# INTRODUCTION TO THE KPZ EQUATION

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## 1. Growth models and interacting particle systems

1.1. **The corner growth model.** Let $p \in (1/2, 1]$ and take $q = 1 - p \in [0, 1/2)$. Let $\mathbb{H}$ be the space of decreasing functions $\zeta : \mathbb{Z} \to \mathbb{Z}$. On the state space $\mathbb{H}$ we define a process, which we call the *corner growth process*, as follows. Let $\{N_t^x; t \geq 0\}_{x \in \mathbb{Z}}$ a sequence of independent, standard Poisson processes. We call the process $\{N_t^x; t \geq 0\}$ the *clock* process at column $x \in \mathbb{Z}$. Let us denote by $\{\sigma_n^x; n \in \mathbb{N}\}$ the sequence of successive jump times of the process $\{N_t^x; t \geq 0\}$. Let $\{\kappa_n^x; n \in \mathbb{N}, x \in \mathbb{Z}\}$ a sequence of i.i.d. random variables, independent of the Poisson processes $\{N_t^x; t \geq 0\}$, such that $P(\kappa_n^x = +1) = p$ and $P(\kappa_n^x = -1) = q$. Let $\{\zeta_t(x); t \geq 0, x \in \mathbb{Z}\}$ be the Markov process with values in $\mathbb{H}$ constructed in the following way. At each jump time $\sigma_n^x$, we read the value of $\kappa_n^x$. If $\kappa_n^x = +1$, then we define

$$\zeta_{\sigma_n^x}(x) = \min\{\zeta_{\sigma_n^x-}(x-1), \zeta_{\sigma_n^x-}(x) + 1\}$$

and we define $\zeta_{\sigma_n^x}(y) = \zeta_{\sigma_n^x-}(y)$ for $y \neq x$. If $\kappa_n^x = -1$, we define

$$\zeta_{\sigma_n^x}(x) = \max\{\zeta_{\sigma_n^x-}(x) - 1, \zeta_{\sigma_n^x-}(x+1)\}$$

and $\zeta_{\sigma_n^x}(y) = \zeta_{\sigma_n^x-}(y)$ for $y \neq x$. The reader with some familiarity on interacting particle systems may recognize in these lines what is called the *graphical construction* of the process $\{\zeta_t; t \geq 0\}$. We call $\zeta_t \in \mathbb{H}$ the *height function* of an interface, and we call $\zeta_t(x)$ the height of the interface at column $x \in \mathbb{Z}$. This model is called the corner growth model, since growth (and decrease) can happen only at corners of the interface. Let us explain the construction of the process $\{\zeta_t; t \geq 0\}$ in an informal way. At each column $x \in \mathbb{Z}$ we wait an exponential time of rate 1. At the end of this exponential time, we decide to move the column up with probability $p$ and to move the column down with probability $q$. The motion is accomplished if and only if the resulting height function is in $\mathbb{H}$. Otherwise nothing happens. This procedure is done independently at each column $x \in \mathbb{Z}$.

We say that a height function $\zeta(x)$ is eventually flat if there exist integers $M_-$, $M_+$ such that $\zeta(x) = \zeta(M_-)$ for any $x \leq M_-$ and such that $\zeta(x) = \zeta(M_+)$ for any $x \geq M_+$. If the initial profile $\zeta_0$ of the corner growth process is eventually flat, it is easy to check that the process $\{\zeta_t; t \geq 0\}$ is well-defined. In fact, in that case, at any time $t \geq 0$ there are at most $M_+ - M_-$ columns on which the process $\zeta_t$ may change. Therefore, $\{\zeta_t; t \geq 0\}$ corresponds to a continuous-time Markov chain in a denumerable state space, on which jumps occur with rate at most $M_+ - M_-$. In particular, there are no explosions to take care of. The situation for an arbitrary initial height $\zeta_0$ is more delicate. One possibility is to verify that the graphical construction described above can be carried out for any initial height $\zeta_0 \in \mathbb{H}$. Another option is to verify that the construction of interacting particle systems due to Liggett (see Chap. 1 of [Lig]) applies in our situation. Both alternatives have advantages and disadvantages. There is an interesting bijection between the corner growth model and the so-called *asymmetric simple exclusion process*. This relation will be described in detail in Section 1.2. At this point, we only point out that Liggett's construction of the exclusion process allows to show that for any initial height $\zeta_0$, the process $\{\zeta_t; t \geq 0\}$ constructed above is a well-defined, strong Markov process with state space $\mathbb{H}$.

In various applications, the corner growth model is defined on the semi-line $\mathbb{N}_0 = \{0,1,...\}$. [1] The graphical construction can be carried out in that case with just one modification: at the column $x = 0$, there is no restriction on growth, that is, if $\kappa_n^0 = +1$, then we put $\eta_{\sigma_n^0}(0) = \eta_{\sigma_n^0-}(0)+1$. The corner growth on the semi-line $\mathbb{N}_0$ can be thought as a corner growth model on $\mathbb{Z}$, on which we define $\zeta_0(x) = +\infty$ for any $x < 0$ (and therefore $\zeta_t(x) = +\infty$ for any $t \geq 0$, and any $x < 0$). The corner growth model on the semi-line $\{..., -1, 0\}$ can be defined in a similar way.

