1^{\otimes d} : V_0] \cdot t^d = \cfrac{1}{1 - \cfrac{t^2}{1 - \cfrac{t^2}{1 - \cfrac{t^2}{1 - \cdots}}}}.$$

4.11. While this alternative proof of Corollary 4.9 may seem complicated, it differs from the original way we obtained that proposition in that it does not depend on knowing previously the multiplicities, and this is quite significant. In principle, the same idea can be used to compute the series

$$h_{TM}(t) = \sum_{d\geq 0} [M^{\otimes d} : V_0] \cdot t^d$$

for *all* finite-dimensional modules M. In practice, though, this requires solving certain equations which are not easy. Let us consider in detail the case of the 3-dimensional simple module V_2.

As before, we consider the series

$$f(q,t) = \sum_{d\geq 0} \chi_{V_2^{\otimes d}}(q) \cdot t^d = \sum_{d\geq 0} (q^2 + 1 + q^{-1})^d t^d = \frac{1}{1 - (q^2 + 1 + q^{-2})t},$$

which converges absolutely and uniformly on every compact subset of

$$\Omega = \{(q,t) \in \mathbb{C}^2 : q \neq 0, |(q^2 + 1 + q^{-2})t| < 1\}.$$

For each t in a neighborhood of 0, we have

$$\sum_{d\geq 0} [V_2^{\otimes d} : V_0] \cdot t^d = \frac{1}{2\pi i} \int_{S^1} (q^{-1} - q) f(q,t) \, \mathrm{d}q,$$

by essentially the same calculation that we did for V_1, and we are left with computing this integral. It is immediately seen to be equal to

$$\frac{1}{2\pi i} \int_{S^1} \frac{q^3 - q}{q^3 t + q^2(t-1) + t} \, \mathrm{d}q. \tag{36}$$

We restrict our attention to small but non-zero t. The denominator of this integrand factors as

$$t(q^2 - A(t))(q^2 - B(t)),$$

with

$$A(t) = \frac{1 - t + \sqrt{1 - 2t - 3t^2}}{2t}, \qquad B(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t}.$$

As $\lim_{t\to 0} A(t) = \infty$, $\lim_{t\to 0} B(t) = 0$, and $B(t) \neq 0$ for $t \neq 0$, for t sufficiently small the integrand has in the interior of S^1 two simple poles at the square roots of $B(t)$ and is holomorphic in the closed unit disc. If we denote one of those two square roots $C(t)$, it follows from all this that the integral (36) is equal to

$$\operatorname*{Res}_{q=C(t)} \frac{q^3 - q}{q^3 t + q^2(t-1) + t} + \operatorname*{Res}_{q=-C(t)} \frac{q^3 - q}{q^3 t + q^2(t-1) + t},$$

which in turn, since the two poles are simple, is the same as

$$\lim_{q\to C(t)} \frac{q^3 - q}{q^3 t + q^2(t-1) + t}(q - C(t)) + \lim_{q\to -C(t)} \frac{q^3 - q}{q^3 t + q^2(t-1) + t}(q + C(t))$$
$$= \frac{C(t)^3 - C(t)}{2C(t)(C(t)^2 - A(t))} + \frac{-C(t)^3 + C(t)}{-2C(t)(C(t)^2 - A(t))} = \frac{B(t) - 1}{B(t) - A(t)}.$$

Simplifying this last expression, we conclude that

$$h_{TV_2}(t) = \sum_{d\geq 0} [V_2^{\otimes d} : V_0] \cdot t^d = \frac{2}{1 + t + \sqrt{1 - 2t - t^3}}.$$

Let us denote R_d the coefficient $[V_2^{\otimes d} : V_0]$ with which the monomial t^d appears in this series, so that $h_{TV_2}(t) = \sum_{d\geq 0} R_d t^d$. A straighforward computation shows that

$$t(t+1)h_{TV_2}(t)^2 - (t+1)h_{TV_2}(t) + 1 = 0.$$

This implies that

$$0 = t h_{TV_2}(t)^2 - h_{TV_2}(t) + \frac{1}{1+t} = 1 - R_0 + \sum_{d\geq 1} \left(\sum_{i+j=d-1} R_i R_j - R_d + (-1)^d \right) t^d,$$

so that $R_0 = 1$ and

$$R_d = \sum_{i+j=d-1} R_i R_j + (-1)^d$$

for all $d \geq 1$. Using this recurrence relation —which is remarkably similar to the one for Catalan numbers— it is very easy to compute these numbers: the sequence starts with

$$1,\ 0,\ 1,\ 1,\ 3,\ 6,\ 15,\ 36,\ 91,\ 232,\ 603,\ 1\,585,\ 4\,213,\ 11\,298,\ 30\,537,\ 83\,097,\ 227\,475,$$
$$625\,992,\ 1\,730\,787,\ 4\,805\,595,\ 13\,393\,689,\ 37\,458\,330,\ 105\,089\,229,\ 295\,673\,994,\ \ldots$$

These are the ***Riordan numbers*** —in honor of the combinatorialist John F. Riordan (1903–1988, United States)— and their sequence appears as the entry A005043 in the OEIS [Slo2017].

Symmetric powers

4.12. Let us now consider symmetric powers. If M and N are finite-dimensional modules, then for all $d \geq 0$ there is an isomorphism of modules

$$S^d(M \oplus N) \cong \bigoplus_{i+j=d} S^i M \otimes S^j N,$$

and the Clebsch–Gordan Formula 4.1 therefore reduces, in principle, the description of the structure of the symmetric powers of an arbitrary module to that of the symmetric powers of simple ones. It makes sense, then, that we concentrate on these.

4.13. Let us fix $d, r \geq 0$, and let V_r be a finite-dimensional simple module of dimension r. We want to describe the dth symmetric power $S^d V_r$. According to Proposition 3.3, there is a basis $\mathscr{B} = \{m_0, \dots, m_r\}$ of V_r such that, among other things,

$$H \cdot m_i = (r - 2i)m_i \text{ for all } i \in \{0, \dots, r\}.$$

Let $I = \{0, \dots, r\}$ and let $I^{(d)}$ be the set of all d-tuples $\mathbf{i} = (i_1, \dots, i_d)$ in I^d that are non-decreasing, so that $1 \leq i_1 \leq \cdots \leq i_d \leq r$. For each $\mathbf{i} = (i_1, \dots, i_d) \in I^{(d)}$, we consider the element $m_\mathbf{i} = m_{i_1} \odot \cdots \odot m_{i_d} \in S^d V_r$. As we know, the set $\mathscr{B}^{(d)} = \{m_\mathbf{i} : \mathbf{i} \in I^{(d)}\}$ is a basis of the dth symmetric power $S^d V_r$. The elements of this basis are weight vectors: indeed, if $\mathbf{i} = (i_1, \dots, i_d) \in I^{(d)}$, then we have

$$\begin{aligned} H \cdot m_\mathbf{i} = H \cdot m_{i_1} \odot \cdots \odot m_{i_d} &= \sum_{j=1}^d m_{i_1} \odot \cdots (H \cdot m_{i_j}) \odot \cdots \odot m_{i_d} \\ &= \sum_{j=1}^d (r - 2i_j) m_{i_1} \odot \cdots m_{i_j} \odot \cdots \odot m_{i_d} = \left(\sum_{j=1}^d (r - 2i_j)\right) m_\mathbf{i} \end{aligned}$$

so that $m_\mathbf{i}$ is a weight vector of weight

$$\sum_{j=1}^d (r - 2i_j) = rd - 2(i_1 + \cdots + i_d).$$

It follows then from Lemma 3.29 that the character of $S^d V_r$ is

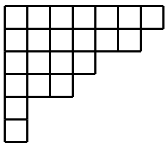
$$\chi_{S^d V_r}(q) = \sum_{\mathbf{i} \in I^{(d)}} q^{rd - 2(i_1 + \cdots + i_d)}. \tag{37}$$

We have to understand this sum, and to do that it will be convenient to use the language of partitions.

4.14. A ***partition*** is an ordered sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of integers such that $\lambda_l \geq \cdots \geq \lambda_1 \geq 1$; if $n = \lambda_1 + \cdots + \lambda_l$ we say that λ is a partition of n. The integers λ_i that appear in the partition are its ***parts*** and the number l of parts is the ***length*** of the partition; the length can be zero, and in that case the partition is necessarily a partition of 0, which is its only partition. For example, $(7,6,4,4,2,1,1)$ is a partition of 25 into 7 parts, and the fifteen partitions of 7 are

(7)	$(6,1)$	$(5,2)$	$(5,1,1)$	$(4,3)$
$(4,2,1)$	$(4,1,1,1)$	$(3,3,1)$	$(3,2,2)$	$(3,2,1,1)$
$(3,1,1,1,1)$	$(2,2,2,1)$	$(2,2,1,1,1)$	$(2,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$

We will represent a partition graphically using ***Young diagrams***: if $(\lambda_1,\dots,\lambda_l)$ is a partition, then the corresponding Young diagram is a finite collection of boxes, arranged in left-justified rows, the length of which are, from top to bottom, the parts of the partition. An example is worth a thousand words: the diagram corresponding to the partition $(7,6,4,3,1,1)$ is



For each choice of r, d, $n \in \mathbb{N}_0$ we let $\Pi(r,d,n)$ be the set of all partitions of n into at most d parts all of which are not larger than r, and write $\pi(r,d,n)$ the cardinal of the set $\Pi(r,d,n)$. In terms of diagrams, this means that a partition is in $\Pi(r,d,n)$ if the corresponding diagram has n boxes in total and fits in a rectangle of height d and width r. Among the partitions of 7 that we listed above, the elements of $\Pi(3,4,7)$ are

$(3,3,1)$	$(3,2,2)$	$(3,2,1,1)$	$(2,2,2,1)$

and therefore $\pi(3,4,7) = 4$. We make the convention that if n is non-integral, then $\Pi(r,d,n)$ is the empty set.

