Chapter 7

The Prime Number Theorem

In this final chapter we will take advantage of an opportunity to apply many of the ideas and results from earlier chapters in order to give an analytic proof of the famous prime number theorem: If $\pi(x)$ is the number of primes less than or equal to x, then $x^{-1}\pi(x)\ln x \to 1$ as $x \to \infty$. That is, $\pi(x)$ is asymptotically equal to $x/\ln x$ as $x \to \infty$. (In the sequel, prime will be taken to mean *positive* prime.)

Perhaps the first recorded property of $\pi(x)$ is that $\pi(x) \to \infty$ as $x \to \infty$, in other words, the number of primes is infinite. This appears in Euclid's "Elements". A more precise result that was established much later by Euler (1737) is that the series of reciprocals of the prime numbers,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots ,$$

is a divergent series. This can be interpreted in a certain sense as a statement about how fast $\pi(x) \to \infty$ as $x \to \infty$. Later, near the end of the 18-th century, mathematicians, including Gauss and Legendre, through mainly empirical considerations, put forth conjectures that are equivalent to the above statement of the prime number theorem (PNT). However, it was not until nearly 100 years later, after much effort by numerous 19-th century mathematicians, that the theorem was finally established (independently) by Hadamard and de la Vallée Poussin in 1896. The quest for a proof led Riemann, for example, to develop complex variable methods to attack the PNT and related questions. In the process, he made a remarkable and as yet unresolved conjecture known as the Riemann hypothesis, whose precise statement will be given later. Now it is not clear on the surface that there is a connection between complex analysis and the distribution of prime numbers. But in fact, every proof of the PNT dating from Hadamard and de la Vallée Poussin, up to 1949 when P. Erdös and A.Selberg succeeded in finding "elementary" proofs, has used the methods of complex variables in an essential way. In 1980, D.J. Newman published a new proof of the PNT which, although still using complex analysis, nevertheless represents a significant simplification of previous proofs. It is Newman's proof, as modified by J. Korevaar, that we present in this chapter.

There are a number of preliminaries that must be dealt with before Newman's method can be applied to produce the theorem. The proof remains far from trivial but the steps along the way are of great interest and importance in themselves. We begin by introducing the Riemann zeta function, which arises via Euler's product formula and forms a key link between the sequence of prime numbers and the methods of complex variables.

7.1 The Riemann Zeta function

The *Riemann zeta function* is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

where $n^z = e^{z \ln n}$. Since $|n^z| = n^{\operatorname{Re} z}$, the given series converges absolutely on $\operatorname{Re} z > 1$ and uniformly on $\{z : \operatorname{Re} z \geq 1 + \delta\}$ for every $\delta > 0$. Let $p_1, p_2, p_3, \dots$ be the sequence 2,3,5, ... of prime numbers and note that for $j = 1, 2, \dots$ and $\operatorname{Re} z > 1$, we have

$$\frac{1}{1 - 1/p_j^z} = 1 + \frac{1}{p_j^z} + \frac{1}{p_j^{2z}} + \cdots .$$

Now consider the partial product

$$\prod_{j=1}^{m} \frac{1}{1 - p_j^{-z}} = \prod_{j=1}^{m} (1 + \frac{1}{p_j^z} + \frac{1}{p_j^{2z}} + \cdots).$$

By multiplying the finitely many absolutely convergent series on the right together, rearranging, and applying the fundamental theorem of arithmetic, we find that the product is the same as the sum $\sum_{n \in P_m} \frac{1}{n^z}$, where P_m consists of 1 along with those positive integers whose prime factorization uses only primes from the set $\{p_1, \dots, p_m\}$. Therefore

$$\prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-z}} = \sum_{n=1}^{\infty} \frac{1}{n^z}, \ \operatorname{Re} z > 1.$$

We now state this formally.

7.1.1 Euler's Product formula

For $\operatorname{Re} z > 1$, the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ is given by the product

$$\prod_{j=1}^{\infty} \left(\frac{1}{1 - p_j^{-z}} \right)$$

where $\{p_j\}$ is the (increasing) sequence of prime numbers.

The above series and product converge uniformly on compact subsets of $\operatorname{Re} z > 1$, hence ζ is analytic on $\operatorname{Re} z > 1$. Furthermore, the product representation of ζ shows that ζ has no zeros in $\operatorname{Re} z > 1$ (Theorem 6.1.7). Our proof of the PNT requires a number of additional properties of ζ. The first result is concerned with extending ζ to a region larger than $\operatorname{Re} z > 1$.

7.1.2 Extension Theorem for Zeta

The function $\zeta(z) - 1/(z-1)$ has an analytic extension to the right half plane $\operatorname{Re} z > 0$. Thus ζ has an analytic extension to $\{z : \operatorname{Re} z > 0, z \neq 1\}$ and has a simple pole with residue 1 at $z = 1$.

