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Departamento de Matemática

**Analysis of Finite Element Methods for Singularly  
Perturbed Problems**

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**Título.** Análisis de Métodos de Elementos Finitos para Problemas Singularmente Perturbados.

**Resumen.** Proponemos y analizamos métodos de elementos finitos para problemas estacionarios singularmente perturbados, tales como problemas de reacción-difusión o de convección-difusión. Es conocido que la técnicas de discretización estándares no producen buenas aproximaciones a la solución de esta clase de problemas si el parámetro de perturbación es pequeño debido a la presencia de capas límites o internas. Estamos interesados en métodos robustos que funcionen adecuadamente para todos los valores del parámetro de perturbación singular.

Consideramos dos técnicas diferentes. Una de ellas se basa en refinamientos locales de la malla cerca de las capas límites. Usamos que la solución está en espacios de Sobolev con peso para probar estimaciones del error de interpolación sobre mallas rectangulares adecuadamente graduadas. Introducimos un operador de interpolación de promedios para el cual probamos estimaciones de error bajo la condición de que elementos vecinos tengan longitudes comparables en cada dirección. Esta condición es verificada por mallas que aparecen naturalmente en la aproximación de capas límites. También consideramos la aproximación de funciones que se anulan en el borde por funciones con la misma propiedad. Finalmente, nuestras estimaciones permiten sobre el lado derecho normas de Sobolev con pesos, donde el peso está relacionado con la distancia al borde.

Proponemos también un método de Galerkin Discontinuo (DG) con estabilización de tipo Exponential Fitting para resolver un problema de interés en semiconductores. El método DG considerado es una variante del método de Interior Penalty. Analizamos el método propuesto en las formulaciones mixta y primal, y presentamos ejemplos numéricos que muestran resultados adecuados. Finalmente probamos estimaciones de error óptimas para el método DG introducido en el caso de un problema regular.

**Palabras claves.** Elementos finitos, Elementos anisotrópicos, Problemas singularmente perturbados, Normas con pesos, Galerkin discontinuo.



**Title.** Analysis of Finite Element Methods for Singularly Perturbed Problems.

**Abstract.** We develop and analyze finite element methods for stationary singularly perturbed problems such as reaction-diffusion or convection-diffusion problems. It is known that standard discretization techniques do not give good approximations to the solution of this kind of problems when the perturbation parameter is small because of the presence of boundary or internal layers. We are interested in obtaining robust methods that work for all the values of the singular perturbation parameter.

We consider two different finite element techniques. One of them is based on mesh refinements near the boundary layers. We use the fact that the solution is in weighted Sobolev spaces in order to prove interpolation error estimates on suitably graded rectangular meshes. We prove our error estimates for a mean interpolation operator under the mild condition that neighboring elements have comparable sizes in each direction. This condition is verified for the meshes that appear naturally in the approximation of boundary layers. Also we consider the approximation of function vanishing on the boundary by functions with the same property. Finally, our estimates allow on the right hand side some weighted Sobolev norms where the weight is related with the distance to the boundary.

We also propose a Discontinuous Galerkin (DG) method with stabilization of Exponential Fitting type to approximate the solution of a problem of interest in semiconductors. The DG method considered here is a modification of the Interior Penalty method. We analyze the proposed method in mixed and primal formulation paying attention to the presence of “overflow”, and we present some numerical examples showing adequate results. Finally we prove optimal error estimates for the DG method introduced here for a regular problem.

**Key words.** Finite elements, Anisotropic elements, Singularly perturbed problems, Weighted norms, Discontinuous Galerkin.