4.15. Let us go back to the expresion (37) for the character of $S^d V_r$. Grouping terms in the sum according to the exponent of q, we find immediately that

$$\chi_{S^d V_r}(q) = \sum_{\mathbf{i}\in I^{(d)}} q^{rd-2(i_1+\cdots+i_d)} = \sum_{n\geq 0} \#I^{(d)}(n)\cdot q^{rd-2n}$$

with $I^{(d)}(n) = \{\mathbf{i}\in I^{(d)} : i_1+\cdots+i_d = n\}$ for each $n\in\mathbb{N}_0$. Now, the sets $I^{(d)}(n)$ and $\Pi(r,d,n)$ are in bijection. If $\mathbf{i} = (i_1,\dots,i_d)$ is an element of $I^{(d)}(n)$, then dropping all the trailing zeros from the sequence $(i_d,\dots,i_1)$ obtained by reversing $\mathbf{i}$ we obtain a partition of n which has at most d parts, all of which are not larger than r, and such a thing is an element of $\Pi(r,d,n)$. Conversely, if $\lambda = (\lambda_1,\dots,\lambda_l)$ is an element of $\Pi(r,d,n)$, then adding $d-l$ zeros to end of the reversed sequence $(\lambda_l,\dots,\lambda_1)$ we obtain an element $(0,\dots,0,\lambda_l,\dots,\lambda_1)$ of $I^{(d)}(n)$. It is clear that these two constructions are mutually inverse.

We conclude in this way that

$$\chi_{S^d V_r}(q) = \sum_{n\geq 0} \pi(r,d,n)\cdot q^{rd-2n}. \tag{38}$$

r \ d	0	1	2	3	4
0	0^1	0^1	0^1	0^1	0^1
1	0^1	1^1	2^1	3^1	4^1
2	0^1	2^1	$0^1 4^1$	$2^1 6^1$	$0^1 4^1 8^1$
3	0^1	3^1	$2^1 6^1$	$3^1 5^1 9^1$	$0^1 4^1 6^1 8^1 12^1$
4	0^1	4^1	$0^1 4^1 8^1$	$0^1 4^1 6^1 8^1 12^1$	$0^1 4^2 8^2 10^1 12^1 16^1$
5	0^1	5^1	$2^1 6^1 10^1$	$3^1 5^1 7^1 9^1 11^1 15^1$	$0^1 4^2 6^1 8^2 10^1 12^2 14^1 16^1 20^1$
6	0^1	6^1	$0^1 4^1 8^1 12^1$	$2^1 6^2 8^1 10^1 12^1 14^1 18^1$	$0^2 4^2 6^1 8^3 10^1 12^3 14^1 16^2 18^1 20^1 24^1$

Table 2. The composition factors of the symmetric powers $S^d V_r$. Here the expression $0^1 2^3 4^1 5^2$ denotes the module $V_0 \oplus 3V_1 \oplus V_4 \oplus 2V_5$, and so on.

4.16. From this it is a simple matter to obtain the multiplicities of the composition factors of $S^d V_r$. The following result was announced by Arthur Cayley in 1856 [Cay1889] and proved for the first time by Joseph Sylvester in 1878 [Syl1974].

Proposition. *If d, r and s are non-negative integers, then the multiplicity of V_s as a composition factor of $S^d V_r$ is*

$$[S^d V_r : V_s] = \pi(r, d, \tfrac{1}{2}(rd - s)) - \pi(r, d, \tfrac{1}{2}(rd - s) - 1).$$

Proof. Let $d, r, s \in \mathbb{N}_0$. The multiplicity $[S^d V_r : V_s]$ is the coefficient of q^{s+1} in the the product

$$\begin{aligned}(q - q^{-1})\chi_{S^d V_r}(q) &= (q - q^{-1}) \sum_{n \geq 0} \pi(r, d, n) \cdot q^{rd-2n} \\ &= \sum_{n \geq 0} \pi(r, d, n) \cdot q^{rd-2n+1} - \sum_{n \geq 0} \pi(r, d, n) \cdot q^{rd-2n-1} \\ &= \sum_{n \geq 0} \big(\pi(r, d, \tfrac{1}{2}(rd - n + 1)) - \pi(r, d, \tfrac{1}{2}(rd - n - 1))\big) q^n\end{aligned}$$

which is $\pi(r, d, \frac{1}{2}(rd - s)) - \pi(r, d, \frac{1}{2}(rd - s) - 1)$, as the proposition claims. □

4.17. Using Proposition 4.16 we can compute —somewhat laboriously— the composition factors for all $S^d V_r$. Looking at the results some patterns become evident, and with some ingenuity they can be proved. For example, looking at the column with $d = 2$ in that table we notice that all the multiplicites are zero or one and we can actually prove this:

Proposition. *If $r \geq 0$, then*

$$S^2 V_r \cong \bigoplus_{0 \leq k \leq \frac{r}{2}} V_{2r-4k}.$$

Proof. Let $s \geq 0$ and let us compute the multiplicity $[S^2 V_r : V_s]$, which according to Proposition 4.16 is equal to

$$\pi(r, 2, \tfrac{1}{2}(2r - s)) - \pi(r, 2, \tfrac{1}{2}(2r - s) - 1). \tag{39}$$

If s is odd or if $s > 2r$ this is clearly equal to zero. We suppose that is not the case, so that s is even and $0 \leq s \leq 2r$. In that case we have $\frac{1}{2}(2r - s) - 1 < r$: it follows from this that if λ is a partition in $\Pi(r, 2, \frac{1}{2}(2r-s)-1)$, then we can add 1 to the biggest part of λ and obtain an element of $\Pi(r, 2, \frac{1}{2}(2r-s))$. This defines a function $\phi : \Pi(r, 2, \frac{1}{2}(2r-s)-1) \to \Pi(r, 2, \frac{1}{2}(2r-s))$ which is obviously injective. Moreover, a partition in $\Pi(r, 2, \frac{1}{2}(2r-s)))$ is in the image of this function exactly when it does not have two parts of the same length: only in that case we cannot shorten the biggest part by 1 to obtain a preimage. Now $\Pi(r, 2, \frac{1}{2}(2r-s))$ contains a partition with two parts of equal length if and only if the number $\frac{1}{2}(2r-s)$ is even, that is, if $2r - s$ is a multiple of 4, and when it does it obviously contains exactly one. We see in this way that

- if $2r - s$ is not divisible by 4, then the function ϕ is a bijection, and therefore its domain and codomain have the same cardinal: this means precisely that the number (39) is 0.
- On the other hand, if $2r - s$ is divisible by 4, there is exactly one element in the codomain of ϕ which is not in the image of that function. As the function is injective, this implies that (39) is equal to 1.

In other words, the number (39) is equal to zero unless there exists a $k \in \mathbb{N}_0$ with $0 \leq k \leq \frac{1}{2}r$ such that $s = 2r - 4k$, in which case it equals 1. The result follows at once from this. □

4.18. Looking at the Table 2 we easily notice that it is symmetric with respect to its diagonal. This result was obtained originally by Charles Hermite (1822–1901, France) in his work [Her1854] on the invariant theory of binary forms, and is usually called *Hermite reciprocity*.

Proposition. *If r and d are non-negative integers, there is an isomorphism of modules*

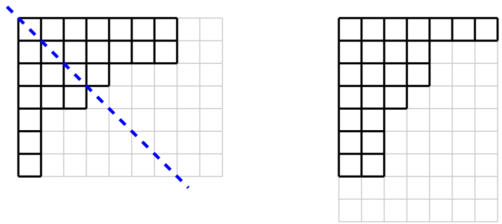
$$S^r V_d \cong S^d V_r.$$

Proof. To show this, and thanks to Proposition 4.16, it is enough to show that $\pi(r, d, n) = \pi(d, n, r)$ for all $r, d, n \geq 0$, and to do that, that there is a bijection $\Pi(r, d, n) \to \Pi(d, n, r)$.

Suppose that $\lambda = (\lambda_1, \dots, \lambda_l)$ is an element of $\Pi(r, d, n)$ of length l and largest part $m = \lambda_l$. For each $i \in \{1, \dots, m\}$ we write λ'_i the number of parts of λ which are not smaller than i, that is,

$$\lambda'_i = \#\{j \in \{1, \dots, l\} : \lambda_j \geq i\}.$$

Clearly, we have $\lambda'_1 \geq \dots \geq \lambda'_m \geq 1$, so that $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ is a partition. Counting the elements of the set $\{(i, j) \in \{1, \dots, m\} \times \{1, \dots, l\} : \lambda_j \geq i\}$ in two ways, we find that $\sum_{i=1}^m \lambda'_i = \sum_{j=1}^l \lambda_j = n$, so that λ' is a partition of n. It has m parts and its largest part is $\lambda_1 = l$, and this means that $\lambda' \in \Pi(d, r, n)$. We call λ' the ***transpose*** of λ. For example, the transpose of the partition $(7, 7, 4, 3, 1, 1, 1) \in \Pi(9, 7, 24)$ is $(7, 4, 4, 2, 2, 2) \in \Pi(7, 9, 24)$. In terms of Young diagrams, this construction corresponds exactly to transposition, that is, reflection with respect to the main diagonal:



One sees easily that $\lambda'' = \lambda$ for all $\lambda \in \Pi(r,d,n)$, so that transposition is in fact a bijection $\Pi(r,d,n) \to \Pi(d,n,r)$, like we wanted. □

4.19. A special case of Hermite reciprocity is:

Corollary. *For all* $r \geq 0$ *we have* $V_r \cong S^r V_1$.

This is another manifestation of the idea that the whole finite-dimensional representation of out Lie algebra is "contained" in its 2-dimensional simple module V_1, as we noticed in 4.8.

Proof. If $r \geq 0$, then Hermite reciprocity tells us that $S^r V_1 \cong S^1 V_r$, and this last module is obviously isomorphic to V_r itself. □

Gaussian polynomials

4.20. An immediate consequence of Proposition 4.16 is that for all $d, r, s \geq 0$ the difference

$$\pi(r,d,\tfrac{1}{2}(rd-s)) - \pi(r,d,\tfrac{1}{2}(rd-s)-1)$$

is a non-negative number: the proposition states that it is equal to the multiplicity with which a simple module appears in another module, so it cannot be a negative number! While this seems at first sight a rather inconsequential observation, it is the key ingredient of a proof of an important result which we now describe.

4.21. As usual, we fix a variable q. For each $n \geq 0$ the ***quantum integer*** $[n]_q$ is the polynomial

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q],$$

and we define the ***quantum factorial*** to be the product

$$[n]_q! = [1]_q[2]_q \cdots [n]_q.$$

Since $[n]_q$ has degree $n-1$, $[n]_q!$ has degree $(1-1)+(2-1)+\cdots+(n-1) = \frac{1}{2}n(n-1)$.