Proof. For $\operatorname{Re} z > 1$, apply the summation by parts formula (Problem 2.2.7) with $a_n = n$ and $b_n = 1/n^z$ to obtain

$$\sum_{n=1}^{k-1} n \left[\frac{1}{(n+1)^z} - \frac{1}{n^z} \right] = \frac{1}{k^{z-1}} - 1 - \sum_{n=1}^{k-1} \frac{1}{(n+1)^z}.$$

Thus

$$1 + \sum_{n=1}^{k-1} \frac{1}{(n+1)^z} = \frac{1}{k^{z-1}} - \sum_{n=1}^{k-1} n \left[\frac{1}{(n+1)^z} - \frac{1}{n^z} \right].$$

But

$$n \left[\frac{1}{(n+1)^z} - \frac{1}{n^z} \right] = -nz \int_n^{n+1} t^{-z-1}\, dt = -z \int_n^{n+1} [t] t^{-z-1}\, dt$$

where $[t]$ is the largest integer less than or equal to t. Hence we have

$$\sum_{n=1}^{k} \frac{1}{n^z} = 1 + \sum_{n=1}^{k-1} \frac{1}{(n+1)^z} = \frac{1}{k^{z-1}} + z \sum_{n=1}^{k-1} \int_n^{n+1} [t] t^{-z-1}\, dt$$
$$= \frac{1}{k^{z-1}} + z \int_1^k [t] t^{-z-1}\, dt.$$

Letting $k \to \infty$, we obtain the integral formula

$$\zeta(z) = z \int_1^\infty [t] t^{-z-1}\, dt \tag{1}$$

for $\operatorname{Re} z > 1$. Consider, however, the closely related integral

$$z \int_1^\infty t t^{-z-1}\, dt = z \int_1^\infty t^{-z}\, dt = \frac{z}{z-1} = 1 + \frac{1}{z-1}.$$

Combining this with (1) we can write

$$\zeta(z) - \frac{1}{z-1} = 1 + z \int_1^\infty ([t] - t) t^{-z-1}\, dt.$$

Now fix $k > 1$ and consider the integral $\int_1^k ([t] - t) t^{-z-1}\, dt$. By (3.3.3), this integral is an entire function of z. furthermore, if $\operatorname{Re} z > 0$, then

$$\left| \int_1^k ([t] - t) t^{-z-1}\, dt \right| \leq \int_1^k t^{-\operatorname{Re}(z+1)}\, dt \leq \int_1^\infty t^{-1-\operatorname{Re} z}\, dt = \frac{1}{\operatorname{Re} z}.$$

This implies that the sequence $f_k(z) = \int_1^k ([t]-t)t^{-z-1}\,dt$ of analytic functions on $\mathrm{Re}\,z > 0$ is uniformly bounded on compact subsets. Hence by Vitali's theorem (5.1.14), the limit function

$$f(z) = \int_1^\infty ([t]-t)t^{-z-1}\,dt$$

(as the uniform limit on compact subsets of $\mathrm{Re}\,z > 0$) is analytic, and thus the function

$$1 + z\int_1^\infty ([t]-t)t^{-z-1}\,dt$$

is also analytic on $\mathrm{Re}\,z > 0$. But this function agrees with $\zeta(z) - \frac{1}{z-1}$ for $\mathrm{Re}\,z > 1$, and consequently provides the required analytic extension of ζ to $\mathrm{Re}\,z > 0$. This completes the proof of the theorem. ♣

We have seen that Euler's formula (7.1.1) implies that ζ has no zeros in the half plane $\mathrm{Re}\,z > 1$, but how about zeros of (the extension of) ζ in $0 < \mathrm{Re}\,z \le 1$? The next theorem asserts that ζ has no zeros on the line $\mathrm{Re}\,z = 1$. This fact is crucial to our proof of the PNT.

7.1.3 Theorem

The Riemann zeta function has no zeros on $\mathrm{Re}\,z = 1$, so $(z-1)\zeta(z)$ is analytic and zero-free on a neighborhood of $\mathrm{Re}\,z \ge 1$.

Proof. Fix a real number $y \neq 0$ and consider the auxiliary function

$$h(x) = \zeta^3(x)\zeta^4(x+iy)\zeta(x+i2y)$$

for x real and $x > 1$. By Euler's product formula, if $\mathrm{Re}\,z > 1$ then

$$\ln|\zeta(z)| = -\sum_{j=1}^\infty \ln|1-p_j^{-z}| = -\mathrm{Re}\sum_{j=1}^\infty \mathrm{Log}(1-p_j^{-z}) = \mathrm{Re}\sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nz}$$

where we have used the expansion $-\mathrm{Log}(1-w) = \sum_{n=1}^\infty w^n/n$, valid for $|w| < 1$. Hence

$$\begin{aligned}
\ln|h(x)| &= 3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+i2y)| \\
&= 3\,\mathrm{Re}\sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nx} + 4\,\mathrm{Re}\sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nx-iny} \\
&\quad + \mathrm{Re}\sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nx-i2ny} \\
&= \sum_{j=1}^\infty\sum_{n=1}^\infty \frac{1}{n}p_j^{-nx}\,\mathrm{Re}(3 + 4p_j^{-iny} + p_j^{-i2ny}).
\end{aligned}$$

But $p_j^{-iny} = e^{-iny\ln p_j}$ and $p_j^{-i2ny} = e^{-i2ny\ln p_j}$. Thus $\mathrm{Re}(3 + 4p_j^{-iny} + p_j^{-i2ny})$ has the form