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# Introducción

En esta Tesis proponemos y analizamos métodos de elementos finitos para problemas estacionarios singularmente perturbados, tales como problemas de reacción-difusión o de convección-difusión. Es conocido que las técnicas clásicas de discretización no producen aproximaciones adecuadas de la solución de esta clase de problemas cuando el parámetro de perturbación es pequeño debido a la presencia de capas límites o internas (boundary or internal layers). Los resultados de esta Tesis tienen como fin obtener métodos robustos que funcionen correctamente para todos los valores del parámetro de perturbación singular. Básicamente estudiamos dos técnicas de elementos finitos. Una de ellas se basa en convenientes refinamientos de la malla de acuerdo con estimaciones *a priori* para la solución. Por lo tanto, para esta aproximación, resulta necesario tener algún conocimiento de la solución y en particular, la ubicación exacta de sus singularidades. Por otro lado se estudia una técnica de estabilización conocida como “Exponential Fitting”. En este caso no se necesita ningún conocimiento preciso de propiedades de la solución.

La primera técnica trata con mallas altamente no uniformes, donde la medida de los elementos es mucho menor cerca de las singularidades. En el caso de presencia de capas límites, dichas mallas resultan anisotrópicas, es decir, contienen elementos cuyos lados tienen longitudes de distintos órdenes.

Siguiendo las ideas de [20], introducimos un nuevo operador de interpolación sobre funciones de clase  $Q_1$  (seccionalmente bilineales en 2d o trilineales en 3d) y obtenemos estimaciones del error de interpolación válidas para mallas rectangulares (en 2d o 3d) bastante generales conteniendo elementos arbitrariamente anisotrópicos. El análisis de error clásico se basa en la hipótesis de regularidad, excluyendo esta clase de elementos (ver por ejemplo [12, 17]). Sin embargo se sabe que esta hipótesis no es necesaria. En efecto, existen muchos artículos en donde se obtienen estimaciones de error bajo condiciones más generales. En particular, para elementos rectangulares cabe mencionar [1, 20, 38] y sus referencias.

La interpolación considerada aquí, es una interpolación de promedios. Hay dos razones para trabajar con esta clase de aproximación en lugar de la clásica interpolación de Lagrange. La primera es la posibilidad de aproximar funciones que no son tan suaves, para las cuales la interpolación de Lagrange no está definida, de hecho, esta es la razón que motivó la introducción de interpolaciones de promedios (ver [18]). Por otro lado, en el caso tridimensional, las interpolaciones de promedios tienen mejores propiedades de aproximación que la interpolación de Lagrange, aún para funciones suaves, cuando se usan elementos anisotrópicos (ver [1, 20]).

Para motivar los principales resultados de esta Tesis, exponemos aquí el siguiente ejemplo. Referimos al final de esta sección para notaciones generales y definiciones. En lo que sigue la letra  $C$  denotará una constante, que puede variar en las distintas apariciones, pero que será siempre independiente de la función  $u$  y de las mallas. Sea  $\Omega \subset \mathbb{R}^2$  un dominio poligonal y  $u \in H^2(\Omega)$  la solución de una ecuación general de segundo orden, cuya formulación débil es: Hallar  $u \in V$  tal que

$$a(u, v) = (F, v) \quad \forall v \in V$$

donde  $V$  es un subespacio de  $H^1(\Omega)$ ,  $F \in L^2(\Omega)$  y  $a : V \times V \rightarrow \mathbb{R}$  es una forma bilineal continua y coerciva. Dada una familia de mallas triangulares (o rectangulares)  $\mathcal{T}_h$ , con  $h = \max\{\text{diam } K, K \in \mathcal{T}_h\}$ , sea  $u_h$  la aproximación conforme por elementos finitos lineales (o bilineales) de  $u$  asociada con  $\mathcal{T}_h$ . Por el Lema de Cea sabemos que

$$|u - u_h|_{1,\Omega} \leq C|u - u_I|_{1,\Omega}$$

donde  $C$  depende solamente de la forma bilineal  $a$ , y  $u_I \in V$  es una aproximación de  $u$  seccionalmente lineal (o bilineal), por ejemplo una interpolación. Luego, para probar la convergencia del método cuando  $h \rightarrow 0$  debemos acotar  $|u - u_I|_{1,\Omega}$ . Si la familia de mallas  $\mathcal{T}_h$  es regular, es decir, si

$$\frac{h_K}{\rho_K} \leq C \quad \forall K \in \mathcal{T}_h, h > 0. \quad (1)$$