1.2. **The asymmetric simple exclusion process.** As in Sect. 1.1, let $p \in (1/2, 1]$ and take $q = 1-p \in [0,1/2)$. Let $\Omega$ be the space of binary sequences $\eta : \mathbb{Z} \to \{0,1\}$. In this model we say that the elements $x \in \mathbb{Z}$ are *sites*, and we interpret the 1's as *particles*. We say that the elements $\eta \in \Omega$ are *configurations* of particles. We say that, in the configuration $\eta \in \Omega$ there is a particle at site $x \in \mathbb{Z}$ if $\eta(x) = 1$. If $\eta(x) = 0$ we say that the site $x \in \mathbb{Z}$ is *empty*. Let $\{\mathscr{N}_t^x; t \geq 0\}_{x\in\mathbb{Z}}$ be a sequence of independent, standard Poisson processes. Let $\{s_n^x; n \in \mathbb{N}\}$ the sequence of successive jumps of the process $\{\mathscr{N}_t^x; t \geq 0\}$. And let $\{k_n^x; n \in \mathbb{N}, x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables, independent of the Poisson processes $\{\mathscr{N}_t^x; t \geq 0\}_{x\in\mathbb{Z}}$. An attentive reader may have noticed that the processes and random variables we have just defined are exactly the same ones defined at the beginning of Sect. 1.1. Later on we will comment on why we have chosen different notations for the same objects. Let $\{\eta_t; t \geq 0\}$ be the Markov process with values in $\Omega$ constructed in the following way. At each jump time $s_n^x$ we check the value of $\eta_{s_n^x-}(x)$. If $\eta_{s_n^x-}(x) = 0$, nothing happens. If $\eta_{s_n^x-}(x) = 1$, we read the value of $k_n^x$. If $k_n^x = +1$, we check the value of $\eta_{s_n^x-}(x+1)$. If $\eta_{s_n^x-}(x+1) = 1$, nothing happens. If $\eta_{s_n^x-}(x+1) = 0$, we define

$$\eta_{s_n^x}(y) = \begin{cases} 0, & y = x \\ 1, & y = x+1 \\ \eta_{s_n^x-}(y), & y \neq x, x+1. \end{cases}$$

If $k_n^x = -1$, we check the value of $\eta_{s_n^x-}(x-1)$. If $\eta_{s_n^x-}(x-1) = 1$, nothing happens. If $\eta_{s_n^x-}(x-1) = 0$, we define

$$\eta_{s_n^x}(y) = \begin{cases} 0, & y = x \\ 1, & y = x-1 \\ \eta_{s_n^x-}(y), & y \neq x, x-1. \end{cases}$$

What we have described above is the *graphical construction* of the asymmetric simple exclusion process (which we call ASEP for ease of notation). The evolution $\{\eta_t; t \geq 0\}$ can be described as follows. At each site $x \in \mathbb{Z}$ we wait an exponential time of rate 1. At the end of this exponential time we decide to move the particle at site $x$; nothing happens if there is no particle at site $x$ at that time. The particle attempts to jump to $x+1$ with probability $p$, and it attempts to jump to $x-1$ with probability $q$. The jump is accomplished only if the destination site is empty at the time of the jump. Otherwise nothing happens. This procedure is done independently at each site $x \in \mathbb{Z}$. Liggett's result quoted in the previous section (Chap. 1 of [Lig]) is actually formulated in terms of the exclusion process $\{\eta_t; t \geq 0\}$, and it shows that the process $\eta_t; t \geq 0\}$ is well defined for any initial configuration $\eta_0 \in \Omega$.

It turns out that there is a beautiful connection between the corner growth model $\{\zeta_t; t \geq 0\}$ described in Sect. 1.1 and the ASEP process described above. Let us say that the corner

[1] We use $\mathbb{N}_0$ for the set of non-negative integers $\{0,1,...\}$ and $\mathbb{N}$ for the set of positive integers $\{1,2,...\}$.

growth model $\{\zeta_t; t \geq 0\}$ is given. Each column on the corner growth model will represent a particle on the ASEP. Let us start constructing the initial configuration of particles $\eta_0$. Let us define $x_0 = 0$ and for $\ell \in \mathbb{N}$, let us define $x_\ell$ in an inductive way: $x_\ell = x_{\ell-1} + 1 + \zeta_0(\ell - 1) - \zeta(\ell)$. For $\ell < 0$, we use a similar inductive procedure: $x_\ell = x_{\ell+1} - 1 + \zeta(\ell+1) - \zeta(\ell)$. Then we define $\eta_0(x_\ell) = 1$ for $\ell \in \mathbb{Z}$ and $\eta_0(x) = 0$ if $x \neq x_\ell$ for any $\ell \in \mathbb{Z}$. In other words, the height differences between consecutive columns on the corner growth model represent the distance between consecutive particles on the ASEP. Define $x_\ell(t) = x_\ell + \zeta_0(\ell) - \zeta_t(\ell)$, and define

$$\eta_t(x) = \begin{cases} 1, & \text{if } x = x_\ell(t) \text{ for some } \ell \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

A careful checking shows that the process $\{\eta_t; t \geq 0\}$ defined in this way has the dynamics described above for the ASEP. Notice that this relation defines a different graphical construction of the ASEP: in the construction coming from this relation with the corner growth model, the Poisson processes $\{N_t^x; t \geq 0\}$ are associated to the particles and not to the sites, like in the graphical construction of this Section.