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1+\cos\theta)^2 \ge 0.$$

Therefore $\ln|h(x)| \geq 0$ and consequently

$$|h(x)| = |\zeta^3(x)||\zeta^4(x+iy)||\zeta(x+i2y)| \geq 1.$$

Thus

$$\frac{|h(x)|}{x-1} = |(x-1)\zeta(x)|^3 \left|\frac{\zeta(x+iy)}{x-1}\right|^4 |\zeta(x+i2y)| \geq \frac{1}{x-1}.$$

But if $\zeta(1+iy) = 0$, then the left hand side of this inequality would approach a finite limit $|\zeta'(1+iy)|^4|\zeta(1+i2y)|$ as $x \to 1^+$ since ζ has a simple pole at 1 with residue 1. However, the right hand side of the inequality contradicts this. We conclude that $\zeta(1+iy) \neq 0$. Since y is an arbitrary nonzero real number, ζ has no zeros on $\operatorname{Re} z = 1$. ♣

Remark

The ingenious introduction of the auxiliary function h is due to Mertens (1898). We now have shown that any zeros of ζ in $\operatorname{Re} z > 0$ must lie in the strip $0 < \operatorname{Re} z < 1$. The study of the zeros of ζ has long been the subject of intensive investigation by many mathematicians. Riemann had stated in his seminal 1859 paper that he considered it "very likely" that all the zeros of ζ in the above strip, called the *critical strip*, lie on the line $\operatorname{Re} z = 1/2$. This assertion is now known as the *Riemann hypothesis*, and remains as yet unresolved. However, a great deal *is* known about the distribution of the zeros of ζ in the critical strip, and the subject continues to capture the attention of eminent mathematicians. To state just one such result, G.H. Hardy proved in 1915 that ζ has infinitely many zeros on the line $\operatorname{Re} z = 1/2$. Those interested in learning more about this fascinating subject may consult, for example, the book *Riemann's Zeta Function* by H.M. Edwards. Another source is http://mathworld.wolfram.com/RiemannHypothesis.html.

We turn next to zeta's logarithmic derivative ζ'/ζ, which we know is analytic on $\operatorname{Re} z > 1$. In fact, more is true, for by (7.1.3), ζ'/ζ is analytic on a neighborhood of $\{z : \operatorname{Re} z \geq 1 \text{ and } z \neq 1\}$. Since ζ has a simple pole at $z = 1$, so does ζ'/ζ, with residue $\operatorname{Res}(\zeta'/\zeta, 1) = -1$. [See the proof of (4.2.7).] We next obtain an integral representation for ζ'/ζ that is similar to the representation (1) above for ζ. [See the proof of (7.1.2).] But first, we must introduce the *von Mangoldt function* Λ, which is defined by

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\Lambda(n)$ is $\ln p$ if n is a power of the prime p, and is 0 if not. Next define ψ on $x \geq 0$ by

$$\psi(x) = \sum_{n \leq x} \Lambda(n). \tag{2}$$

An equivalent expression for ψ is

$$\psi(x) = \sum_{p \leq x} m_p(x) \ln p,$$

where the sum is over primes $p \le x$ and $m_p(x)$ is the largest integer such that $p^{m_p(x)} \le x$. (For example, $\psi(10.4) = 3\ln 2 + 2\ln 3 + \ln 5 + \ln 7$.) Note that $p^{m_p(x)} \le x$ iff $m_p(x)\ln p \le \ln x$ iff $m_p(x) \le \frac{\ln x}{\ln p}$. Thus $m_p(x) = \left[\frac{\ln x}{\ln p}\right]$ where as before, [ ] denotes the greatest integer function. The function ψ will be used to obtain the desired integral representation for ζ'/ζ.

7.1.4 Theorem

For $\operatorname{Re} z > 1$,

$$-\frac{\zeta'(z)}{\zeta(z)} = z\int_1^\infty \psi(t)t^{-z-1}\,dt \tag{3}$$

where ψ is defined as above.

Proof. In the formulas below, p and q range over primes. If $\operatorname{Re} z > 1$, we have $\zeta(z) = \prod_p (1-p^{-z})^{-1}$ by (7.1.1), hence

$$\begin{aligned}\zeta'(z) &= \sum_p \frac{-p^{-z}\ln p}{(1-p^{-z})^2}\prod_{q\ne p}\frac{1}{1-q^{-z}}\\ &= \zeta(z)\sum_p \frac{-p^{-z}\ln p}{(1-p^{-z})^2}(1-p^{-z})\\ &= \zeta(z)\sum_p \frac{-p^{-z}\ln p}{1-p^{-z}}.\end{aligned}$$

Thus

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_p \frac{p^{-z}\ln p}{1-p^{-z}} = \sum_p\sum_{n=1}^\infty p^{-nz}\ln p.$$