donde  $h_K$  y  $\rho_K$  son los diámetros exterior e interior del elemento  $K$ , y si  $u_I$  es la interpolación de Lagrange de  $u$  seccionalmente lineal (o bilineal), entonces vale (ver por ejemplo [17])

$$|u - u_I|_{1,\Omega} \leq Ch|u|_{2,\Omega}$$

que implica la convergencia. En el caso de mallas rectangulares, la condición (1) es equivalente a la existencia de constantes  $C_1$  y  $C_2$  tales que

$$C_1 \leq \frac{h_{1,K}}{h_{2,K}} \leq C_2 \quad \forall K \in \mathcal{T}_h, h > 0. \quad (2)$$

( $h_{i,K}$  es la longitud de  $K$  en la  $i$ -ésima dirección). Pero, es sabido que la condición de regularidad (1) no es necesaria para obtener estimaciones de error (que implican la convergencia). Esto ha sido estudiado, por ejemplo, por Apel y Dobrowolski [4], Babuska y Aziz [9], Jamet [25], Krížek [28], para el caso de mallas triangulares, y por Jamet [25] y Al Shenk [38] en el caso de mallas rectangulares. En el primer caso, la condición de regularidad puede ser reemplazada por una condición de “ángulo máximo” (ángulos acotados por una constante estrictamente menor que  $\pi$ ). Para elementos rectangulares podemos proceder como sigue. Sea  $\hat{K}$  un elemento rectangular de referencia, y consideremos una función  $f \in H^2(\hat{K})$ . Sea  $p \in \mathcal{P}_1$  tal que

$$\left\| \frac{\partial}{\partial x_1}(f - p) \right\|_{0,\hat{K}} \leq C \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0,\hat{K}}$$

(por ejemplo,  $p$  puede ser el polinomio de Taylor promediado de grado 1, ver [11]). Sea  $f_I \in \mathcal{Q}_1$  la interpolación de Lagrange seccionalmente bilineal de  $f$  sobre  $\hat{K}$ . Entonces tenemos

$$\left\| \frac{\partial}{\partial x_1}(f - f_I) \right\|_{0,\hat{K}} \leq \left\| \frac{\partial}{\partial x_1}(f - p) \right\|_{0,\hat{K}} + \left\| \frac{\partial}{\partial x_1}(p - f_I) \right\|_{0,\hat{K}}.$$

Basta estimar  $\left\| \frac{\partial}{\partial x_1}(p - f_I) \right\|_{0,\hat{K}}$ . Si  $v = p - f_I \in \mathcal{Q}_1$ , es fácil ver que (ver Figura 1)

$$\left\| \frac{\partial v}{\partial x_1} \right\|_{0,\hat{K}} \sim |v(\text{B}) - v(\text{A})|^2 + |v(\text{D}) - v(\text{C})|^2.$$

Tenemos

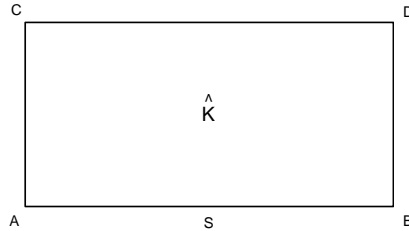


Figure 1: Un elemento de referencia.

$$\begin{aligned}
|v(\mathbf{B}) - v(\mathbf{A})| &= |(p(\mathbf{B}) - f(\mathbf{B})) - (p(\mathbf{A}) - f(\mathbf{A}))| \\
&= \left| \int_S \frac{\partial}{\partial x_1} (p - f) ds \right| \\
&\leq C \left\{ \left\| \frac{\partial}{\partial x_1} (p - f) \right\|_{0, \hat{K}} + \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0, \hat{K}} \right\}
\end{aligned}$$

donde hemos usado la desigualdad de trazas

$$\|w\|_{0,S} \leq C \|w\|_{1, \hat{K}} \quad \forall w \in H^1(\hat{K}). \quad (3)$$