For the reader who understand what a generator of a Markov process is, we now describe the generator of the ASEP. We say that a function $f : \Omega \to \mathbb{R}$ is *local* if there exists a finite set $A \subseteq \mathbb{Z}$ such that $f(\eta) = f(\xi)$ whenever $\eta(x) = \xi(x)$ for every $x \in A$. For each local function $f : \Omega \to \mathbb{R}$ we define $Lf : \Omega \to \mathbb{R}$ as

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \big\{ p\eta(x)\big(1 - \eta(x+1)\big) + q\eta(x+1)\big(1 - \eta(x)\big) \big\} \nabla_{x,x+1} f(\eta),$$

where $\nabla_{x,x+1} f(\eta) = f(\eta^{x,x+1}) - f(\eta)$ and for $\eta \in \Omega$ we define $\eta^{x,x+1} \in \Omega$ as

$$\eta^{x,x+1}(z) = \begin{cases} \eta(x+1), & z = x \\ \eta(x), & z = x+1 \\ \eta(z), & z \neq x, x+1. \end{cases}$$

The operator $L$ defined in this way can be extended in a unique way to a Markov generator, which turns out to be the generator of the ASEP $\{\eta_t; t \geq 0\}$.

## 2. THE STOCHASTIC HEAT EQUATION

Fix $T > 0$. Let $(\mathbb{X}, \mathscr{F}, P)$ be a probability space endowed with a white noise $\mathscr{W}(dxdt)$ defined on $\mathbb{R} \times [0,T]$. We say that a measurable function $z : \mathbb{X} \times \mathbb{R} \times [0,T]$ is a *random function*, and we usually denote it by $z(x,t)$, without making explicit mention of the random variable $\omega \in \mathbb{X}$. Fix $a > 0$. On the space of random functions $z$ we define

$$|||z|||_T^2 = \sup_{\substack{x \in \mathbb{R} \\ t \in [0,T]}} e^{-a|x|} E\big[z(x,t)^2\big]$$

and we denote by $\mathscr{P}_T$ the space of *predictable* random functions (see the definition in the Appendix E) with $|||z|||_T < +\infty$. The space $\mathscr{P}_T$ turns out to be a Banach space with respect to the norm $|||\cdot|||_T$.

Let $K_t(x) = (2\pi t)^{-1/2} e^{-x^2/2t}$ be the fundamental solution of the heat equation $\partial_t u = \frac{1}{2}\Delta u$ in $\mathbb{R} \times [0,T]$. We say that a random function $z(x,t)$ is a *mild solution* of the stochastic heat equation

$$\partial_t z = \frac{1}{2}\Delta z + z\mathscr{W}$$

with initial condition $z_0$ if $\sup_x e^{-a|x|}E[z_0(x)^2] < +\infty$ and

$$z(t,x) = \int_{\mathbb{R}} K_t(x-y)z_0(y)dy + \int_0^t \int K_{t-s}(x-y)z(s,y)\mathscr{W}(dyds)$$

for any $x \in \mathbb{R}$, and any $t \in [0,T]$. Our objective is to prove the following proposition:

**Proposition 2.1.** *Let $\{z_0(x), x \in \mathbb{R}\}$ be a random function satisfying*

$$\sup_x e^{-a|x|}E[z_0(x)^2] < +\infty.$$

*Then there exist a unique mild solution on the space $\mathscr{P}$ of the stochastic heat equation*

$$\partial_t z = \frac{1}{2}\Delta z + z\mathscr{W}$$

*with initial condition $z_0$.*

*Proof.* The proof of this proposition follows closely Picard's method in ordinary differential equations. Let us define the operator $\Lambda^0 : \mathscr{P}_T \to \mathscr{P}_T$ defined by

$$\Lambda^0 z(x,t) = \int_0^t \int K_{t-s}(x-y)z(y,s)\mathscr{W}(dyds).$$

At this point we have not shown that $\Lambda^0 z \in \mathscr{P}_T$, but this fact will be a simple consequence of the computations below. By (E.2),

$$E\big[\big(\Lambda^0 z(x,t)\big)^2\big] = \int_0^t \int K_{t-s}(x-y)^2 E[z(y,s)^2]dyds.$$

Notice that $K_t(x)^2 = (4\pi t)^{-1/2}K_{t/2}(x)$. Therefore the last integral is equal to

$$\int_0^t \frac{1}{\sqrt{4\pi s}} \int K_{s/2}(x-y)E[z(t-s,y)^2]dyds. \tag{2.1}$$

Notice that by the definition of the triple norm $\|\!|\cdot|\!\|_T$,

$$E[z(y,t-s)^2] \le e^{a|x|}\|\!|z|\!\|_T^2. \tag{2.2}$$

We have the elementary estimate $e^{a|x|} \le e^{ax} + e^{-ax}$. Making a simple change of variables, we obtain the identity $\int e^{ax}e^{-x^2/t}dx = e^{a^2t/4}\int e^{-x^2/t}dx$. Using these two facts plus the estimate (2.2) into (2.1) we see that (2.1) is bounded by

$$e^{a|x|}\|\!|z|\!\|_T^2 \int_0^t \frac{e^{a^2 s/4}}{\sqrt{4\pi s}}ds. \tag{2.3}$$