The iterated sum is absolutely convergent for $\operatorname{Re} z > 1$, so it can be rearranged as a double sum

$$\sum_{(p,n),n\ge 1}(p^n)^{-z}\ln p = \sum_k k^{-z}\ln p$$

where $k = p^n$ for some n. Consequently,

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{k=1}^\infty k^{-z}\Lambda(k) = \sum_{k=1}^\infty k^{-z}(\psi(k)-\psi(k-1))$$

by the definitions of Λ and ψ. But using partial summation once again we obtain, with $a_k = k^{-z}$, $b_{k+1} = \psi(k)$, and $b_1 = \psi(0) = 0$ in Problem 2.2.7,

$$\sum_{k=1}^M k^{-z}(\psi(k)-\psi(k-1)) = \psi(M)(M+1)^{-z} + \sum_{k=1}^M \psi(k)(k^{-z}-(k+1)^{-z}).$$

Now from the definition (2) of $\psi(x)$ we have $\psi(x) \leq x \ln x$, so if $\operatorname{Re} z > 1$ we have $\psi(M)(M+1)^{-z} \to 0$ as $M \to \infty$. Moreover, we can write

$$\begin{aligned}
\sum_{k=1}^{M} \psi(k)(k^{-z} - (k+1)^{-z}) &= \sum_{k=1}^{M} \psi(k) z \int_{k}^{k+1} t^{-z-1}\, dt \\
&= \sum_{k=1}^{M} z \int_{k}^{k+1} \psi(t) t^{-z-1}\, dt \\
&= z \int_{1}^{M} \psi(t) t^{-z-1}\, dt
\end{aligned}$$

because ψ is constant on each interval $[k, k+1)$. Taking limits as $M \to \infty$, we finally get

$$-\frac{\zeta'(z)}{\zeta(z)} = z \int_{1}^{\infty} \psi(t) t^{-z-1}\, dt, \ \operatorname{Re} z > 1. \ \clubsuit$$

7.2 An Equivalent Version of the Prime Number Theorem

The function ψ defined in (2) above provides yet another connection, through (3), between the Riemann zeta function and properties of the prime numbers. The integral that appears in (3) is called the *Mellin transform* of ψ and is studied in its own right. We next establish a reduction, due to Chebyshev, of the prime number theorem to a statement involving the function ψ.

7.2.1 Theorem

The prime number theorem holds, that is, $x^{-1}\pi(x) \ln x \to 1$, iff $x^{-1}\psi(x) \to 1$ as $x \to \infty$.

Proof. Recall that

$$\begin{aligned}
\psi(x) &= \sum_{p \leq x} \left[\frac{\ln x}{\ln p}\right] \ln p \\
&\leq \sum_{p \leq x} \frac{\ln x}{\ln p} \ln p \qquad (1) \\
&= \ln x \sum_{p \leq x} 1 \\
&= (\ln x)\pi(x).
\end{aligned}$$

However, if $1 < y < x$, then

$$\begin{aligned}\pi(x) &= \pi(y) + \sum_{y<p\leq x} 1 \\ &\leq \pi(y) + \sum_{y<p\leq x} \frac{\ln p}{\ln y} \\ &< y + \frac{1}{\ln y} \sum_{y<p\leq x} \ln p \\ &\leq y + \frac{1}{\ln y}\psi(x).\end{aligned} \tag{2}$$

Now take $y = x/(\ln x)^2$ in (2), and we get

$$\pi(x) \leq \frac{x}{(\ln x)^2} + \frac{1}{\ln x - 2\ln\ln x}\psi(x).$$

Thus

$$\pi(x)\frac{\ln x}{x} \leq \frac{1}{\ln x} + \frac{\ln x}{\ln x - 2\ln\ln x}\frac{\psi(x)}{x}. \tag{3}$$

It now follows from (1) and (3) that

$$\frac{\psi(x)}{x} \leq \frac{\ln x}{x}\pi(x) \leq \frac{1}{\ln x} + \frac{\ln x}{\ln x - 2\ln\ln x}\frac{\psi(x)}{x}$$

and from this we can see that $x^{-1}\psi(x) \to 1$ iff $x^{-1}\pi(x)\ln x \to 1$ as $x \to \infty$. ♣

The goal will now be to show that $\psi(x)/x \to 1$ as $x \to \infty$. A necessary intermediate step for our proof is to establish the following weaker estimate on the asymptotic behavior of $\psi(x)$.

7.2.2 Lemma

There exists $C > 0$ such that $\psi(x) \leq Cx, x > 0$. For short, $\psi(x) = O(x)$.

Proof. Again recall that $\psi(x) = \sum_{p\leq x}[\frac{\ln x}{\ln p}]\ln p,\ x > 0$. Fix $x > 0$ and let m be an integer such that $2^m < x \leq 2^{m+1}$. Then

$$\begin{aligned}\psi(x) &= \psi(2^m) + \psi(x) - \psi(2^m) \\ &\leq \psi(2^m) + \psi(2^{m+1}) - \psi(2^m) \\ &= \sum_{p\leq 2^m}\left[\frac{\ln 2^m}{\ln p}\right]\ln p + \sum_{2^m<p\leq 2^{m+1}}\left[\frac{\ln 2^{m+1}}{\ln p}\right]\ln p.\end{aligned}$$