Siendo que  $|v(\mathbf{D}) - v(\mathbf{C})|$  puede ser acotado análogamente, tenemos

$$\left\| \frac{\partial}{\partial x_1} (f - f_I) \right\|_{0, \hat{K}} \leq C \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0, \hat{K}} \quad (4)$$

Notamos que la derivada  $\frac{\partial^2 f}{\partial x_2^2}$  no aparece en el lado derecho de esta desigualdad. Este hecho es importante, pues, si  $K$  es un rectángulo con lados de longitudes  $h_1$  y  $h_2$  (ver Figura 2), la desigualdad (4) nos permite obtener, para la solución  $u$  de nuestro problema, a través de un cambio de variables

$$\left\| \frac{\partial}{\partial x_1} (u - u_I) \right\|_{0,K} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,K} + h_2 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,K} \right\}. \quad (5)$$

( $u_I$  es ahora la interpolación bilineal de  $u$  sobre  $K$ ). Claramente, también tenemos

$$\left\| \frac{\partial}{\partial x_2} (u - u_I) \right\|_{0,K} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,K} + h_2 \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,K} \right\}. \quad (6)$$

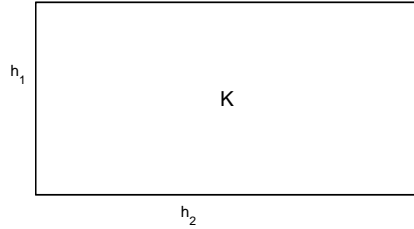


Figure 2: Un elemento genérico.

Las dos desigualdades anteriores implican la convergencia del método de elementos finitos, aún si las mallas  $\mathcal{T}_h$  contienen elementos anisotrópicos, o sea, si la condición (2) no se verifica.

La situación es diferente si  $\Omega$  es un dominio en  $\mathbb{R}^3$ . En este caso las mallas  $\mathcal{T}_h$  están formadas por 3d-rectángulos (paralelepípedos). No es posible repetir el argumento que acabamos de describir debido a que la desigualdad de trazas (3) no es válida si  $\hat{K}$  es un paralelepípedo en  $\mathbb{R}^3$  y  $S$  es una de sus aristas. En efecto, sea  $\hat{K} = [0, 1]^3$  y  $S = [0, 1] \times \{0\} \times \{0\}$  una de sus aristas, y para cada  $\varepsilon > 0$  consideremos la función

$$w_\varepsilon(x) = \min\left(1, \varepsilon \log\left|\log\frac{r}{e}\right|\right) \in H^1(\hat{K}),$$

con  $r = \sqrt{x_2^2 + x_3^2}$ . Se sigue que  $w_\varepsilon|_S \equiv 1$ , y luego  $\|w_\varepsilon\|_{0,S} = 1$ , pero, por otro lado, siendo que  $w_\varepsilon \searrow 0$  para casi todo punto cuando  $\varepsilon \rightarrow 0$ , tenemos  $\|w_\varepsilon\|_{0,\hat{K}} \rightarrow 0$ , y

$$\int_{\hat{K}} |\nabla w_\varepsilon|^2 dx \leq C\varepsilon^2 \int_0^2 r^{-1} \left|\log\frac{r}{e}\right|^{-2} dr \rightarrow 0,$$

esto es,  $\|w_\varepsilon\|_{1,\hat{K}} \rightarrow 0$ , lo que muestra que la desigualdad de trazas (3) no vale si  $\hat{K} \subset \mathbb{R}^3$ . Este ejemplo es una modificación trivial de uno dado por Apel y Dobrowolski [4] para mostrar que, efectivamente, no es posible obtener una desigualdad análoga a (4) si se reemplaza  $\hat{K}$  por el 3-simplex unitario  $\Lambda_1$ . Como ahora mostramos, el mismo ejemplo funciona si  $\hat{K} = [0, 1]^3$ . Consideremos para cada  $\varepsilon > 0$  la función

$$u_\varepsilon(x) = \left(1 - \min\left(1, \varepsilon \log\left|\log\frac{r}{e}\right|\right)\right) x_1 \in H^2(\hat{K}).$$