We conclude that for any $t \in [0,T]$,

$$\sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} e^{-a|x|}E\big[\big(\Lambda^0 z(x,s)\big)^2\big] \le C(t)\|\!|z|\!\|_T^2,$$

where $C(t)$ is equal to the integral appearing in (2.3). This bound shows that $\|\!|\Lambda^0 z|\!\|$ is finite. In order to see that $\Lambda^0 z$ is predictable, it is enough to observe that for any predictable random function $z$ and any continuous, bounded function $F_t(x,y)$ with enough decay at infinity, the stochastic integral

$$\int_0^t \int F(x,y)z(s,y)\mathscr{W}(dyds)$$

is predictable. In fact, this property follows from the measurability of the stochastic integral with respect to $\mathscr{F}_t$ and from approximating $F(x,y)$ by simple functions. Notice that $C(t) \to$

0 if $t \to 0$. In particular, there exists $t > 0$ such that $\Lambda^0$ is a contraction when restricted to the space $\mathscr{P}_t$ (the space $\mathscr{P}_t$ is constructed changing $T$ by $t$ in the definition of $\mathscr{P}_T$). Notice that the function

$$z^0(x,s) = \int_{\mathbb{R}} K_s(x-y) z_0(y) dy$$

belongs to $\mathscr{P}_t$. Therefore a random function $z$ is a solution of the stochastic heat equation with initial condition $z_0$ if and only if it solves the fixed point equation $z = z^0 + \Lambda^0 z$. Since $\Lambda^0$ is a contraction, this fixed point equation has a unique solution. Repeating the argument $\lceil T/t \rceil$ times, we prove the proposition. $\square$

## APPENDIX A. THE POISSON PROCESS

We say that a real-valued random variable $\tau$ has an *exponential distribution* of rate $\lambda$ if $P(\tau > 0) = 1$ and $P(\tau > t) = e^{-\lambda t}$ for any $t \geq 0$. The multiplicative property of the exponential function has the following consequence: for any $s,t > 0$, $P(\tau > s+t | \tau > s) = e^{-\lambda t}$. This property is known as the loss of memory of the exponential distribution: if we interpret $\tau$ as a waiting time of some event, then the distribution of this waiting time, given that the event has not yet occurred, is exponential with the same rate.

The exponential distribution has other remarkable properties: if $\tau_1$, $\tau_2$ are two independent, exponential random variables of rates $\lambda_1$, $\lambda_2$, then $\tau =: \min\{\tau_1, \tau_2\}$ has an exponential distribution of rate $\lambda_1 + \lambda_2$. Moreover, $P(\tau = \tau_1) = \lambda_1(\lambda_1 + \lambda_2)^{-1}$ and the event $\{\tau = \tau_1\}$ is independent of $\tau$. Similar properties hold for arbitrary sequences $\{\tau_i\}$ of independent random variables with exponential distributions.

A collection $\{N_t; t \geq 0\}$ of random variables is said to be a (homogeneous) *Poisson process* if the following holds:

i) With probability 1, $N_0 = 0$ and the trajectory $t \mapsto N_t$ is right-continuous with left limits. In that case, we say that the process $\{N_t; t \geq 0\}$ is *càdlàg*.
ii) For any $t \geq 0$, $N_t \in \mathbb{N}_0$ with probability 1.
iii) For any $s,t \geq 0$, the increment $N_{t+s} - N_t$ is independent of $\sigma(N_u; u \leq t)$ and it has the same distribution of $N_s$.

Notice that, given i), condition ii) implies that, with probability 1, the path $t \mapsto N_t$ assume values in $\mathbb{N}_0$ for every $t \geq 0$.

It turns out that these three properties characterize the distribution of $\{N_t; t \geq 0\}$ on the space of càdlàg trajectories $\mathscr{D}([0,\infty); \mathbb{N}_0)$. In fact, if the process $\{N_t; t \geq 0\}$ satisfies i), ii), iii), then there exists a constant $\lambda$ such that for any $0 \leq s < t$, we have

$$P(N_t - N_s = \ell) = e^{-\lambda(t-s)} \frac{\big(\lambda(t-s)\big)^\ell}{\ell!}.$$

This fact, together with the independence of increments stated in iii) characterize the finite-dimensional distributions of $\{N_t; t \geq 0\}$. This fact plus the right-continuity of trajectories stated in i), characterize the distribution of $\{N_t; t \geq 0\}$ in $\mathscr{D}([0,\infty); \mathbb{N}_0)$. Notice that the trajectories $t \mapsto N_t$ are non-decreasing with probability 1, since the increments are non-negative. The constant $\lambda$ is called the *rate* of the Poisson process $\{N_t; t \geq 0\}$. If $\lambda = 1$, we say that $\{N_t; t \geq 0\}$ is a *standard* Poisson process.

Notice that, according to the previous formula, $P(N_t = 0) = e^{-\lambda t}$. Therefore, the time of the first of $\{N_t; t \geq 0\}$ has an exponential distribution of rate $\lambda$. The same is true for the time between successive jumps. Moreover, these inter-jump times are independent. This observation provides us with a way to construct the process $\{N_t; t \geq 0\}$. Let $\{\tau_n; n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with exponential distribution of rate $\lambda$. Define $S_0 = 0$ and for $n \in \mathbb{N}$ define

$$S_n = \sum_{i=1}^{n} \tau_i.$$

For $t \geq 0$ define $N_t$ as the unique number $n \in \mathbb{N}_0$ such that $S_n \leq t < S_{n+1}$. The random variable $N_t$ is well-defined, since with probability 1, $\tau_n > 0$ for any $n \in \mathbb{N}$. By definition, the process $\{N_t; t \geq 0\}$ defined in this way is right-continuous with left limits, with values in $\mathbb{N}_0$. The independence of increments follows from the independence of the sequence $\{\tau_n; n \in \mathbb{N}\}$ and the loss of memory of the exponential distribution.