Consider, for any positive integer n,

$$\sum_{n<p\leq 2n}\ln p = \ln\prod_{n<p\leq 2n} p.$$

Now for any prime p such that $n < p \leq 2n$, p divides $(2n)!/n! = n!\binom{2n}{n}$. Since such a p does not divide $n!$, it follows that p divides $\binom{2n}{n}$. Hence

$$\prod_{n<p\leq 2n} p \leq \binom{2n}{n} < (1+1)^{2n} = 2^{2n},$$

and we arrive at

$$\sum_{n<p\leq 2n} \ln p < 2n \ln 2.$$

Therefore

$$\sum_{p\leq 2^m} \ln p = \sum_{k=1}^{m} \left(\sum_{2^{k-1}<p\leq 2^k} \ln p \right) < \sum_{k=1}^{m} 2^k \ln 2 < 2^{m+1} \ln 2$$

and

$$\sum_{2^m<p\leq 2^{m+1}} \ln p < 2^{m+1} \ln 2.$$

But if $p \leq x$ is such that $\left[\frac{\ln x}{\ln p}\right] > 1$, then $\frac{\ln x}{\ln p} \geq 2$ and hence $x \geq p^2$ so that $\sqrt{x} \geq p$. Thus those terms in the sum $\sum_{p\leq x} \left[\frac{\ln x}{\ln p}\right] \ln p$ where $\left[\frac{\ln x}{\ln p}\right] > 1$ occur only when $p \leq \sqrt{x}$, and the sum of terms of this form contribute no more than

$$\sum_{p\leq\sqrt{x}} \frac{\ln x}{\ln p} \ln p = \pi(\sqrt{x}) \ln x.$$

It follows from the above discussion that if $2^m < x \leq 2^{m+1}$, then

$$\begin{aligned}
\psi(x) &\leq 2^{m+1} \ln 2 + 2^{m+1} \ln 2 + \pi(\sqrt{x}) \ln x \\
&= 2^{m+2} \ln 2 + \pi(\sqrt{x}) \ln x \\
&< 4x \ln 2 + \pi(\sqrt{x}) \ln x \\
&\leq 4x \ln 2 + \sqrt{x} \ln x \\
&= (4 \ln 2 + \frac{1}{\sqrt{x}} \ln x)x.
\end{aligned}$$

Since $\frac{1}{\sqrt{x}} \ln x \to 0$ as $x \to \infty$, we conclude that $\psi(x) = O(x)$, which proves the lemma. ♣

7.3 Proof of the Prime Number Theorem

Our approach to the prime number theorem has been along traditional lines, but at this stage we will apply D.J. Newman's method (*Simple Analytic Proof of the Prime Number Theorem*, American Math. Monthly 87 (1980), 693-696) as modified by J. Korevaar (*On*

Newman's Quick Way to the Prime Number Theorem, Math. Intelligencer 4 (1982), 108-115). Korevaar's approach is to apply Newman's ideas to obtain properties of certain Laplace integrals that lead to the prime number theorem.

Our plan is to deduce the prime number theorem from a "Tauberian" theorem (7.3.1) and its corollary (7.3.2). Then we will prove (7.3.1) and (7.3.2).

7.3.1 Auxiliary Tauberian Theorem

Let F be bounded and piecewise continuous on $[0, +\infty)$, so that its Laplace transform

$$G(z) = \int_0^\infty F(t)e^{-zt}\,dt$$

exists and is analytic on $\operatorname{Re} z > 0$. Assume that G has an analytic extension to a neighborhood of the imaginary axis, $\operatorname{Re} z = 0$. Then $\int_0^\infty F(t)\,dt$ exists as an improper integral and is equal to $G(0)$. [In fact, $\int_0^\infty F(t)e^{-iyt}\,dt$ converges for every $y \in \mathbb{R}$ to $G(iy)$.]

Results like (7.3.1) are named for A. Tauber, who is credited with proving the first theorem of this type near the end of the 19th century. The phrase "Tauberian theorem" was coined by G.H. Hardy, who along with J.E. Littlewood made a number of contributions in this area. Generally, Tauberian theorems are those in which some type of "ordinary" convergence (e.g., convergence of $\int_0^\infty F(t)e^{-iyt}\,dt$ for each $y \in \mathbb{R}$), is deduced from some "weaker" type of convergence (e.g., convergence of $\int_0^\infty F(t)e^{-zt}\,dt$ for each z with $\operatorname{Re} z > 0$) provided additional conditions are satisfied (e.g., G has an analytic extension to a neighborhood of each point on the imaginary axis). Tauber's original theorem can be found in *The Elements of Real Analysis* by R.G. Bartle.