Entonces, para  $\varepsilon$  suficientemente pequeño,  $u_{\varepsilon,I}$  es la función trilineal definida por  $u_{\varepsilon,I}(0, 0, 0) = u_{\varepsilon,I}(0, 1, 0) = u_{\varepsilon,I}(0, 0, 1) = u_{\varepsilon,I}(0, 1, 1) = u_{\varepsilon,I}(1, 0, 0) = 0$ ,  $u_{\varepsilon,I}(1, 1, 0) = u_{\varepsilon,I}(1, 0, 1) = 1$  y  $u_{\varepsilon,I}(1, 1, 1) = 1 - \varepsilon \log\left|\log\frac{\sqrt{2}}{e}\right|$ . Notemos que  $u_{\varepsilon,I}(1, 1, 1) \rightarrow 1$  if  $\varepsilon \rightarrow 0$ . También,  $\frac{\partial u_\varepsilon}{\partial x_1} \nearrow 1$  para casi todo punto cuando  $\varepsilon \rightarrow 0$ . Por lo tanto, se sigue fácilmente que

$$\int_{\hat{K}} \left|\frac{\partial}{\partial x_1}(u_\varepsilon - u_{\varepsilon,I})\right|^2 dx \rightarrow 0$$

mientras

$$\int_{\hat{K}} \left|\nabla \frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx = \int_{\hat{K}} |\nabla w_\varepsilon|^2 dx \rightarrow 0$$

como queríamos.

En vista de estos resultados, una pregunta natural es si es posible obtener estimaciones óptimas como (5) y (6) válidas uniformemente para elementos anisotrópicos para otras interpolaciones y para funciones más singulares. En esta dirección, R. Durán [20] introdujo un operador de interpolación de promedios para el cual obtuvo estimaciones de error válidas

para mallas anisotrópicas. Dicha interpolación está definida aún para funciones no continuas, como por ejemplo, funciones en  $H^1(\Omega)$  para un dominio  $\Omega$  en dos o tres dimensiones. En el Capítulo 2 de esta tesis probamos estimaciones de error para una interpolación de promedios análoga a la definida por Durán. Nuestras estimaciones extienden resultados previamente conocidos en los siguientes aspectos:

Primero, nuestras hipótesis incluyen mallas más generales que aquellas permitidas en artículos previos. En [20] se requiere que las mallas sean casi-uniformes en cada dirección. Esta hipótesis fué relajada en [1] pero no lo suficiente para incluir las mallas que aparecen naturalmente en la aproximación de capas límites, las cuales verifican nuestros requerimientos. Para probar nuestras estimaciones de error requerimos solamente que elementos vecinos tengan medidas comparables en cada dirección, y por lo tanto, nuestros resultados son válidos para una familia bastante general de mallas anisotrópicas.

Segundo, consideramos la aproximación de funciones que se anulan en la frontera del dominio por funciones con la misma propiedad. Este aspecto no fué considerado en los artículos [1, 20].

Por último, generalizamos las estimaciones de error permitiendo normas más débiles sobre el lado derecho. Esas normas corresponden a espacios de Sobolev con un peso que está relacionado con la distancia a la frontera.

El uso de normas con peso para diseñar mallas apropiadas para la aproximación de problemas singulares es un procedimiento bien conocido. En particular, en varios artículos (ver por ejemplo [3, 6, 10, 23]) se obtienen estimaciones del error de interpolación para funciones en espacios de Sobolev con peso. Los pesos considerados en esos trabajos están relacionados con la distancia a un punto o a una arista (en el caso tridimensional), en cambio aquí consideraremos pesos relacionados con la distancia al borde.