If n and m are non-negative integers such that $0 \leq m \leq n$, we call the quotient

$$\binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \tag{40}$$

a ***Gaussian polynomial*** —we will show below that it is indeed a polynomial— or, for obvious reasons, a ***quantum binomial coefficient***. If $m < 0$ or $m > n$, we make the convention that $\binom{n}{m}_q = 0$.

4.22. These "quantum" versions of classical constructions behave similarly to their classical counterparts in many ways. The simplest observation that one can make in that direction is that the evaluating at $q = 1$ the polynomials $[n]_q$, $[n]_q!$ and $\binom{n}{m}_q$ we obtain the usual integers n, $n!$ and $\binom{n}{m}$. This is the tip of an iceberg. A few more interesting results are contained in the following proposition:

Proposition. (*i*) *If $n, m \in \mathbb{N}_0$, then*

$$[n+m]_q = q^m [n]_q + [m]_q.$$

(*ii*) *If $n, m \in \mathbb{N}_0$ such that $0 \le m \le n$, then*

$$\binom{n}{0}_q = \binom{n}{n}_q = 1, \tag{41}$$

$$\binom{n}{m}_q = \binom{n}{n-m}_q \tag{42}$$

and

$$\binom{n+1}{m+1}_q = \binom{n}{m}_q + q^{m+1}\binom{n}{m+1}_q = q^{n-m}\binom{n}{m}_q + \binom{n}{m+1}_q. \tag{43}$$

Proof. Let $n, m \in \mathbb{N}_0$. We have

$$\begin{aligned} q^m [n]_q + [m]_q &= q^m(1 + q + \cdots + q^{n-1}) + (1 + q + \cdots + q^{m-1}) \\ &= q^m + q^{m+1} + \cdots q^{n+m-1} + 1 + q + \cdots q^{m-1} = [n]_q \end{aligned}$$

and this proves the first claim of the proposition. The equalities (41) and (42) follow immediately from the definition of the Gaussian polynomials. If $m = n$, then the first equality in (43) follows from (41), and if instead we have that $0 \le m < n$, then

$$\begin{aligned} \binom{n}{m}_q + q^{m+1}\binom{n}{m+1}_q &= \frac{[n]_q!}{[m]_q![n-m]_q!} + q^{m+1}\frac{[n]_q!}{[m+1]_q![n-m-1]_q!} \\ &= \frac{[n]_q!}{[m]_q![n-m-1]_q!}\left(\frac{1}{[n-m]_q} + q^{m+1}\frac{1}{[m+1]_q}\right) \\ &= \frac{[n]_q!}{[m]_q![n-m-1]_q!}\,\frac{[m+1]_q + q^{m+1}[n-m]_q}{[n-m]_q[m+1]_q} \\ &= \frac{[n]_q!}{[m]_q![n-m-1]_q!}\,\frac{[n+1]_q}{[n-m]_q[m+1]_q} = \binom{n+1}{m+1}_q. \end{aligned}$$

The second equality in (43) can be proved in exactly the same way, or deduced from the first one using (42). □

4.23. An important corollary of this result is that Gaussian polynomials are polynomials with integer coefficients:

Corollary. *If $n, m \in \mathbb{N}_0$, then $\binom{n}{m}_q$ is an element of $\mathbb{Z}[q]$ of degree nm and its coefficients are non-negative.*

Proof. That $\binom{n}{m}_q$ is a polynomial with non-negative coefficients follows by an obvious induction with respect to n, using the equalities (41) and (43) of Proposition 4.22 above. Once we know that it is a polynomial, its defining formula (40) implies that

$$\begin{aligned}\deg\binom{n}{m}_q &= \deg[n]_q! - \deg[m]_q! - \deg[n-m]_q! \\ &= \tfrac{1}{2}n(n-1) - \tfrac{1}{2}m(m-1) - \tfrac{1}{2}(n-m)(n-m-1) = nm.\end{aligned}$$ □

4.24. Gaussian polynomials and partitions are closely related:

Proposition. *If* m, $n \geq 0$, *then*

$$\binom{n+m}{m}_q = \sum_{i\geq 0} \pi(n,m,i)\cdot q^i.$$

Proof. We know that if n, $m \geq 0$, then there exist non-negative integers $a(n,m,i)$, almost all of which are zero and which are zero if $i < 0$, such that

$$\binom{n+m}{m}_q = \sum_{i\in\mathbb{Z}} a(n,m,i)q^i.$$

Proposition 4.22 tells us that $\binom{n}{0}_q = 1$ for all $n \geq 0$, and this means that

$$a(n,0,i) = a(0,n,i) = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{if not.}\end{cases} \tag{44}$$

On the other hand, that proposition tells us that $\binom{n+m+1}{m+1}_q = \binom{n+m}{m}_q + q^{m+1}\binom{n+m}{m+1}_q$ for all m, $n \geq 0$, and in terms of the coefficients this means that for all $i \geq 0$ we have

$$a(n,m+1,i) = a(n,m,i) + a(n-1,m+1,i-m-1).$$

We claim that we also have, for all m, n, $i \geq 0$, that

$$\pi(n,0,i) = \pi(0,n,i) = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{if not;}\end{cases} \tag{45}$$

and

$$\pi(n,m+1,i) = \pi(n,m,i) + \pi(n-1,m+1,i-m-1). \tag{46}$$

The first equality is immediate: there are no partitions of a non-negative integer i with zero parts or with parts of size zero, unless i itself is equal to zero. To prove (46) we observe that if λ is an element of $\Pi(n,m+1,i)$, then

- either it has at most m parts, so that it is an element of $\Pi(n,m,i)$,
- or it has exactly $m+1$ parts, and then subtracting 1 to each part of λ and removing any leading zero that result from that we obtain a partition of $i-m-1$ with at most $m+1$ parts all of which are at most equal to $n-1$, that is, an element of $\Pi(n-1,m+1,i-m-1)$.

Clearly, we account in this way for all elements of $\Pi(n, m, i)$ and all those of $\Pi(n-1, m+1, i-m-1)$ and, looking at the cardinals of these sets, we find that (46) holds.

We can now prove that $a(n, m, i) = \pi(n, m, i)$ for all $n, m, i \geq 0$, and with that the proposition. Suppose, in fact, that this is not true, and let (n, m) be the smallest element in $\mathbb{N}_0 \times \mathbb{N}_0$ with respect to the lexicographic order on this set such that there exists an $i \in \mathbb{N}_0$ with $a(n, m, i) \neq \pi(n, m, i)$. In view of (44) and (45), we have $n > 0$ and $m > 0$, and the two pairs $(n-1, m)$ and $(n, m-1)$ both belong to $\mathbb{N}_0 \times \mathbb{N}_0$. Since they are strictly lexicographically smaller than (n, m), the way we chose the latter implies that

$$\begin{aligned} a(n, m, i) &= a(n, m-1, i) + a(n-1, m, i-m) = \pi(n, m-1, i) + \pi(n-1, m, i-m) \\ &= \pi(n, m, i), \end{aligned}$$

and this is absurd. This completes the proof. □

4.25. A consequence of Proposition 4.24 is that we can express the characters of symmetric powers in terms of Gaussian polynomials:

Corollary. *If r, $d \geq 0$, then*

$$\chi_{S^d V_r}(q) = q^{rd} \binom{r+d}{d}_{q^{-2}}.$$

Proof. To prove this, we need only compare the expression (38) that we found in 4.15 for the character of $S^d V_r$ with the expression of Gaussian polynomials in terms of partitions given by Proposition 4.24. □

4.26. We say that a finite sequence of real numbers $a_1, a_2, \dots, a_n$ is ***unimodal*** if it first increases and then decreases, that is, if there is an index $t \in \{0, \dots, n\}$ such that

$$a_1 \leq a_2 \leq \cdots \leq a_{t-1} \leq a_t \geq a_{t+1} \geq \cdots \geq a_n.$$

Similarly, we say that a polynomial with real coefficients is unimodal if the sequence of its coefficients ordered accoding to degree is unimodal.

Proposition. *If n, $m \in \mathbb{N}_0$, then the Gaussian polynomial $\binom{n+m}{m}_q$ is unimodal.*

In view of Corollary 4.25, the character of $S^d V_r$ is also an unimodal polynomial for all r, $d \geq 0$.

Proof. As we have shown above in Proposition 4.24, we have that

$$\binom{n+m}{m}_q = \sum_{i \geq 0} \pi(n, m, i) \cdot q^i.$$

It is clear this has degree nm and that it is a symmetric polynomial, since for all $i \in \{0, \dots, nm\}$ we have that $\pi(n, m, i) = \pi(n, m, nm - i)$. To see that this polynomial is unimodal, it suffices to show that $\pi(n, m, i) - \pi(n, m, i-1) \geq 0$ for all integers i such that $1 \leq i \leq \frac{1}{2}nm$. But for such an i the number $s = nm - 2i$ is non-negative and, using Proposition 4.16, we have that

$$\begin{aligned} \pi(n, m, i) - \pi(n, m, i-1) &= \pi(n, m, \tfrac{1}{2}(nm - s)) - \pi(n, m, \tfrac{1}{2}(nm - s) - 1) \\ &= [S^m V_n : V_s] \geq 0. \end{aligned}$$

This proves what we want. □

4.27. The first proof of Proposition 4.26 was given by E. B. Elliott in 1895 in his book [Ell1895] on the theory of invariants of forms, building up on the work Sylvester [Syl1974] on the subject —who had proved, as we mentioned above, Proposition 4.16— and his argument was essentially the same one we used. Since then, many alternative proofs have been provided, of analytical or geometrical nature —the most elementary one being that of Robert Proctor presented in [Pro1982], which depends only on linear algebra. Proctor's paper describes the history of the problem and explains why it is an important one. The first purely combinatorial proof of this result was given by Kathleen O'Hara in [O'H1990], in a celebrated *tour de force*; an explanation of her argument with some simplifications can be found in Doron Zeilberger's paper [Zei1989].

The idea of using the representation theory of Lie algebras in order to prove unimodality results is a very fuitful one. There is a whole family of results in this direction, which includes Proposition 4.26 as its simplest example, starting from work of Eugene Dynkin [Dyn1950]. The survey [Sta1980] of Richard Stanley explains this.