Let us denote by $\mathscr{F}_t$ the $\sigma$-algebra generated by the random variables $\{N_s; s \leq t\}$. The family of increasing $\sigma$-algebras $\{\mathscr{F}_t; t \geq 0\}$ is called the *natural filtration* associated to the process $\{N_t; t \geq 0\}$. Condition iii) in the definition of the Poisson process $\{N_t; t \geq 0\}$ imply that the process $\{M_t; t \geq 0\}$ defined by

$$M_t = N_t - \lambda t$$

is a martingale with respect to the filtration $\{\mathscr{F}_t; t \geq 0\}$. The process $\{M_t^2 - \lambda t; t \geq 0\}$ is also a martingale with respect to $\{\mathscr{F}_t; t \geq 0\}$. We say in that case that $\lambda t$ is the *quadratic variation* of the martingale $\{M_t; t \geq 0\}$.

## APPENDIX B. INHOMOGENEOUS POISSON PROCESSES

We say that a function $\lambda : [0, \infty) \to \mathbb{R}$ is *càglàd* if it is left-continuous with right limits. Let $\{\lambda(t); t \geq 0\}$ be a càglàd, non-negative function. We say that a collection of random variables $\{N_t; t \geq 0\}$ is an *inhomogeneous Poisson process* of rate $\{\lambda(t); t \geq 0\}$ if:

i) With probability 1, $N_0 = 0$ and the trajectory $t \mapsto N_t$ is càdlàg.
ii) For any $t \geq 0$, $N_t \in \mathbb{N}_0$ with probability 1.
iii) For any $s, t \geq 0$, the increment $N_{t+s} - N_t$ is independent of $\sigma(N_u; u \leq t)$ and it has a characteristic function given by

$$E\big[\exp\{i\theta(N_{t+s} - N_t)\}\big] = \exp\Big\{(e^{i\theta} - 1)\int_t^{t+s} \lambda(u)du\Big\}.$$

Let us assume that $\lambda^* = \sup_{t \geq 0} \lambda(t) < +\infty$. In that case, the process $\{N_t; t \geq 0\}$ can be constructed in a simple way. Let $\{N_t^*; t \geq 0\}$ be an homogeneous Poisson point process of rate $\lambda^*$. Let $\{\sigma_n; n \in \mathbb{N}\}$ be the successive jump times of the process $\{N_t^*; t \geq 0\}$. Let $\{\kappa_n; n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with common distribution $\mathscr{U}([0,1])$, that is, $\kappa_n$ is a number chosen uniformly on the interval $[0,1]$. We define

$$\mathbf{N}_n = \sum_{i=1}^{n} \mathbf{1}\big(\kappa_n \lambda^* \leq \lambda(\sigma_n)\big),$$

we define

$$S_\ell = \inf\{\sigma_n; \mathbf{N}_n = \ell\}$$

and finally we define $N_t$ as the unique number $\ell \in \mathbb{N}_0$ such that $S_\ell \leq t < S_{\ell+1}$. A formal description of the construction is the following. The process $\{N_t; t \geq 0\}$ can jump only when the process $\{N_t^*; t \geq 0\}$ jumps. Each time the process $\{N_t^*; t \geq 0\}$ jumps, the process $\{N_t; t \geq 0\}$ jumps with probability $\lambda(t)/\lambda^*$, where $t$ is the jump time. Any similarity with the graphical construction of the corner growth model is *not* a coincidence.

The process $\{N_t; t \geq 0\}$ also has some martingales associated to it. Let $\{\mathscr{F}_t; t \geq 0\}$ the natural filtration associated to the process $\{N_t; t \geq 0\}$. Then, the process $\{M_t; t \geq 0\}$ given by

$$M_t = N_t - \int_0^t \lambda(s)ds$$

is a martingale with respect to $\{\mathscr{F}_t; t \geq 0\}$. The process given by

$$M_t^2 - \int \lambda(s)ds$$

is also a martingale with respect to $\{\mathscr{F}_t; t \geq 0\}$.

Notice that in the construction of the process $\{N_t; t \geq 0\}$ works with no modification in the case on which $\{\lambda(t); t \geq 0\}$ is a collection of random variables such that the trajectories $t \mapsto \lambda(t)$ are càglàd with probability 1.

## APPENDIX C. THE BASIC COUPLING

One of the main advantages of the graphical construction of the corner growth model and the ASEP is that we can construct various processes starting from different initial configurations, using the same Poisson processes. Let us consider the ASEP; what we are going to describe can also be done for the corner growth model. Let $\eta, \xi \in \Omega$ be two initial configurations of particles. Let $\{\eta_t; t \geq 0\}$ be the ASEP constructed in Sect. 1.2 with initial configuration $\eta$, and let $\{\xi_t; t \geq 0\}$ the ASEP with initial configuration $\xi$ and constructed using *the same* random variables $\{s_n^x; n \in \mathbb{N}, x \in \mathbb{Z}\}$, $\{k_n^x; n \in \mathbb{N}, x \in \mathbb{Z}\}$. We call the pair $\{(\eta_t, \xi_t); t \geq 0\}$ a *basic coupling* between these two processes. This basic coupling has a remarkable property: if $\eta(x) \geq \xi(x)$ for every $x \in \mathbb{Z}$, then $\eta_t(x) \geq \xi_t(x)$ for every $t \geq 0$ and every $x \in \mathbb{Z}$. In fact, we can check, in a case by case basis, that this ordering is preserved at each jump time $s_n^x$. This property is locally conserved in the following sense. Take $x \in \mathbb{Z}$ and let us assume that $\eta(y) \geq \xi(y)$ for any $y \geq x$. Take