Nuestro interés en trabajar es esos espacios aparece en la aproximación de capas límites. Para muchos problemas singularmente perturbados es posible probar que la solución tiene primeras y segundas derivadas acotadas, uniformemente respecto del parámetro de perturbación, en apropiadas normas de Sobolev con pesos. Por ejemplo, sea  $u \in H_0^1([0, 1])$  la solución de la ecuación

$$-\varepsilon u'' + b(x)u' + c(x)u = f \quad \text{en } (0, 1) \tag{7}$$

donde  $b$ ,  $c$ , y  $f$  son funciones regulares definidas en  $[0, 1]$ , y  $\varepsilon \in (0, 1]$  es un parámetro positivo pequeño. Si existe una constante  $b_0$  tal que  $b(x) < b_0 < 0$  entonces  $u$  presenta, en



general, una capa límite de tipo exponencial cerca de  $x = 0$ . Es fácil probar (ver Sección 1.1) que

$$\varepsilon^\beta \|x^\alpha u'\|_{0,[0,1]} \leq C \quad \text{si } \alpha + \beta \geq \frac{1}{2}$$

y

$$\varepsilon^\beta \|x^\alpha u''\|_{0,[0,1]} \leq C \quad \text{if } \alpha + \beta \geq \frac{3}{2}$$

con la constante  $C$  independiente de  $\varepsilon$ . En el Capítulo 1 usamos esas estimaciones para probar el siguiente resultado. Sean  $h, \delta < 1$  parámetros positivos, y consideremos la grilla  $\{x_i\}$  definida por  $x_0 = 0$ ,  $x_1 = h^{\frac{1}{1-\delta}}$ ,  $x_{i+1} = x_i + hx_i^\delta$  for  $i = 1, \dots, N-1$  donde  $N-1$  es el primer entero tal que  $x_{N-1} + hx_{N-1}^\delta \geq 1$ , y finalmente redefinimos  $x_N = 1$ . Sea  $\Pi u$  la interpolación de Lagrange seccionalmente lineal de  $u$  asociada con la grilla recién definida. Entonces, si  $\delta = 1 - \frac{1}{\log \frac{1}{\varepsilon}}$  tenemos las siguientes estimaciones del error de interpolación (ver Teorema 1.3.1)

$$\|u - \Pi u\|_{0,[0,1]} \leq Ch \log \frac{1}{\varepsilon} \quad \text{y} \quad \varepsilon^{\frac{1}{2}} \|(u - \Pi u)'\|_{0,[0,1]} \leq Ch$$

donde  $C$  es una constante dependiente de los datos  $b, c$  y  $f$ , pero que es independiente de  $h$  y  $\varepsilon$ . Siendo, por otro lado, que puede probarse que

$$h \leq C \frac{1}{N} \log N \log \frac{1}{\varepsilon}$$

obtenemos la estimación casi-óptima (con respecto al número de nodos) de error

$$\|u - \Pi u\|_\varepsilon \leq C \frac{1}{N} \log N \log^2 \frac{1}{\varepsilon} \quad (8)$$

en la norma de la energía  $\|\cdot\|_\varepsilon$  definida por  $\|v\|_\varepsilon^2 = \|v\|_{0,[0,1]}^2 + \varepsilon \|v'\|_{0,[0,1]}^2$ . Decimos que la estimación de arriba es casi-óptima porque para un problema regular, por ejemplo cuando  $\varepsilon = 1$ , y para una grilla casi-uniforme con  $N$  nodos puede probarse que

$$\|u - \Pi u\|_{1,[0,1]} \leq C \frac{1}{N} |u|_{2,[0,1]}.$$

Debemos remarcar que, en general, en nuestro caso tenemos  $|u|_{2,[0,1]} \sim \varepsilon^{-\frac{3}{2}}$ , y entonces, usando esta estimación, se obtiene la siguiente acotación del error de interpolación en la norma de la energía (ver Sección 1.1)

$$\|u - \Pi u\|_\varepsilon \leq C \frac{1}{N} \left( \frac{1}{N} \varepsilon^{-\frac{3}{2}} + \varepsilon^{-1} \right)$$

que no es adecuada en la práctica, pues si  $\varepsilon$  es pequeño se necesita tomar  $N$  demasiado grande para alcanzar una cierta tolerancia de error.