Invariants of symmetric powers

4.28. We now want to study the invariants of the symmetric powers of our simple modules. We fix a non-negative integer r and intend to describe the invariant subspace of $S^d V_r$ for all $d \geq 0$ and, in particular, the series

$$h_r(t) = \sum_{d\geq 0} \dim(S^d V_r)^{\mathfrak{g}} \cdot t^d. \tag{47}$$

As we did for tensor powers, we consider first the formal series

$$f_r(q,t) = \sum_{d\geq 0} \chi_{S^d V_r}(q) \cdot t^d.$$

This converges absolutely whenever (q,t) belongs to the set

$$\Omega = \{(q,t) \in \mathbb{C}^2 : |q| < 1, |t| < \tfrac{1}{2(r+1)}\}$$

and does so uniformly on compact subsets contained in Ω. Indeed, since $\chi_{S^d V_r}(q)$ is a polynomial with non-negative coefficients, we have for all q with $|q| < 1$ that

$$|\chi_{S^d V_r}(q)| \leq \chi_{S^d V_r}(1) = \binom{r+d}{d}.$$

If additionally $|t| < \frac{1}{2(r+1)}$, then we have that

$$\sum_{d\geq 0} |\chi_{S^d V_r}(q) \cdot t^d| \leq \sum_{d\geq 0} \binom{r+d}{d} \frac{1}{2^d (r+1)^d}$$

and this last numerical series converges, as can be seen by an easy application of d'Alembert's ratio test. Our claim about the convergence of the series $f_r(q,t)$ follows then from Weierstrass's M-test. We can in fact sum the series:

Lemma. *For each $r \geq 0$ we have*

$$f_r(q,t) = \prod_{i=0}^{r} \frac{1}{1-q^{r-2i}t} \tag{48}$$

for all (q,t) belonging to the set $\Omega' = \{(q,t) \in \mathbb{C}^2 : \frac{1}{2} < |q| < 1, |t| < \frac{1}{2^r(r+1)}\}$.

Notice that Ω' is contained in the set Ω described above, so that the equality makes sense. Since f_r is a holomorphic function, it follows form the lemma that the equality (48) holds in fact throughout Ω and that we can in fact continue analytically f_r to a rational function on the whole of $\mathbb{C}^2$.

Proof. Let us fix (q,t) in Ω', then we have $|q^{r-2i}t| < 1$ for all $i \in \{0,\dots,r\}$ and we can therefore expand each factor appearing on the right in (48) into a geometric series, obtaining the equality

$$\prod_{i=0}^{r} \frac{1}{1-q^{r-2i}t} = \prod_{i=0}^{r} \sum_{d\geq 0} q^{(r-2i)d} t^d.$$

Each of the series appearing in this product converges absolutely, so we can distribute the product, finding

$$\sum_{d_0,\dots,d_r \geq 0} q^{\sum_{i=0}^{r}(r-2i)d_i} t^{d_0+\cdots+d_r} = \sum_{d_0,\dots,d_r \geq 0} q^{r(d_0+\cdots+d_r)-2(0d_0+1d_1+\cdots+rd_r)} t^{d_0+\cdots+d_r}, \tag{49}$$

with the series converging absolutely. Grouping terms according to the values of the sums $d_0+\cdots+d_r$ and $0d_0+1d_1+\cdots+rd_r$, we see that this is

$$\sum_{n,d\geq 0} a_{n,d} q^{rd-2n} t^d$$

with $a_{n,d}$ the number of $(r+1)$-tuples $(d_0,\dots,d_r)$ of non-negative integers such that $d = d_0+\cdots+d_r$ and $n = 0d_0 + 1d_1 + \cdots + rd_r$. Now, from such an $(r+1)$-tuple we can construct a partition with d_1 parts equal to 1, d_2 parts equal to 2 and so on, all the way to d_r parts equal to r: this partitition has at most d parts, all of which are at most equal to r, and sums to n: it is therefore an element of $\Pi(r,d,n)$. It is clear that all the elements of this set are obtained in this way, each of them exactly once: it follows from this that $a_{n,d} = \pi(r,d,n)$, and thus

$$\prod_{i=0}^{r} \frac{1}{1-q^{r-2i}t} = \sum_{n,d\geq 0} \pi(r,d,d) q^{rd-2n} t^d.$$

This series was obtained by grouping terms in (49), so it also converges absolutely, and we can can again associate its terms in whatever way we want. In particular, its sum equals that of

$$\sum_{d\geq 0} \left(\sum_{n\geq 0} \pi(r,d,n) q^{rd-2n} \right) \cdot t^d,$$

which is, according to the formula (38) that we obtained in 4.15, the same as $f_r(q,t)$. This proves the lemma. □

4.29. Just as in 4.11, the series h_r of (47) that we are trying to compute can be expressed as an integral. Indeed, let us fix $t \in \mathbb{C}$ such that $0 < |t| < \frac{1}{2(r+1)}$ and let γ be the circle of radius $\frac{1}{2}$ around the origin in $\mathbb{C}$. We know that

$$\begin{aligned} h_r(t) &= \sum_{d\geq 0} \dim(S^d V_r)^{\mathfrak{g}} \cdot t = \sum_{d\geq 0} \operatorname*{Res}_{q=0}(q - q^{-1})\chi_{S^d V_r}(q) \cdot t^d \\ &= \sum_{d\geq 0} \frac{1}{2\pi i} \int_\gamma (q - q^{-1})\chi_{S^d V_r}(q)\, \mathrm{d}q \cdot t^d, \end{aligned}$$

because the Laurent polynomial $(q^{-1} - q)\chi_{S^d V_r}(q)$ is meromorphic in a open set containing the closed interior of γ, continuous on γ and with exactly one pole in the interior, at 0. The last series is equal to

$$\frac{1}{2\pi i} \int_\gamma (q^{-1} - q) \sum_{d\geq 0} \chi_{S^d V_r}(q) \cdot t^d \,\mathrm{d}q$$

because the series appearing here converges absolutely and uniformly on γ. In view of lemma, we therefore have that for all t such that $0 < |t| < \frac{1}{2(r+1)}$ the function h_r is given by

$$h_r(t) = \tfrac{1}{2\pi i} \int_\gamma \frac{q^{-1} - q}{\prod_{i=0}^r (1 - q^{r-2i} t)}\, \mathrm{d}q.$$

We are left with evaluating this integral. The poles of the integrand which are in the interior of γ are the numbers

Suppose first that r is odd and at least 3, and that $s \in \mathbb{N}$ is such that $r = 2s + 1$. We have that

$$\begin{aligned} \prod_{i=0}^{r} (1 - q^{r-2i} t) &= \prod_{i=0}^{s} (1 - q^{r-2i} t) \prod_{i=s+1}^{2s+1} (1 - q^{r-2i} t) = \prod_{i=0}^{s} (1 - q^{r-2i} t) \prod_{i=0}^{s} (1 - q^{-r+2i} t) \\ &= q^{\sum_{i=0}^{s}(-r+2i)} \prod_{i=0}^{s} (1 - q^{r-2i} t) \prod_{i=0}^{s} (q^{r-2i} - t) \\ &= q^{-(s+1)^2} \prod_{i=0}^{s} (1 - q^{r-2i} t) \prod_{i=0}^{s} (q^{2i+1} - t) \end{aligned}$$

and it follows from this that

$$\frac{q^{-1} - q}{\prod_{i=0}^r (1 - q^{r-2i} t)} = \frac{q^{(s+1)^2 - 1} - q^{(s+1)^2 + 1}}{\prod_{i=0}^s (1 - q^{r-2i} t) \prod_{i=0}^s (q^{2i+1} - t)}.$$

For each $n \in \mathbb{N}$ let $\omega_n = \exp(2\pi\sqrt{-1}/n)$ and let $t^{1/n}$ be any one of the nth roots of t. The poles of this rational function inside the circle γ are the numbers

$$\omega_{2i+1}^j t^{1/(2i+1)}, \qquad 0 \leq i \leq s,\ 0 \leq j < 2i + 1$$

and they are all simple. It follows that

$$h_r(t) = \sum_{i=0}^{s} \sum_{j=0}^{2i} \operatorname*{Res}_{q=\omega_{2i+1}^j t^{1/(2i+1)}} \frac{q^{(s+1)^2 - 1} - q^{(s+1)^2 + 1}}{\prod_{i=0}^s (1 - q^{r-2i} t) \prod_{i=0}^s (q^{2i+1} - t)}$$

The integrand has poles at the points of the set

$$\{q \in \mathbb{C} : q \neq 0 \text{ and } q^{r-2i} = t \text{ for some } i \in \{0, \dots, r\}\}$$

and $h_r(t)$ is equal to the sum of the residules of that integrand at the poles which are in the interior of γ. Let us consider the first possible values of r.

- If $r = 0$, we have can compute the integral immediately:

$$h_0(t) = \tfrac{1}{2\pi i} \int \gamma \frac{q^{-1} - q}{1 - t} \, \mathrm{d}q = \tfrac{1}{1-t} = \sum_{d \geq 0} t^d.$$

  This tells us that $\dim(S^d V_0)^{\mathfrak{g}} = 1$ for al $d \geq 0$. Of course, we already knew this, since $S^d V_0 \cong V_0$ for all $d \geq 0$.

- Let now $r = 1$. The formula for $h_r(t)$ tells us that

$$h_1(t) = \frac{1}{2\pi i} \int_\gamma \frac{q^{-1} - q}{(1 - qt)(1 - q^{-1}t)} \, \mathrm{d}q = \frac{1}{2\pi i} \int_\gamma \frac{1 - q^2}{(1 - qt)(q - t)} \, \mathrm{d}q.$$

  The only pole of the last integrand in the interior of γ is at $q = t$, and then

$$h_1(t) = \operatorname*{Res}_{q=t} \frac{1 - q^2}{(1 - qt)(q - t)}$$

  and, since the pole at $q = t$ is simple, this is

$$= \lim_{q \to t} \frac{1 - q^2}{(1 - qt)} = 1.$$

  This means that $\sum_{d \geq 0} \dim(S^d V_1)^{\mathfrak{g}} \cdot t^d = 1$, so that

$$\dim(S^d V_1)^{\mathfrak{g}} = \begin{cases} 1, & \text{if } d = 0; \\ 0, & \text{if } d > 0. \end{cases}$$

  Again, we already knew this: as $S^d V_1 \cong V_r$, the invariant subspace of $S^d V_1$ is isomorphic to $V_d^{\mathfrak{g}}$, which is the zero space if $d > 0$ and 1-dimensional if $d = 0$.