7.3.2 Corollary

Let f be a nonnegative, piecewise continuous and nondecreasing function on $[1, \infty)$ such that $f(x) = O(x)$. Then its *Mellin transform*

$$g(z) = z\int_1^\infty f(x)x^{-z-1}\,dx$$

exists for $\operatorname{Re} z > 1$ and defines an analytic function g. Assume that for some constant c, the function

$$g(z) - \frac{c}{z-1}$$

has an analytic extension to a neighborhood of the line $\operatorname{Re} z = 1$. Then as $x \to \infty$,

$$\frac{f(x)}{x} \to c.$$

As stated earlier, we are first going to see how the prime number theorem follows from (7.3.1) and (7.3.2). To this end, let ψ be as above, namely

$$\psi(x) = \sum_{p \le x} \left[\frac{\ln x}{\ln p}\right] \ln p.$$

Then ψ is a nonnegative, piecewise continuous, nondecreasing function on $[1, \infty)$. Furthermore, by (7.2.2), $\psi(x) = O(x)$, so by (7.3.2) we may take $f = \psi$ and consider the Mellin transform

$$g(z) = z \int_1^\infty \psi(x) x^{-z-1}\, dx.$$

But by (7.1.4), actually $g(z) = -\zeta'(z)/\zeta(z)$, and by the discussion leading up to the statement of (7.1.4), $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ has an analytic extension to a neighborhood of each point of $\operatorname{Re} z = 1$, hence so does $g(z) - \frac{1}{z-1}$. Consequently, by (7.3.2), we can conclude that $\psi(x)/x \to 1$, which, by (7.2.1), is equivalent to the PNT. Thus we are left with the proof of (7.3.1) and its corollary (7.3.2).

Proof of (7.3.1)

Let F be as in the statement of the theorem. Then it follows just as in the proof of (7.1.2), the extension theorem for zeta, that F's Laplace transform G is defined and analytic on $\operatorname{Re} z > 0$. Assume that G has been extended to an analytic function on a region containing $\operatorname{Re} z \geq 0$. Since F is bounded we may as well assume that $|F(t)| \leq 1, t \geq 0$. For $0 < \lambda < \infty$, define

$$G_\lambda(z) = \int_0^\lambda F(t) e^{-zt}\, dt.$$

By (3.3.3), each function G_λ is entire, and the conclusion of our theorem may be expressed as

$$\lim_{\lambda \to \infty} G_\lambda(0) = G(0).$$

That is, the improper integral $\int_0^\infty F(t)\, dt$ exists and converges to $G(0)$. We begin the analysis by using Cauchy's integral formula to get a preliminary estimate of $|G_\lambda(0) - G(0)|$. For each $R > 0$, let $\delta(R) > 0$ be so small that G is analytic inside and on the closed path

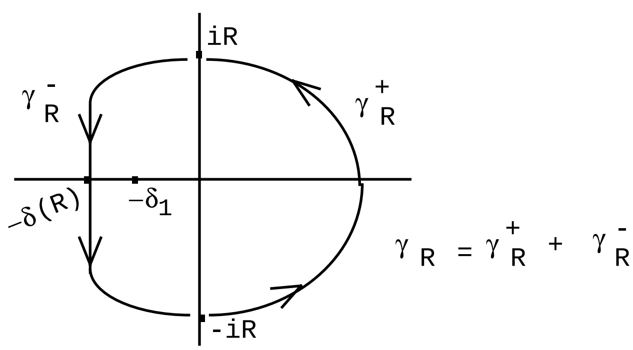

Figure 7.3.1

γ_R in Figure 7.3.1. (Note that since G is analytic on an open set containing $\operatorname{Re} z \geq 0$, such a $\delta(R) > 0$ must exist, although it may well be the case that $\delta(R) \to 0$ as $R \to +\infty$.) Let γ_R^+ denote that portion of γ_R that lies in $\operatorname{Re} z > 0$, and γ_R^- the portion that lies in $\operatorname{Re} z < 0$. By Cauchy's integral formula,

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i}\int_{\gamma_R} (G(z) - G_\lambda(z))\frac{1}{z}\, dz. \tag{1}$$

Let us consider the consequences of estimating $|G(0) - G_\lambda(0)|$ by applying the usual M-L estimates to the integral on the right hand side of (1) above. First, for $z \in \gamma_R^+$ and $x = \operatorname{Re} z$, we have

$$\begin{aligned} \left|\frac{G(z) - G_\lambda(z)}{z}\right| &= \frac{1}{R}\left|\int_\lambda^\infty F(t)e^{-zt}\, dt\right| \\ &\le \frac{1}{R}\int_\lambda^\infty |F(t)|e^{-xt}\, dt \\ &\le \frac{1}{R}\int_\lambda^\infty e^{-xt}\, dt \\ &= \frac{1}{R}\frac{e^{-\lambda x}}{x} \\ &\le \frac{1}{R}\frac{1}{x} = \frac{1}{R}\frac{1}{\operatorname{Re} z}. \end{aligned} \tag{2}$$

But $1/\operatorname{Re} z$ is unbounded on γ_R^+, so we see that a more delicate approach is required to shows that $G(0) - G_\lambda(0) \to 0$ as $\lambda \to \infty$. Indeed, it is here that Newman's ingenuity comes to the fore, and provides us with a modification of the above integral representation for $G(0) - G_\lambda(0)$. This *will* furnish the appropriate estimate. Newman's idea is to replace the factor $1/z$ by $(1/z) + (z/R^2)$ in the path integral in (1). Since $(G(z) - G_\lambda(z))z/R^2$ is analytic, the value of the path integral along γ_R remains unchanged. We further modify (1) by replacing $G(z)$ and $G_\lambda(z)$ by their respective products with $e^{\lambda z}$. Since $e^{\lambda z}$ is entire and has the value 1 at $z = 0$, we can write