Para problemas en más dimensiones necesitamos usar otra aproximación, una interpolación de promedios, para obtener estimaciones de error en espacios de Sobolev con pesos. Además, tal operador de interpolación está definido también para funciones para las cuales no está definida la interpolación de Lagrange.

Para el problema mencionado aquí no hemos obtenido una estimación como (8) con  $\Pi u$  reemplazado por la solución discreta  $u_h$  por elementos finitos continuos seccionalmente lineales. Sin embargo, algunos experimentos numéricos nos hacen pensar que tal estimación puede ser válida. Por otro lado, para un problema de reacción-difusión podemos efectivamente obtener tal estimación.

Como observamos al comienzo de esta Introducción, necesitamos conocer estimaciones *a priori* para la solución, y la ubicación de sus singularidades para diseñar las mallas como hicimos más arriba. Sin embargo, es bien sabido, que existen muchos métodos para problemas singularmente perturbados que permanecen estables aún si se usan mallas que no son localmente refinadas. Esas técnicas son conocidas como métodos estabilizados. En esta Tesis proponemos una de estas técnicas para resolver un problema de interés en semiconductores. Más precisamente, dada una función continua  $\psi$  definida sobre un dominio poligonal  $\Omega \subset \mathbb{R}^2$  consideramos la aproximación del siguiente problema

$$\begin{aligned} -\operatorname{div}(\varepsilon \nabla u - \nabla \psi u) &= f \quad \text{en } \Omega \\ u &= g \quad \text{sobre } \Gamma_D \\ (\varepsilon \nabla u - \nabla \psi u) \cdot n &= 0 \quad \text{sobre } \Gamma_N. \end{aligned}$$

Para  $\varepsilon \ll 1$  la solución puede exhibir capas límite o internas. Introduciendo la variable

$$\rho = u e^{-\frac{\psi}{\varepsilon}} \tag{9}$$

reescribimos el problema como

$$\begin{aligned} -\operatorname{div}\left(\varepsilon e^{\frac{\psi}{\varepsilon}} \nabla \rho\right) &= f \quad \text{en } \Omega \\ \rho &= \chi \quad \text{sobre } \Gamma_D \\ \varepsilon e^{\frac{\psi}{\varepsilon}} \frac{\partial \rho}{\partial n} &= 0 \quad \text{sobre } \Gamma_N \end{aligned} \tag{10}$$

con  $\chi = ge^{-\frac{\psi}{\varepsilon}}$ . Esta formulación del problema es conveniente para la aproximación por elementos finitos. En tal caso, llegaríamos a un sistema lineal para la aproximación  $\rho_h$  de  $\rho$  como

$$\widetilde{M}\rho_h = b.$$

El coeficiente  $\varepsilon e^{\frac{\psi}{\varepsilon}}$  en el lado izquierdo de la primera línea de (10), puede ser una fuente de problemas numéricos al construir la matriz  $\widetilde{M}$ . Por esta razón, y siendo que estamos interesados en obtener una aproximación de  $u$ , efectuamos, a nivel matricial, una transformación discreta  $u_h = \mathcal{R}(\rho_h)$ , obteniendo entonces la forma deseable

$$Mu_h = b.$$

Este procedimiento es conocido como estabilización de tipo Exponential Fitting y referimos a [13] para un extenso estudio del tema. En dicho artículo, los autores analizan diferentes aproximaciones por elementos finitos de (10) como elementos finitos o volúmenes finitos mixtos. En esta Tesis analizamos un método de Galerkin discontinuo, más precisamente, una modificación del método de Interior Penalty introducido en [7]. Se sigue de nuestra discusión en la Sección 4.2 que el método de Interior Penalty no es conveniente para ser combinado con la estabilización de tipo exponential fitting en la formulación mixta, por lo cual necesitamos introducir algunas modificaciones. Presentamos algunos experimentos numéricos (ver la Sección 4.4) donde se puede observar la efectividad del esquema estabilizado propuesto para capturar capas límites e internas sin producir oscilaciones espurias.