- Consider now the case in which $r = 2$. We have that

$$\begin{aligned} h_2(t) &= \frac{1}{2\pi i} \int_\gamma \frac{q^{-1} - q}{(1 - q^2 t)(1 - t)(1 - q^{-2} t)} \, \mathrm{d}q \\ &= \frac{1}{2\pi i} \int_\gamma \frac{q - q^3}{(1 - q^2 t)(1 - t)(q^2 - t)} \, \mathrm{d}q \end{aligned}$$

  The poles of the integrand which are in the interior of γ are at the square roots of t, so that if we denote s on of them we have

$$h_2(t) = \operatorname*{Res}_{q=s} \frac{q - q^3}{(1 - q^2 t)(1 - t)(q^2 - t)} + \operatorname*{Res}_{q=-s} \frac{q - q^3}{(1 - q^2 t)(1 - t)(q^2 - t)}$$

and since the two poles are simple, this is equal to

$$\begin{aligned}
&\lim_{q\to s}\frac{q-q^3}{(1-q^2t)(1-t)(q^2-t)}(q-s)+\lim_{q\to -s}\frac{q-q^3}{(1-q^2t)(1-t)(q^2-t)}(q+s)\\
&\quad=\frac{s-s^3}{(1-s^2t)(1-t)(s+s)}+\frac{-s-s^3}{(1-s^2t)(1-t)(-s-s)}\\
&\quad=\frac{s-s^3}{(1-s^2t)(1-t)s}=\frac{1}{1-t}
\end{aligned}$$

We see in this way that

$$\dim(S^dV_1)^{\mathfrak{g}}=\begin{cases}1, & \text{if } d \text{ is even;}\\ 0, & \text{if it is odd.}\end{cases}$$

- Let $r=3$, so that

$$\begin{aligned}
h_3(t)&=\frac{1}{2\pi i}\int_\gamma\frac{q^{-1}-q}{(1-q^3t)(1-qt)(1-q^{-1}t)(1-q^{-3}t)}\,\mathrm{d}q\\
&=\frac{1}{2\pi i}\int_\gamma\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}\,\mathrm{d}q
\end{aligned}$$

The integrand has simple poles at t and at the three cube roots of t. At t the residue is

$$\begin{aligned}
&\operatorname*{Res}_{q=t}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}\\
&\quad=\lim_{q\to t}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}(q-t)=\frac{t^3-t^5}{(1-t^4)(1-t^2)(t^3-t)}\\
&\quad=\frac{t^2}{(1-t^4)(t^2-1)}.
\end{aligned}$$

On the other hand, if s is a cube root of t, then

$$\begin{aligned}
&\operatorname*{Res}_{q=s}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}\\
&\quad=\lim_{q\to s}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}(q-s)\\
&\quad=\frac{s^3-s^5}{(1-s^3t)(1-st)(s-t)(1+s+s^2)}=\frac{s^2}{(1-s^6)(1-s^4)(1+s+s^2)},
\end{aligned}$$

and if ω is one of the non-real cubic roots of 1,

$$\begin{aligned}
&\operatorname*{Res}_{q=\omega s}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}\\
&\quad=\lim_{q\to\omega s}\frac{q^3-q^5}{(1-q^3t)(1-qt)(q-t)(q^3-t)}(q-\omega s)\\
&\quad=\frac{s^3-\omega^2s^5}{(1-s^3t)(1-\omega st)(\omega s-t)(\omega s-t)(\omega s-\omega^2 s)}
\end{aligned}$$

Exterior powers

The Grothendieck ring

4.30. If M is a module, we denote $[M]$ its isomorphism class, and we write $\mathscr{M}$ the set of all isomorphism classes of finite-dimensional modules —it can be shown that this is indeed a set.

Let $G(\mathfrak{g})$ be the free abelian group with basis $\mathscr{M}$. Its elements are formal linear combinations of elements of $\mathscr{M}$ with coefficients in $\mathbb{Z}$. As $G(\mathfrak{g})$ is free, it is easy to show that there is a unique $\mathbb{Z}$-bilinear function $\cdot : G(\mathfrak{g}) \times G(\mathfrak{g}) \to G(\mathfrak{g})$ such that

$$[M] \cdot [N] = [M \otimes N]$$

for all finite-dimensional modules M and N. Endowed with this map as multiplication, the abelian group $G(\mathfrak{g})$ becomes a commutative ring:

- The multiplication distributes over addition simply because the map $\cdot$ is $\mathbb{Z}$-bilinear.
- To show that $\cdot$ is an associative operation it is enough —thanks to distributivity and the fact that every element of $G(\mathfrak{g})$ is a $\mathbb{Z}$-linear combination of elements of $\mathscr{M}$— to show that for all finite-dimensional modules M, N and P we have $[M] \cdot ([N] \cdot [P]) = ([M] \cdot [N]) \cdot [P]$. The definition of the product implies that the left and right hand sides of this equations are $[M \otimes (N \otimes P)]$ and $[(M \otimes N) \otimes P]$, respectively, and these two classes are equal because, according to Proposition 2.14*(i)*, there is an isomorphism $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$
- The isomorphism class $[\Bbbk]$ of the trivial module is a unit element in $G(\mathfrak{g})$. Indeed, if M is a finite-dimensional module, then Proposition 2.14*(iii)* tells us that there are isomorphisms of modules $M \otimes \Bbbk \cong M \cong \Bbbk \otimes M$, so that

  $$[M] \cdot [\Bbbk] = [M \otimes \Bbbk] = [M] = [\Bbbk \otimes M] = [\Bbbk] \cdot [M],$$

  and then bilinearity of the product implies that $c \cdot [\Bbbk] = c = [\Bbbk] \cdot c$ for all $c \in G(\mathfrak{g})$.
- Finally, if M and N are finite-dimensional modules, Proposition 2.14*(ii)* tells us that there is an isomorphisms of modules $M \otimes N \cong N \otimes M$, so that

  $$[M] \cdot [N] = [M \otimes N] = [N \otimes M] = [N] \cdot [M],$$

  and it follows from this that in fact $c \cdot d = d \cdot c$ for all $c, d \in G(\mathfrak{g})$.

4.31. If $\mathscr{E}$ is a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$$

we consider the element

$$c_{\mathscr{E}} = [M] - [E] + [N] \in G(\mathfrak{g}),$$

and let $I(\mathfrak{g})$ be the subgroup of $G(\mathfrak{g})$ generated by all elements of this form. This subgroup is in fact an ideal of $G(\mathfrak{g})$. To see this, we notice that if $\mathscr{E}$ is a short exact sequence of modules as above and P is a finite-dimensional module, then we can construct a new exact sequence of modules

$$0 \longrightarrow P \otimes M \xrightarrow{\mathrm{id}_P \otimes f} P \otimes E \xrightarrow{\mathrm{id}_P \otimes g} P \otimes N \longrightarrow 0$$

which we denote $P \otimes \mathscr{E}$, whose corresponding element in $G(\mathfrak{g})$ is

$$\begin{aligned} c_{P\otimes\mathscr{E}} &= [P \otimes M] - [P \otimes E] + [P \otimes N] \\ &= [P] \cdot [M] - [P] \cdot [E] + [P] \cdot [N] \\ &= [P] \cdot ([M] - [E] + [N]) \\ &= [P] \cdot c_{\mathscr{E}}, \end{aligned}$$

so that $[P] \cdot c_{\mathscr{E}} \in I(\mathfrak{g})$. As the elements of $\mathscr{M}$ generate $G(\mathfrak{g})$ as an abelian group, this is enough to prove that $I(\mathfrak{g})$ is an ideal.

We may therefore consider the quotient

$$K(\mathfrak{g}) = G(\mathfrak{g})/I(\mathfrak{g}),$$

which is a ring and which we call the ***Grothendieck ring*** of our Lie algebra. If M is a finite-dimensional module, we will write $[\![M]\!]$ the class $[M] + I(\mathfrak{g})$ of $[M]$ in $K(\mathfrak{g})$.

4.32. As an immediate consequence of the way we constructed the Grothendieck ring, we have the following result:

Proposition. (*i*) *If*

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$$

is a short exact sequence of finite-dimensional modules, then in $K(\mathfrak{g})$ we have that

$$[\![E]\!] = [\![M]\!] + [\![N]\!]. \tag{50}$$

(*ii*) *If M and N are finite-dimensional modules, we have that*

$$[\![M \oplus N]\!] = [\![M]\!] + [\![N]\!] \tag{51}$$

and

$$[\![M \otimes N]\!] = [\![M]\!] \cdot [\![N]\!]. \tag{52}$$

Proof. (*i*) If we have a short exact sequence of modules as in the statement, then we know that $[M] - [E] + [N]$ is an element of the ideal $I(\mathfrak{g})$, so that its image in $K(\mathfrak{g})$ is equal to zero. As this image is clearly $[\![M]\!] - [\![E]\!] + [\![N]\!]$, we have the equality (50).

(*ii*) Let M and N be two finite-dimensional modules. As we have the split short exact sequence

$$0 \longrightarrow M \xrightarrow{\binom{\mathrm{id}_M}{0}} M \oplus N \xrightarrow{(0\ \mathrm{id}_N)} N \longrightarrow 0$$

the first part of the proposition tells us that the equality (51) holds. On the other hand, the equality (52) follows immediately form the fact that the projection function $p : G(\mathfrak{g}) \to K(\mathfrak{g})$ is a morphism of rings. □

4.33. Since $K(\mathfrak{g})$ is defined as a quotient of a free abelian group, it is not obvious what its group structure is. A direct consequence of our results on semisimplicity, we can prove that it is in fact free.