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i}\int_{\gamma_R} (G(z) - G_\lambda(z))e^{\lambda z}(\frac{1}{z} + \frac{z}{R^2})\, dz.$$

Note that for $|z| = R$ we have $(1/z) + (z/R^2) = (\overline{z}/|z|^2) + (z/R^2) = (2\operatorname{Re} z)/R^2$, so that if $z \in \gamma_R^+$, (recalling (2) above),

$$|(G(z) - G_\lambda(z))e^{\lambda z}(\frac{1}{z} + \frac{z}{R^2})| \le \frac{1}{\operatorname{Re} z}e^{-\lambda \operatorname{Re} z}e^{\lambda \operatorname{Re} z}\frac{2\operatorname{Re} z}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\frac{1}{2\pi i}\int_{\gamma_R^+} (G(z) - G_\lambda(z))e^{\lambda z}(\frac{1}{z} + \frac{z}{R^2})\, dz\right| \le \frac{1}{R}$$

by the M-L theorem. Note that this estimate of the integral along the path γ_R^+ is independent of λ. Now let us consider the contribution to the integral along γ_R of the integral along γ_R^-. First we use the triangle inequality to obtain the estimate

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\gamma_R^-}(G(z)-G_\lambda(z))e^{\lambda z}(\frac{1}{z}+\frac{z}{R^2})\,dz\right| \\
&\leq \left|\frac{1}{2\pi i}\int_{\gamma_R^-}G(z)e^{\lambda z}(\frac{1}{z}+\frac{z}{R^2})\,dz\right| + \left|\frac{1}{2\pi i}\int_{\gamma_R^-}G_\lambda(z)e^{\lambda z}(\frac{1}{z}+\frac{z}{R^2})\,dz\right| \\
&= |I_1(R)| + |I_2(R)|.
\end{aligned}$$

First consider $I_2(R)$. Since $G_\lambda(z)$ is an entire function, we can replace the path of integration γ_R^- by the semicircular path from iR to $-iR$ in the left half plane. For z on this semicircular arc, the modulus of the integrand in $I_2(R)$ is

$$|(\int_0^\lambda F(t)e^{-zt}\,dt)e^{\lambda z}\frac{2\operatorname{Re} z}{R^2})| \leq \frac{1}{|\operatorname{Re} z|}\frac{2|\operatorname{Re} z|}{R^2} = \frac{2}{R^2}.$$

(Note that $|F| \leq 1$, we can replace the upper limit of integration by ∞, and $e^{\lambda x} \leq 1$ for $x \leq 0$.) This inequality also holds if $\operatorname{Re} z = 0$ (let $z \to iy$). Thus by the M-L theorem we get $|I_2(R)| \leq (1/2\pi)(2/R^2)(\pi R) = 1/R$, again.

Finally, we consider $|I_1(R)|$. This will be the trickiest of all since we only know that on γ_R^-, G is an analytic *extension* of the explicitly defined G in the right half plane. To deal with this case, first choose a constant $M(R) > 0$ such that $|G(z)| \leq M(R)$ for $z \in \gamma_R^-$. Choose δ_1 such that $0 < \delta_1 < \delta(R)$ and break up the integral defining $I_1(R)$ into two parts, corresponding to $\operatorname{Re} z < -\delta_1$ and $\operatorname{Re} z \geq -\delta_1$. The first contribution is bounded in modulus by

$$\frac{1}{2\pi}M(R)e^{-\lambda\delta_1}(\frac{1}{\delta(R)}+\frac{1}{R})\pi R = \frac{1}{2}RM(R)(\frac{1}{\delta(R)}+\frac{1}{R})e^{-\lambda\delta_1},$$

which for fixed R and δ_1 tends to 0 as $\lambda \to \infty$. On the other hand, the second contribution is bounded in modulus by

$$\frac{1}{2\pi}M(R)(\frac{1}{\delta(R)}+\frac{1}{R})2R\arcsin\frac{\delta_1}{R},$$

the last factor arising from summing the lengths of two short circular arcs on the path of integration. Thus for fixed R and $\delta(R)$ we can make the above expression as small as we please by taking δ_1 sufficiently close to 0. So at last we are ready to establish the conclusion of this theorem. Let $\epsilon > 0$ be given. Take $R = 4/\epsilon$ and fix $\delta(R), 0 < \delta(R) < R$, such that G is analytic inside and on γ_R. Then as we saw above, for all λ,

$$\left|\frac{1}{2\pi i}\int_{\gamma_R^+}(G(z)-G_\lambda(z))e^{\lambda z}(\frac{1}{z}+\frac{z}{R^2})\,dz\right| \leq \frac{1}{R} = \frac{\epsilon}{4}$$

and also

$$\left|\frac{1}{2\pi i}\int_{\gamma_R^-}(G_\lambda(z)e^{\lambda z}(\frac{1}{z}+\frac{z}{R^2})\,dz\right| \leq \frac{1}{R} = \frac{\epsilon}{4}.$$