$$\sigma = \min\{s_n^x; n \in \mathbb{N}, s_n^x > s_1^{x-1}\}.$$

Then, $\eta_t(y) \geq \xi_t(y)$ for any $t \leq \sigma$ and any $y \geq x+1$. In fact, the time $\sigma$ represents the first time that the value of $\eta(y)$ outside $\{x-1, x, \dots\}$ influences the value of $\eta_t(z)$ for $z \geq x$. Since $s_1^{x-1}$ has an exponential distribution of rate 1, we can prove that $P(\sigma \leq t) = \mathscr{O}(t^2)$ when $t \to 0$. Therefore, for any $y \geq x+1$ we have that $P(\eta_t(y) \geq \xi_t(y)) \leq \mathscr{O}(t^2)$. Similar statements can be shown for initial configurations of particles coinciding on a given interval.

## APPENDIX D. THE GENERATOR OF AN INTERACTING PARTICLE SYSTEM

Let us consider $\Omega = \{0,1\}^{\mathbb{Z}}$, that is, $\Omega$ is the set of binary sequences $\eta = \{\eta(x), x \in \mathbb{Z}\}$. It can be shown that the product topology on $\Omega$ coincides with the topology generated by the metric $d : \Omega \times \Omega \to [0, \infty)$ defined by

$$d(\eta, \xi) = \sum_{x \in \mathbb{Z}} \frac{|\eta(z) - \xi(z)|}{2^{|z|}}.$$

Let $\mathscr{C}(\Omega)$ be the space of functions $\Omega \to \mathbb{R}$ which are continuous with respect to this metric. Notice that the space $(\Omega, d)$ is compact, and therefore any function $f \in \mathscr{C}(\Omega)$ is bounded. We say that a function $f : \Omega \to \mathbb{R}$ is *local* if there exists a finite set $A \subseteq \mathbb{Z}$ such that $f(\eta) = f(\xi)$ whenever $\eta(x) = \xi(x)$ for every $x \in A$. We denote by $\mathrm{supp}(F)$ the smallest of these sets $A$. We observe that any local function $f$ is continuous. Moreover, the set of local functions is dense on $\mathscr{C}(\Omega)$. Let $\mathscr{D}([0,\infty); \Omega)$ denote the set of càdlàg paths from $[0,\infty)$ to $\Omega$, that is, the sets of right-continuous trajectories with well-defined let limits at any time $t \in [0, \infty)$. Let $\{\eta_t; t \geq 0\}$ be a Markov process with values in $\Omega$. For any $\eta \in \Omega$, let us denote by $P^\eta$ the distribution in $\mathscr{D}([0,\infty); \Omega)$ of the process $\{\eta_t; t \geq 0\}$ satisfying $\eta_0 = \eta$. We denote by $E^\eta$ the expectation with respect to $P^\eta$. For any function $f \in \mathscr{C}(\Omega)$, let us define $P_t f : \Omega \to \mathbb{R}$ as

$$P_t f(\eta) = E^\eta[f(\eta_t)]$$

for any $\eta \in \Omega$. Notice that $P_t f$ is well-defined, since any function $f \in \mathscr{C}(\Omega)$ is bounded. We say that a a function $f \in \mathscr{C}(\Omega)$ belongs to $\mathrm{Dom}(L)$ if the limit

$$\lim_{t \to 0} \frac{P_t f - f}{t}$$

exists with respect to the uniform topology of $\mathscr{C}(\Omega)$. For $f \in \mathrm{Dom}(L)$ we define $Lf : \Omega \to \mathbb{R}$ as

$$Lf(\eta) = \lim_{t \to 0} \frac{P_t f(\eta) - f(\eta)}{t}. \tag{D.1}$$

It turns out that $(L, \mathrm{Dom}(L))$ is a densely defined operator in $\mathscr{C}(\Omega)$. This operator is called the *generator* of the Markov process $\{\eta_t; t \geq 0\}$.

Let us consider the ASEP $\{\eta_t; t \geq 0\}$ defined in Sect. 1.2. Let $f : \Omega \to \mathbb{R}$ be a local function, and let us try to compute the limit (D.1). Take $\ell \in \mathbb{N}_0$ such that $\mathrm{supp}(f) \subseteq \{-\ell, ..., \ell\}$. Let $\eta, \xi \in \Omega$ be such that $\eta(x) = \xi(x)$ for any $x \in \{-(\ell+1), ..., \ell+1\}$. Let $\{(\eta_t, \xi_t); t \geq 0\}$ the basic coupling described in Appendix C. Since $P(\eta_t(x) \neq \xi_t(x)) = \mathscr{O}(t^2)$ for any $x \in \{-\ell, ..., \ell\}$, we see that $Lf(\eta) = Lf(\xi)$, if the limit defining any of the two quantities $Lf(\eta), Lf(\xi)$ exist. Since $f$ is a bounded function, events with probability $\mathscr{O}(t^2)$ does not influence the value of $Lf(\eta)$. Therefore, if we look at the process constructed by turning off all the Poisson processes $\{N_t^x; t \geq 0\}$ for $x \notin \{-(\ell+1), ..., \ell+1\}$. In that case we are leading with a finite number of Poisson point processes, for which we can compute the limit above in an explicit way, obtaining that

$$Lf(\eta) = \sum_{x=-(\ell+1)}^{\ell} \big\{ p\eta(x)\big(1 - \eta(x+1)\big) + q\eta(x+1)\big(1 - \eta(x)\big) \big\} \nabla_{x,x+1} f(\eta).$$