Proposition. *The set $\mathscr{S} = \{[\![V_r]\!] : r \in \mathbb{N}_0\}$ of classes in $K(\mathfrak{g})$ corresponding to the simple modules is a basis of $K(\mathfrak{g})$ as an abelian group.*

Proof. If M is a finite-dimensional module, Theorem 3.14 tells us that there exist $n \in \mathbb{N}_0$ and simple modules $S_1, \dots, S_n$ such that $M \cong \bigoplus_{i=1}^n S_i$ and using Proposition 4.32*(ii)* we see from that that $[\![M]\!] = \sum_{i=1}^n [\![S_i]\!]$, which is in the subgroup of $K(\mathfrak{g})$ generated by $\mathscr{S}$. This clearly implies that the set $\mathscr{S}$ generates $K(\mathfrak{g})$ as an abelian group. To prove the proposition, then, we need to show that the set $\mathscr{S}$ is linearly independent.

Let P be a finite-dimensional module. Since the abelian group $G(\mathfrak{g})$ is free in the set $\mathscr{M}$, there exists a morphism of groups $\bar{\epsilon}_P : G(\mathfrak{g}) \to \mathbb{Z}$ such that $\bar{\epsilon}_P([M]) = \dim \hom_{\mathfrak{g}}(P, M)$ for all finite-dimensional modules M. If now $\mathscr{E}$ is a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$$

of finite-dimensional modules, then it follows from Proposition 5.11 that we also have a short exact sequence of finite-dimensional vector spaces

$$0 \longrightarrow \hom_{\mathfrak{g}}(P, M) \xrightarrow{f_*} \hom_{\mathfrak{g}}(P, E) \xrightarrow{g_*} \hom_{\mathfrak{g}}(P, N) \longrightarrow 0$$

and then, of course, we have that

$$\dim \hom_{\mathfrak{g}}(P, M) - \dim \hom_{\mathfrak{g}}(P, E) + \dim \hom_{\mathfrak{g}}(P, N) = 0.$$

The left hand side in this equality is $\bar{\epsilon}_P(c_E)$: we see in this way that the morphism $\bar{\epsilon}_P$ maps the ideal $I(\mathfrak{g})$ to zero, because it maps each of its generators to zero. This implies that there is a morphism of groups $\epsilon_P : K(\mathfrak{g}) \to \mathbb{Z}$ such that

$$\epsilon_P([\![M]\!]) = \bar{\epsilon}_P([M]) = \dim \hom_{\mathfrak{g}}(P, M)$$

for all finite-dimensional modules M. In particular, if S and T are finite-dimensional simple modules, we have —in view of Lemma 3.15— that

$$\epsilon_S([\![T]\!]) = \begin{cases} 1, & \text{if } S \text{ and } T \text{ are isomorphic;} \\ 0, & \text{if not.} \end{cases} \tag{53}$$

With this at hand, we can easily prove that the set $\mathscr{S}$ is linearly independent. Suppose that $n \in \mathbb{N}$, that $r_1, \dots, r_n \in \mathbb{N}_0$ are n distinct non-negative integers, and that $a_1, \dots, a_r \in \mathbb{Z}$ are such that $\sum_{i=1}^n a_i [\![V_{r_i}]\!] = 0$ in $K(\mathfrak{g})$. If j is an element of $\{1, \dots, n\}$, applying the morphism $\epsilon_{V_{r_j}}$ to both sides of this equality and using (53) we find at once that $a_j = 0$. This establishes the linear independence of $\mathscr{S}$, and completes the proof. □

§5. Appendix: Extensions of modules

Extensions

5.1. If P and N are modules, an ***extension of P by N*** is a short exact sequence

$$\mathscr{E}: \qquad 0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

of modules and morphisms of modules which starts at P and ends at N. If

$$\mathscr{E}': \qquad 0 \longrightarrow N \xrightarrow{f'} M' \xrightarrow{g'} P \longrightarrow 0$$

is another extension of P by N, then we say that a morphism of modules $\phi : M \to M'$ is a ***morphism of extensions from $\mathscr{E}$ to $\mathscr{E}'$*** if the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & P & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{f'} & M' & \xrightarrow{g'} & P & \longrightarrow & 0 \end{array}$$

is commutative.

5.2. An important observation is the following result, known as the *Short Five Lemma*:

Lemma. *If $\mathscr{E}$ and $\mathscr{E}'$ are extensions of P by N as above and the morphism of modules $\phi : M \to M'$ is a morphism of extensions from $\mathscr{E}$ to $\mathscr{E}'$, then ϕ is an isomorphism of modules and its inverse morphism $\phi^{-1} : M' \to M$ is a morphism of extensions from $\mathscr{E}'$ to $\mathscr{E}$.*

In view of this, we say that the two extensions $\mathscr{E}$ and $\mathscr{E}'$ of P by N are ***isomorphic*** if there is a morphism of extensions from $\mathscr{E}$ to $\mathscr{E}'$. It is easy to see, using the lemma, that this defines an equivalence relation among extensions of P by N.

Proof. If $m \in M$ is such that $\phi(m) = 0$, then $g(m) = g'(\phi(m)) = 0$ and therefore there exists an $n \in N$ such that $m = f(n)$. Now $f'(n) = \phi(f(n)) = \phi(m) = 0$ and, since f' is injective, $n = 0$. Of course, this tells us that $m = f(n) = 0$ and we see that the morphism ϕ is injective.

Let now $m' \in M'$. Since the morphism g is surjective, there exists an $m \in M$ such that $g(m) = g'(m')$ and we have

$$g'(m' - \phi(m)) = g'(m') - g'(\phi(m)) = g(m) - g'(\phi(m)) = 0.$$

By exactness of $\mathscr{E}'$, there exists an $n \in N$ with $f'(n) = m' - \phi(m)$. Since

$$\phi(m + f(n)) = \phi(m) + \phi(f(n)) = \phi(m) + f'(n) = m',$$

we see that the element m' is in the image of ϕ. This tells us that the map ϕ is surjective and we may conclude that it is in fact an isomorphism. Finally, since ϕ is a morphism of extensions from $\mathscr{E}$ to $\mathscr{E}'$, we have $g = g' \circ \phi$ and $\phi \circ f = f'$, and composing on the right with ϕ^{-1} in the first equality and on the right in the second one we see that $g \circ \phi^{-1} = g'$ and $f = \phi^{-1} \circ f'$. These two equations tell us that ϕ^{-1} is a morphism of extensions from $\mathscr{E}'$ to $\mathscr{E}$. □

Split extensions

5.3. If P and N are modules, The ***split extension of*** P ***by*** N is the extension

$$0 \longrightarrow N \xrightarrow{\binom{\mathrm{id}_N}{0}} N \oplus P \xrightarrow{(\,0\ \mathrm{id}_P\,)} P \longrightarrow 0$$

and, more generally, we say that an extension of P by N ***splits*** if it is isomorphic to the split extension of P by N. Of course, the split extension of P by N is split.

5.4. The main reason that explains our interest in split extensions is the following:

Proposition. *Let P and N be modules. If*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

is an extension of P by N which is split, then there exists an isomorphism $M \cong N \oplus P$.

Proof. Let $\mathscr{E}$ be an extension of P by N as in the statement. If it is split, then there exists a morphism $\phi : M \to N \oplus M$ of modules which is a morphism of extensions from $\mathscr{E}$ to the split extension $\mathscr{E}_\oplus$ of N by N. According to Lemma 5.2, this morphism ϕ is then an isomorphism. □

5.5. It is useful to be able to recognize split extensions easily, and the following proposition helps in doing that:

Proposition. *Let*

$$\mathscr{E}: \qquad 0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

be an extension. The following three statements are equivalent:

(a) *The extension $\mathscr{E}$ is split.*
(b) *There exists a morphism of modules $r : M \to N$ such that $r \circ f = \mathrm{id}_N$.*
(c) *There exists a morphism of modules $s : P \to M$ such that $g \circ s = \mathrm{id}_P$.*

If they hold, then the morphisms s and r appearing in (b) *and* (c)*, respectively, can be chosen in such a way that $r \circ s = 0$.*

We call a morphism r with the property described in (*b*) a ***retraction*** of the morphism f, and a morphism s with the property described in (*c*) a ***section*** of the morphism g.

Proof. Let us write $\mathscr{E}_\oplus$ the split extension of P by N.

Suppose first that the extension $\mathscr{E}$ is split, so that there is a morphism of extensions $\phi : \mathscr{E} \to \mathscr{E}_\oplus$, that is, a morphism $\phi : M \to N \oplus P$ making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & P & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow[\binom{\mathrm{id}_N}{0}]{} & N \oplus P & \xrightarrow[(\,0\ \mathrm{id}_P\,)]{} & P & \longrightarrow & 0
\end{array}$$

commutative. The map $r = \left(\begin{smallmatrix} \mathrm{id}_M & 0 \end{smallmatrix}\right) \circ \phi : M \to N$ is a retraction of f, since

$$r \circ f = \begin{pmatrix} \mathrm{id}_M & 0 \end{pmatrix} \circ \phi \circ f = \begin{pmatrix} \mathrm{id}_M & 0 \end{pmatrix} \circ \left(\begin{smallmatrix} id_N \\ 0 \end{smallmatrix}\right) = \mathrm{id}_N,$$

and the map $s = \phi^{-1} \circ \left(\begin{smallmatrix} 0 \\ \mathrm{id}_P \end{smallmatrix}\right) : P \to M$ is a section of s, since

$$g \circ s = g \circ \phi^{-1} \circ \begin{pmatrix} 0 \\ \mathrm{id}_P \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_P \end{pmatrix} \circ \begin{pmatrix} 0 \\ \mathrm{id}_P \end{pmatrix} = \mathrm{id}_P.$$

This shows that the condition (a) implies both (b) and (c). Moreover, since we have

$$r \circ s = \begin{pmatrix} \mathrm{id}_M & 0 \end{pmatrix} \circ \phi \circ \phi^{-1} \circ \begin{pmatrix} 0 \\ \mathrm{id}_P \end{pmatrix} = \begin{pmatrix} \mathrm{id}_M & 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ \mathrm{id}_P \end{pmatrix} = 0,$$

when the condition (a) holds we can choose the retraction r of f and the section s of g so that $r \circ s = 0$, as the last sentence of the proposition claims.