Now choose δ_1 such that $0 < \delta_1 < \delta(R)$ and such that

$$\frac{1}{2\pi}M(R)(\frac{1}{\delta(R)}+\frac{1}{R})2R\arcsin\frac{\delta_1}{R} < \frac{\epsilon}{4}.$$

Since

$$\frac{1}{2}RM(R)(\frac{1}{\delta(R)}+\frac{1}{R})e^{-\lambda\delta_1} < \frac{\epsilon}{4}$$

for all λ sufficiently large, say $\lambda \geq \lambda_0$, it follows that

$$|G_\lambda(0) - G(0)| < \epsilon,\ \lambda \geq \lambda_0$$

which completes the proof. ♣

Proof of (7.3.2)

Let $f(x)$ and $g(z)$ be as in the statement of the corollary. Define F on $[0, +\infty)$ by

$$F(t) = e^{-t}f(e^t) - c.$$

Then F satisfies the first part of the hypothesis of the auxiliary Tauberian theorem, so let us consider its Laplace transform,

$$G(z) = \int_0^\infty (e^{-t}f(e^t) - c)e^{-zt}\,dt,$$

which via the change of variables $x = e^t$ becomes

$$\begin{aligned} G(z) &= \int_1^\infty (\frac{1}{x}f(x) - c)x^{-z}\frac{dx}{x} \\ &= \int_1^\infty f(x)x^{-z-2}\,dx - c\int_1^\infty x^{-z-1}\,dx \\ &= \int_1^\infty f(x)x^{-z-2}\,dx - \frac{c}{z} \\ &= \frac{g(z+1)}{z+1} - \frac{c}{z} \\ &= \frac{1}{z+1}[g(z+1) - \frac{c}{z} - c]. \end{aligned}$$

It follows from the hypothesis that $g(z+1) - (c/z)$ has an analytic extension to a neighborhood of the line $\operatorname{Re} z = 0$, and consequently the same is true of the above function G. Thus the hypotheses of the auxiliary Tauberian theorem are satisfied, and we conclude that the improper integral $\int_0^\infty F(t)\,dt$ exists and converges to $G(0)$. In terms of f, this says that $\int_0^\infty (e^{-t}f(e^t) - c)\,dt$ exists, or equivalently (via the change of variables $x = e^t$ once more) that

$$\int_1^\infty (\frac{f(x)}{x} - c)\frac{dx}{x}$$

exists. Recalling that f is nondecreasing, we can infer that $f(x)/x \to c$ as $x \to \infty$. For let $\epsilon > 0$ be given, and suppose that for some $x_0 > 0, [f(x_0)/x_0] - c \geq 2\epsilon$. It follows that

$$f(x) \geq f(x_0) \geq x_0(c + 2\epsilon) \geq x(c + \epsilon) \text{ for } x_0 \leq x \leq \frac{c + 2\epsilon}{c + \epsilon} x_0.$$

Hence,

$$\int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon} x_0} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} \geq \int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon} x_0} \frac{\epsilon}{x}\, dx = \epsilon \ln\left(\frac{c + 2\epsilon}{c + \epsilon}\right).$$

But $\int_{x_1}^{x_2} (\frac{f(x)}{x} - c)\frac{dx}{x} \to 0$ as $x_1, x_2 \to \infty$, because the integral from 1 to ∞ is convergent. Thus for all x_0 *sufficiently large*,

$$\int_{x_0}^{\frac{c+2\epsilon}{c+\epsilon} x_0} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} < \epsilon \ln\left(\frac{c + 2\epsilon}{c + \epsilon}\right).$$

However, reasoning from the assumption that $[f(x_0)/x_0] - c \geq 2\epsilon$, we have just deduced the opposite inequality. We must conclude that for all x_0 sufficiently large, $[f(x_0)/x_0] - c < 2\epsilon$. Similarly, $[f(x_0)/x_0] - c > -2\epsilon$ for all x_0 sufficiently large. [Say $[f(x_0)/x_0] - c \leq -2\epsilon$. The key inequality now becomes

$$f(x) \leq f(x_0) \leq x_0(c - 2\epsilon) \leq x(c - \epsilon) \text{ for } \left(\frac{c - 2\epsilon}{c - \epsilon}\right)x_0 \leq x \leq x_0$$

and the limits of integration in the next step are from $\frac{c-2\epsilon}{c-\epsilon} x_0$ to x_0.] Therefore $f(x)/x \to c$ as $x \to \infty$, completing the proof of both the corollary and the prime number theorem. ♣

The prime number theorem has a long and interesting history. We have mentioned just a few of the many historical issues related to the PNT in this chapter. There are several other *number theoretic* functions related to $\pi(x)$, in addition to the function $\psi(x)$ that was introduced earlier. A nice discussion of some of these issues can be found in Eric W. Weisstein, "Prime Number Theorem", from MathWorld—A Wolfram Web Resource, http://mathworld.wolfram.com/PrimeNumberTheorem.html. This source also includes a number of references on PNT related matters.

References

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2. Julian Havil, "Gamma-Exploring Euler's Constant," Princeton University Press, Princeton and Oxford, 2003, Chapter 15.