## APPENDIX E. THE WHITE NOISE AND STOCHASTIC INTEGRATION

Let $\mathbb{H}$ be a (real) Hilbert space. Let $\{u_n; n \in \mathbb{N}\}$ be a orthonormal basis of $\mathbb{H}$. Let $\{\varphi_n; n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables of common distribution $\mathscr{N}(0,1)$, that is, each random variable $\varphi_n$ has Gaussian distribution of unit variance. The white noise in $\mathbb{H}$ is formally defined as $\mathscr{W} = \sum_n \varphi_n u_n$. The definition is just formal, because the series does not convergent in $\mathbb{H}$ with probability 1. It can be shown that the white noise $\mathscr{W}$ can be defined as a random element of some Sobolev space of negative index associated to the basis $\{u_n; n \in \mathbb{N}\}$, but this description will not be useful for our purposes. A simpler definition, which will be enough for us is the following. Let $\mathbb{H}$ be a Hilbert space. Denote by $\|\cdot\|$ the norm in $\mathbb{H}$ and by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{H}$. Let $\mathbb{F} \subseteq \mathbb{H}$ a vector space, and assume that $\mathbb{F}$ is dense in $H$. We say that a family of real-valued random variables $\mathscr{W} = \{\mathscr{W}(f); f \in \mathbb{F}\}$ defined on some probability space $(\mathbb{X}, \mathscr{F}, P)$ is a *white noise* if:

i) for any finite subset $\{f_1, ..., f_\ell\}$ of $\mathbb{F}$, the vector $(\mathscr{W}(f_1), ..., \mathscr{W}(f_\ell))$ is Gaussian,
ii) for any function $f \in \mathbb{F}$, $E[\mathscr{W}(f)^2] = \|f\|^2$.

Two important consequences of this definition are the following. By the polarization identity, we have $E[\mathscr{W}(f)\mathscr{W}(g)] = \langle f, g \rangle$ for any $f, g \in \mathbb{F}$. In particular, $\mathscr{W}(f)$ and $\mathscr{W}(g)$ are independent if and only if $f$ and $g$ are orthogonal. Take $f \in \mathbb{H}$ and let $\{f_n; n \in \mathbb{N}\}$ be in $\mathbb{F}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$. Then, there exists a random variable $\mathscr{W}(f)$ such that $E[(\mathscr{W}(f_n) - \mathscr{W}(f))^2] \to 0$ as $n \to \infty$. The collection of random variables $\{\mathscr{W}(f); f \in \mathbb{H}\}$ obtained in this way also satisfies the conditions i), ii) of the definition of the white noise $\mathscr{W}$, this time for $\mathbb{F} = \mathbb{H}$.

The existence of the white noise $\mathscr{W}$ can be obtained starting from the formal definition $\mathscr{W} = \sum_n \varphi_n u_n$. In fact, it is enough to define

$$\mathscr{W}(f) = \sum_{n \in \mathbb{N}} \varphi_n \langle u_n, f \rangle.$$

The series in convergent in $L^2(P)$, due to the fact that $\sum_n \langle u_n, f \rangle^2 < +\infty$ and the independence of the random variables $\{\varphi_n; n \in \mathbb{N}\}$. Notice that the choice of the basis $\{u_n; n \in \mathbb{N}\}$

is not relevant: if $\{v_n; n \in \mathbb{N}\}$ is another orthonormal basis of $\mathbb{H}$, the sequence $\{\varphi'_n; n \in \mathbb{N}\}$ defined by $\varphi'_n = \mathscr{W}(v_n)$ is i.i.d. with common distribution $\mathscr{N}(0,1)$, and $\mathscr{W}$ admits the formal representation $\sum_n \varphi'_n v_n$.

Let us consider the case $\mathbb{H} = L^2(\mathbb{R} \times [0,\infty))$, that is, $\mathbb{H}$ is the space of functions $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ which are square-integrable with respect to the Lebesgue measure. It will be useful to look at $\mathscr{W}$ as a random measure. For any set $A \subseteq \mathbb{R} \times [0,\infty)$ of finite Lebesgue measure we define $\mathscr{W}(A) = \mathscr{W}(\mathbf{1}(A))$. Let us denote by $|A|$ the Lebesgue measure of the set $A$. The "measure" $\mathscr{W}$ satisfies the properties

i) for any $A$ such that $|A| < +\infty$, $\mathscr{W}(A)$ has distribution $\mathscr{N}(0,|A|)$,
ii) for any disjoint sets $A$, $B$, $\mathscr{W}(A)$ and $\mathscr{W}(B)$ are independent.

It turns out that these two properties are equivalent to the two properties defining the white noise $\mathscr{W}$.