Let us suppose now that the condition (b) holds, so that there is a retraction $r : M \to N$ of f. The morphism $\phi = \left(\begin{smallmatrix} r \\ g \end{smallmatrix}\right) : M \to N \oplus P$ is such that

$$\begin{pmatrix} 0 & \mathrm{id}_P \end{pmatrix} \circ \phi = \begin{pmatrix} 0 & \mathrm{id}_P \end{pmatrix} \circ \begin{pmatrix} r \\ g \end{pmatrix} = g$$

and

$$\phi \circ f = \begin{pmatrix} r \\ g \end{pmatrix} \circ f = \begin{pmatrix} r \circ f \\ g \circ f \end{pmatrix} = \begin{pmatrix} \mathrm{id}_N \\ 0 \end{pmatrix},$$

and is therefore a morphism of extensions $\phi : \mathscr{E} \to \mathscr{E}_\oplus$. The extension $\mathscr{E}$ is thus split in this situation.

Finally, let us suppose that condition (c) holds, and let $s : P \to M$ be a section of g. The morphism $h = \mathrm{id}_M - s \circ g : M \to M$ is such that

$$g \circ h = g - g \circ s \circ g = g - g = 0$$

and, since f is a kernel of g, this implies that there exists a morphism $r : M \to N$ such that $f \circ r = h = \mathrm{id}_M - s \circ g$. Now, as

$$f \circ r \circ f = f - s \circ g \circ f = f \circ \mathrm{id}_N$$

and the morphism f is injective, we have that $r \circ f = \mathrm{id}_N$, that is, that r is a retraction of f and, then, that the condition (b) holds. This completes the proof. □

Filtrations

5.6. If M is a module and $t \in \mathbb{N}$, a ***filtration*** of length t on M is a sequence

$$M_0 \subseteq M_1 \subset \cdots \subseteq M_t$$

of submodules of M such that $M_0 = 0$ and $M_t = M$, and the quotient modules $M_1/M_0, \dots, M_t/M_{t-1}$ are the ***subquotients*** of the filtration.

5.7. Proposition. *Let M and P be modules and let*

$$M_0 \subseteq M_1 \subset \cdots \subseteq M_t$$

be a filtration of length $t \in \mathbb{N}$ of M. If every extension of O by one of the subquotients of this filtration splits, then every extension of P by M splits.

Proof. We proceed by induction on the length of the filtration. If $t = 1$, then there is nothing to prove, as the unique subquotient of the filtration is then $M_1/M_0 \cong M$, so the conclusion coincides with the hypothesis.

Suppose next that $t > 2$. Since

$$M_0 \subseteq M_1 \subset \cdots \subseteq M_{t-1}$$

is a filtration of length $t - 1$ of M_{t-1} and since every extension of P by one of its subquotients splits —simply because its subquotients are some of the subquotients of the filtration of M— the inductive hypothesis tells us that every extension of P by M_{t-1} splits.

We have a short exact sequence of the form

$$0 \longrightarrow M_{t-1} \longrightarrow M_t \longrightarrow M_t/M_{t-1} \longrightarrow 0$$

and we know that every extension of P by M_{t-1} or by M_t/M_{t-1} splits, so Proposition 5.8 tells us that every extension of P by M_t splits. As $M = M_t$, this proves what we want.

Suppose finally that $t = 2$. In that case, the filtration amounts to a choice of a submodule M_1 in $M_2 = M$, and the hypothesis is that in the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0$$

every extension of P by either M_1 or M/M_1 splits. That every extension of P by M splits is then a consequence of Proposition 5.8 that we will now prove. □

5.8. Proposition. *Let Q be a module and let*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of modules. If every extension of Q by N or by P splits, then every extension of Q by M splits.

Proof. Let us suppose that every extension of Q by N or by P splits, and let

$$0 \longrightarrow M \xrightarrow{u} E \xrightarrow{v} Q \longrightarrow 0 \tag{54}$$

be an extension of Q by M. Let us consider the map $\alpha = \left(\begin{smallmatrix} u \\ g \end{smallmatrix}\right) : M \to E \oplus P$, let E' be its cokernel and let $\beta = \left(\begin{smallmatrix} g' & u' \end{smallmatrix}\right) : E \oplus P \to E'$ be the canonical projection, so that we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha=\left(\begin{smallmatrix} u \\ g \end{smallmatrix}\right)} E \oplus P \xrightarrow{\beta=\left(\begin{smallmatrix} g' & u' \end{smallmatrix}\right)} E' \longrightarrow 0 \tag{55}$$

and, in particular,

$$g' \circ u + u' \circ g = 0. \tag{56}$$

The morphism $\gamma = (\, v \;\; 0\,) : E \oplus P \to Q$ is such that $\gamma \circ \alpha = (\, v \;\; 0\,) \circ \left(\begin{smallmatrix} u \\ g \end{smallmatrix}\right) = v \circ u = 0$, so the universal property of the cokernel tells us that it factors through E', that is, that there exists a morphism $v' : E' \to Q$ such that $\gamma = v' \circ \beta$, which means that

$$v' \circ g' = v \tag{57}$$

and

$$v' \circ u' = 0. \tag{58}$$

Let us check that

$$0 \longrightarrow P \xrightarrow{u'} E' \xrightarrow{v'} Q \longrightarrow 0 \tag{59}$$

is an extension of Q by P.

- Suppose that $p \in P$ is such that $u'(p) = 0$. It follows then that $\beta\left(\begin{smallmatrix} 0 \\ p \end{smallmatrix}\right) = 0$ and the exactness of (55) tells us that there is an $m \in M$ such that

  $$\begin{pmatrix} 0 \\ p \end{pmatrix} = \alpha(n) = \begin{pmatrix} u(n) \\ g(n) \end{pmatrix}.$$

  Since u is an injective function, we see that $n = 0$, and therefore that $p = g(n) = 0$. This means that the morphism u' is injective.
- According to (58), we have $v' \circ u' = 0$. Suppose that $e' \in E'$ is such that $v'(e') = 0$. Since the map β in (55) is surjective, there exist $e \in E$ and $p \in P$ such that

  $$e' = \beta\begin{pmatrix} e \\ p \end{pmatrix} = g'(e) + u'(p). \tag{60}$$

  It follows from this that

  $$0 = v'(e') = v'(g'(e)) + v'(u'(p)) = v(e),$$

  in view of (57) and (58) and, since (54) is exact, that there exists an $m \in M$ such that $u(m) = e$. Now

  $$\begin{aligned} u'(p - g(m)) &= u'(p) - u'(g(m)) \\ &= u'(p) + g'(u(m)) && \text{because of (56)} \\ &= u'(p) + g'(e) && \text{because of (60)} \\ &= e' \end{aligned}$$

  and this shows that e' is in the image of u'. We thus see that the sequence (59) is exact at E'.

- Finally, if $q \in Q$, from the exactness of the sequence (57) it follows that there exists an $e \in E$ such that $v(e) = q$ and then, using (57), that

  $$g = v(e) = v'(g'(e))$$

  is in the image of v'. The morphism v' is thus surjective.

This completes the proof of exactness of the sequence (59), which is therefore an extension of Q by P. The hypothesis then tells us that this extension splits and, according to Proposition 5.5, this implies that there exists morphisms $r : E' \to P$ and $s' : Q \to E'$ such that

$$\begin{aligned} &r \circ u' = \mathrm{id}_P, \\ &v' \circ s = \mathrm{id}_Q \end{aligned} \tag{61}$$

and

$$r \circ s = 0.$$

Let now E'' be the kernel of $r \circ g'$ and let $f' : E'' \to E$ be the inclusion. Of course, we have that

$$r \circ g' \circ f' = 0. \tag{62}$$

As

$$r \circ g' \circ u \circ f = -r \circ u' \circ g \circ f = 0,$$

there exists a morphism $u'' : N \to E'$ such that

$$f' \circ u'' = u \circ f. \tag{63}$$

We put $v'' = v \circ f' : E'' \to Q$ and consider the sequence

$$0 \longrightarrow N \xrightarrow{u''} E'' \xrightarrow{v''} Q \longrightarrow 0 \tag{64}$$

This is also an exact sequence:

- Since u and f are injective morphisms, the equality (63) implies immediately that u'' is also injective.
- Using that same equality (63) we see that

  $$v'' \circ u'' = v \circ f' \circ u'' = v \circ u \circ f = 0.$$

  Let $e'' \in E''$ be such that $v''(e'') = v(f'(e'')) = 0$. Since the sequence (54) is exact, there exists an $m \in M$ such that $u(m) = f'(e'')$. We can now compute that

  $$\begin{aligned} g(m) &= r(u'(g(m))) && \text{because of (61)} \\ &= -r(g'(u(m)) && \text{because of (56)} \\ &= -r(g'(f'(e'')) && \text{in view of the choice of } e'' \\ &= 0 && \text{because of (62).} \end{aligned}$$

Using the exactness of the short exact sequence that appears in the statement of the proposition we see that there exists an $n \in N$ such that $f(n) = m$. Now

$$f'(u''(n)) = u(f(n)) = u(m) = f'(e'')$$

and since the morphism f' is injective this implies that $u''(n) = e''$. The sequence (64) is therefore exact at E''.

- Finally, let $q \in Q$. Since the morphism v is surjective, there is an $e \in E$ such that $v(e) = q$, and since the morphism g is surjective, an $m \in M$ such that $g(m) = r(g'(e))$. As

$$\begin{aligned} r(g'(e - u(m)) &= r(g'(e)) - r(g'(u(m))) \\ &= r(g'(e)) - r(u'(g(m))) \\ &= g(m) - g(m) = 0 \end{aligned}$$

the definition of E'' and the morphism f' implies that there exists an $e'' \in E''$ such that $f'(e'') = e - u(m)$, and then

$$v''(e'') = v(f'(e'')) = v(e) - v(u(m)) = q.$$

As the sequence (64) is an extension of Q by N, it is split by hypothesis and, as before, there exist morphisms $r' : E'' \to N$ and $s' : Q \to E''$ such that

$$\begin{aligned} &r' \circ u'' = \mathrm{id}_N, \\ &v'' \circ s' = \mathrm{id}_Q \end{aligned}$$

and

$$r' \circ s' = 0.$$

We now consider the sequence

$$0 \longrightarrow N \xrightarrow{\phi = \left(\begin{smallmatrix} -u'' \\ f \end{smallmatrix}\right)} E'' \oplus M \xrightarrow{\psi = \left(\begin{smallmatrix} f' & u \end{smallmatrix}\right)} E \longrightarrow 0 \tag{65}$$

and —for the last time!— show that it is exact:

- Since the morphism u'' is injective, it is clear that the morphism ϕ appearing here is injective.
- We have

$$\psi \circ \phi = \begin{pmatrix} f' & u \end{pmatrix} \circ \begin{pmatrix} -u'' \\ f \end{pmatrix} = -f' \circ u'' + f \circ u = 0.$$

Let, on the other hand, $e'' \in E''$ and $m \in M$ be such that $\psi\left(\begin{smallmatrix} e'' \\ m \end{smallmatrix}\right) = f'(e'') + u(m) = 0$. As

$$v''(e'') = v(f'(e'')) = -v(u(m)) = 0,$$

the exactness of the sequence (64) implies that there exists an $n \in N$ such that $u''(n) = -e''$, and then

$$u(f(n)) = f'(u'(n)) = f'(-e'') = u(m),$$

so that $f(n) = m$, because the morphism u is injective. It follows that $\phi(n) = \binom{e''}{m}$ and, as a consequence of this, that the sequence (65) is exact at $E'' \oplus M$.