Since $\mathscr{W}$ can be interpreted as a random measure in $\mathbb{R} \times [0,\infty)$, it is natural to ask which kind of functions can we integrate with respect to $\mathscr{W}$. We will write $\mathscr{W}(dxdt)$ when we want to look at $\mathscr{W}$ as a measure. Integration of functions $f : \mathbb{R} \times [0,\infty)$ is immediate: we just define $\int f \mathscr{W}(dxdt) = \mathscr{W}(f)$. But we would like to integrate *random* functions as well. It turns out that not any random function can be integrated with respect to $\mathscr{W}(dxdt)$. Up to here, time and space are treated in the same way; this will change in a moment. For each $T \geq 0$, let us denote by $\mathscr{F}_T$ the $\sigma$-algebra generated by the random variables $\{\mathscr{W}(A); A \subseteq \mathbb{R} \times [0,T]\}$. We say that a random function $f$ is *elementary* if it is of the form

$$f(x,t,\omega) = X(\omega)\mathbf{1}(a < t \leq b)\mathbf{1}(x \in A)$$

for some $b > a \geq 0$, some measurable set $A \subseteq \mathbb{R}$ with $|A| < +\infty$ and some bounded random variable $X \in \mathscr{F}_a$. The assumption $X \in \mathscr{F}_a$ is the crucial one. We say that a function $f$ is *simple* if it is of the form $f_1 + ... + f_\ell$ for some elementary functions $f_1, ..., f_\ell$, that is,

$$f(x,t,\omega) = \sum_{i=1}^{\ell} X_i(\omega)\mathbf{1}(a_i < t \leq b_i)\mathbf{1}(x \in A_i) \tag{E.1}$$

for some intervals $\{(a_i,b_i]\}_i$, some measurable sets $\{A_i\}_i$ of finite Lebesgue measure and some bounded random variables $\{X_i\}_i$ with $X_i \in \mathscr{F}_{a_i}$. Breaking the intervals $\{(a_i,b_i]\}_i$ and the sets $\{A_i\}_i$ into finite pieces if necessary, we can assume that the intervals $(a_i,b_i]$ are either disjoint or equal, and that if they are equal, then the corresponding sets $A_i$ are disjoint.

For an elementary function $f$ as above, we define the *stochastic integral* $\iint f \mathscr{W}(dxdt)$ as

$$\iint f \mathscr{W}(dxdt) = X(\omega)\mathscr{W}(A \times (a,b]).$$

If $f$ is a simple function like in (E.1), then we define $\int f \mathscr{W}(dxdt)$ by linearity:

$$\iint f \mathscr{W}(dxdt) = \sum_{i=1}^{\ell} X_i(\omega)\mathscr{W}(A_i \times (a_i,b_i]).$$

The terms on the previous sum are not correlated among them. In fact, if $(a_i,b_i] \cap (a_j,b_j] = \varnothing$ and $a_i < a_j$, then

$$\begin{aligned}
E\big[X_i(\omega)\mathscr{W}(A_i \times (a_i,b_i])X_j(\omega)\mathscr{W}(A_j \times (a_j,b_j])\big] &= \\
= E\big[X_i(\omega)\mathscr{W}(A_i \times (a_i,b_i])X_j(\omega)E\big[\mathscr{W}(A_j \times (a_j,b_j])\big|\mathscr{F}_{a_j}\big]\big] &= 0,
\end{aligned}$$

since $\mathscr{W}(A_j \times (a_j, b_j])$ is independent of $\mathscr{F}_{a_j}$ and $X_j$ is $\mathscr{F}_{a_j}$-measurable. If $(a_i, b_i] = (a_j, b_j]$, then $A_i \cap A_j = \varnothing$, and the same argument can be carried out. In particular, we have the formula

$$E\Big[\Big(\iint f\mathscr{W}(dxdt)\Big)^2\Big] = \sum_{i=1}^{\ell} E\big[X_i^2\big](b_i - a_i)|A_i| = E\Big[\iint f^2 dxdt\Big] \tag{E.2}$$

In the same way that Parseval's identity allows to extend the definition of the Fourier transform to functions in $L^2(\mathbb{R})$, this identity allows to extend the definition of the stochastic integral. Let us denote by $\mathscr{P}$ the closure of the set of simple functions in $L^2(P(d\omega)dxdt)$. For a random function $f \in \mathscr{P}$, we define the stochastic integral $\iint f\mathscr{W}(dxdt)$ as

$$\iint f\mathscr{W}(dxdt) = \lim_{n\to\infty} \iint f_n \mathscr{W}(dxdt),$$

where $\{f_n; n \in \mathbb{N}\}$ is a sequence of simple functions such that

$$\lim_{n\to\infty} E\Big[\iint (f - f_n)^2 dxdt\Big] = 0.$$

The identity (E.2) shows that $\iint f\mathscr{W}(dxdt)$ is well defined and it does not depend on the choice of the approximating sequence $\{f_n; n \in \mathbb{N}\}$. The random functions $f \in \mathscr{P}$ are called *predictable* functions.

For any $T \in [0, \infty)$ and any $f \in \mathscr{P}$, we define

$$\int_0^T \int_{\mathbb{R}} f\mathscr{W}(dxdt) = \iint f\mathbf{1}(0 \le t \le T)\mathscr{W}(dxdt).$$

It turns out that this integral is $\mathscr{F}_T$-measurable. This will be important when dealing with stochastic PDE's.

## REFERENCES

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