- Finally, let $e \in E$. Since the morphism g is surjective, there exists an $m \in M$ such that $g(m) = r(g'(e))$, and then

$$r(g'(e-u(m))) = r(g'(e))-r(g'(u(m))) = g(m)-r(u'(g(m)) = g(m)-g(m) = 0,$$

so that there exists an $e'' \in E''$ such that $f'(e'') = e - u(m)$. Then

$$\psi\begin{pmatrix} e'' \\ m \end{pmatrix} = f'(e'') + u(m) = e.$$

This shows that the morphism ψ is surjective.

Let us now consider the morphism

$$w = \begin{pmatrix} f \circ r' & \mathrm{id}_M \end{pmatrix} : E'' \oplus M \to M$$

Since $w \circ \phi = -r \circ r' \circ u'' + f = 0$, the exactness of the sequence (65) implies that there exists a morphism $\sigma : E \to M$ such that $w = \sigma \circ \psi$, and this means that

$$\begin{pmatrix} f \circ r' & \mathrm{id}_M \end{pmatrix} = \begin{pmatrix} \sigma \circ f' & \sigma \circ u \end{pmatrix}$$

and, in particular, $\mathrm{id}_M = \sigma \circ u$. We conclude in this way that the morphism σ is a retraction for the map u appearing in the extension (55) and that the latter is therefore split. This proves the proposition. □

Projectivity

5.9. If $\mathscr{C}$ is a class of modules, we say that a module P is ***projective relative the class $\mathscr{C}$*** if every extension of P by a module belonging to $\mathscr{C}$ is split.

5.10. Projectivity is usually presented in a slightly different but equivalent way:

Proposition. *Let $\mathscr{C}$ be a class of modules. A module P is projective relative to the class $\mathscr{C}$ if and only if for each short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} Q \longrightarrow 0 \tag{66}$$

in which the module M belongs to $\mathscr{C}$ we have that

> *if $\phi : P \to Q$ is a morphism of modules, then there exists a morphism of modules $\bar{\phi} : P \to N$ such that $g \circ \bar{\phi} = \mathfrak{g}$.*

Proof. Let us suppose first that P is projective relative to the class $\mathscr{C}$, let us consider a short exact sequence as in (66) with the the module M belonging to $\mathscr{C}$ and let $\phi : P \to Q$ be a morphism.

Let $\beta = (\, g \; \phi \,) : N \oplus P \to Q$, let E be the kernel of β and let $\alpha = \begin{pmatrix} \phi' \\ g' \end{pmatrix} : E \to N \oplus P$ be the inclusion, so that we have a short exact sequence

$$0 \longrightarrow E \xrightarrow{\alpha = \begin{pmatrix} \phi' \\ g' \end{pmatrix}} N \oplus P \xrightarrow{\beta = (\, g \; \phi \,)} Q \longrightarrow 0 \tag{67}$$

and, in particular,

$$g \circ \phi' + \phi \circ g' = 0.$$

The map $\gamma = \begin{pmatrix} f \\ 0 \end{pmatrix} : M \to N \oplus P$ is such that $\beta \circ \gamma = g \circ f = 0$, so that the universal property of the kernel tells us that there exists a morphism $f' : M \to E$ such that $\alpha \circ f' = \gamma$, that is, such that

$$\phi' \circ f' = f \tag{68}$$

and

$$g' \circ f' = 0. \tag{69}$$

Let us show that the sequence

$$0 \longrightarrow M \xrightarrow{f'} E \xrightarrow{g'} P \longrightarrow 0 \tag{70}$$

is exact.

- Since the morphism f is injective, the equality (68) implies at once that so is the morphism f'.
- The equation (69) tells us that $g' \circ f' = 0$. Suppose that $e \in E$ is such that $g'(e) = 0$. Since $g(\phi'(e)) = -\phi(g'(e)) = 0$, the exactness of the sequence (66) tells us that there exists an $m \in M$ such that $f(m) = \phi'(e)$. Now

  $$\alpha(f'(m) - e) = \begin{pmatrix} \phi'(f'(m)) - \phi'(e) \\ g'(f'(m)) - g'(e) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  and the morphism α is injective, so that $f'(m) = e$.
- If $p \in P$, then —since the morphism g is surjective— there exists an $n \in N$ such that $g(n) = -\phi(p)$ and therefore

  $$\beta \begin{pmatrix} n \\ p \end{pmatrix} = g(n) + \phi(p) = 0.$$

  The exactness of the sequence (67) implies then that there is an $e \in E$ such that

  $$\begin{pmatrix} n \\ p \end{pmatrix} = \alpha(e) = \begin{pmatrix} \phi'(e) \\ g'(e) \end{pmatrix}$$

  and, in particular, $\phi'(e) = p$.

Since M belongs to the class $\mathscr{C}$, the extension (70) if P by M is split, and there exists a section $s : P \to E$ of g', so that $g' \circ s = \mathrm{id}_P$. Using this, we see at once that the morphism $\bar{\phi} = -\phi' \circ s$ satisfies the desired condition:

$$g \circ \bar{\phi} = g \circ (-\phi' \circ s) = -g \circ \phi' \circ s = \phi \circ g' \circ s = \phi.$$

This shows that the condition in the proposition is necessary.

Let us now suppose that that condition is satisfied, and let us show that the module is then projective relative to the class $\mathscr{C}$. Suppose for this that

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} P \longrightarrow 0 \tag{71}$$

is an extension of P by a module M belonging to the class $\mathscr{C}$. By hypothesis, there exists a morphism $s : P \to E$ such that $g \circ s = \mathrm{id}_P$. This means precisely that s is a section of g, and then the extension (71) is split. This shows that P is projective relative to the class $\mathscr{C}$, as we wanted. □

5.11. Proposition. *Let $\mathscr{C}$ be a class of modules and let P be a module which is projective relative to $\mathscr{C}$. If*

$$0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$$

is a short exact sequence of modules in $\mathscr{C}$, then the sequence of vector spaces

$$0 \longrightarrow \hom_{\mathfrak{g}}(P, M) \xrightarrow{f_*} \hom_{\mathfrak{g}}(P, E) \xrightarrow{g_*} \hom_{\mathfrak{g}}(P, N) \longrightarrow 0$$

is also exact.

Proof. HACER □

5.12. Proposition. *Let $\mathscr{C}$ be a class of modules and let $\tilde{\mathscr{C}}$ be the class of all modules which admit a filtration of finite length whose subquotients all belong to the class $\mathscr{C}$. A module is projective relative to the class $\mathscr{C}$ if and only if it is projective relative to the class $\tilde{\mathscr{C}}$.*

Proof. The necessity of the condition is the content of Proposition 5.7 and its sufficiency is evident since $\tilde{\mathscr{C}} \subseteq \mathscr{C}$. □

§6. References

[Bae2002] John C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205, available at https://arxiv.org/abs/math/0105155. MR1886087 ↑6

[Cay1889] Arthur Cayley, *A second memoir upon quantics*, Collected Math. Papers, Vol. 2, Cambridge University Press, New York, 1889, pp. 250–275. ↑54

[CS2003] John H. Conway and Derek A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A K Peters, Ltd., Natick, MA, 2003. MR1957212 ↑6

[Dyn1950] Eugene B. Dynkin, *Some properties of the system of weights of a linear representation of a semisimple Lie group*, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 221–224 (Russian). MR0033813 ↑60

[Ell1895] Edwin Bayley Elliott, *An Introduction to the Algebra of Quantics*, Oxford University Press, 1895. ↑60

[Her1854] Charles Hermite, *Sur la theorie des fonctions homogenes à deux indéterminées*, Cambridge and Dublin Mathematical Journal **9** (1854), 172–217, available at http://resolver.sub.uni-goettingen.de/purl?PPN600493962. ↑55

[O'H1990] Kathleen M. O'Hara, *Unimodality of Gaussian coefficients: a constructive proof*, J. Combin. Theory Ser. A **53** (1990), no. 1, 29–52, DOI 10.1016/0097-3165(90)90018-R. MR1031611 ↑60

[Pro1982] Robert A. Proctor, *Solution of two difficult combinatorial problems with linear algebra*, Amer. Math. Monthly **89** (1982), no. 10, 721–734, DOI 10.2307/2975833. MR683197 ↑60

[Sch1904] Issai Schur, *Neue Begründung der Theorie der Gruppencharaktere*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **1904** (1904), 406–432 (German). ↑26

[Slo2017] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (2017), http://oeis.org/. ↑51

[Sta1980] Richard P. Stanley, *Unimodal sequences arising from Lie algebras*, Combinatorics, representation theory and statistical methods in groups, Lecture Notes in Pure and Appl. Math., vol. 57, Dekker, New York, 1980, pp. 127–136, available at http://math.mit.edu/~rstan/pubs/pubfiles/41.pdf. MR588199 ↑60

[Sta2015] Richard P. Stanley, *Catalan numbers*, Cambridge University Press, New York, 2015. MR3467982 ↑48

[Syl1974] James Joseph Sylvester, *Proof of the hitherto undemonstrated fundamental theory of invariants*, Collected Math. Papers, Vol. 3, Chelsea, New York, 1974. ↑54, 60

[Zei1989] Doron Zeilberger, *Kathy O'Hara's Constructive Proof of the Unimodality of the Gaussian Polynomials*, The American Mathematical Monthly **96** (1989), no. 7, 590-602, available at http://www.jstor.org/stable/2325177. ↑60