El Capítulo 1 trata sobre estimaciones de interpolación para la solución de problemas de convección-difusión singularmente perturbados en una dimensión. Trabajamos allí con la interpolación de Lagrange, y obtenemos estimaciones de error casi-óptimas sobre grillas convenientemente graduadas, como las mencionadas arriba.

En el Capítulo 2 introducimos un operador de interpolación de promedios definido sobre funciones no demasiado suaves y probamos estimaciones de error en normas de Sobolev con peso. También consideramos la aproximación de funciones que se anulan en el borde por funciones con la misma propiedad.

En el Capítulo 3, como una aplicación de los resultados obtenidos en el Capítulo previo, presentamos algunos ejemplos y experimentos numéricos de diferentes ecuaciones singularmente perturbadas. Para un problema de reacción difusión en dos dimensiones probamos estimaciones casi-óptimas del error de aproximación de la solución por elementos finitos si

se usa una malla graduada adecuada. Consideramos también un problema de convección-difusión en dos dimensiones, para el cual presentamos algunos ejemplos numéricos que nos permiten conjeturar que es posible obtener aproximaciones por elementos finitos clásicos (sin necesidad de estabilización) si las mallas se diseñan adecuadamente. Finalmente, estudiamos un problema de cuarto orden también singularmente perturbado, que es considerado también en [33], donde los autores proponen un método no conforme de elementos finitos robusto. Aquí proponemos un esquema de elementos finitos diferente basado en el rectángulo de Adini y probamos que si se usan mallas casi-uniformes, entonces se alcanza el mismo orden de convergencia que el obtenido en [33]. Después damos evidencia numérica de que se pueden obtener mejores aproximaciones si tal método de elementos finitos es usado sobre mallas graduadas.

En el Capítulo 4, consideramos de nuevo la aproximación de problemas singularmente perturbados, pero seguimos una línea diferente. Buscamos métodos de elementos finitos que permanezcan estables aún usando mallas que no son localmente refinadas. Introducimos un método de Galerkin Discontinuo con estabilización de tipo Exponential Fitting. Presentamos algunos ejemplos numéricos, y finalmente, siguiendo las técnicas introducidas en [8] analizamos las propiedades de convergencia del método de elementos discontinuos (sin estabilización) considerado aquí.

Usaremos las siguientes notaciones y definiciones generales. Dado un dominio  $\Omega \subset \mathbb{R}^n$  y un entero positivo  $k$ ,  $H^k(\Omega)$  denota el espacio de Sobolev de funciones en  $L^2(\Omega)$  cuyas derivadas de ordenes menores o iguales que  $k$  pertenecen al espacio  $L^2(\Omega)$ . Denotamos por  $\|\cdot\|_{0,\Omega}$  la norma  $L^2$ , y la norma de una función  $v \in H^k(\Omega)$  está dada por

$$\|v\|_{k,\Omega}^2 = \sum_{|\alpha| \leq k} \left\| \frac{\partial^{|\alpha|} v}{\partial x^\alpha} \right\|_{0,\Omega}^2,$$

donde, para un multi-índice  $\alpha = (\alpha_1, \dots, \alpha_n)$  ponemos  $|\alpha| = \alpha_1 + \dots + \alpha_n$  y  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . En  $H^k(\Omega)$  también consideramos la seminorma

$$|v|_{k,\Omega}^2 = \sum_{|\alpha|=k} \left\| \frac{\partial^{|\alpha|} v}{\partial x^\alpha} \right\|_{0,\Omega}^2.$$

A veces usaremos también la notación  $D^\alpha$  para la derivada  $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ . Si no da lugar a confusión, omitiremos el subíndice  $\Omega$  en las notaciones de arriba. La clausura en  $H^k(\Omega)$  del subespacio

$C_0^\infty(\Omega)$  de todas las funciones infinitamente diferenciables con soporte compacto contenido en  $\Omega$  es denotada por  $H_0^k(\Omega)$ . Si  $\Omega$  es un dominio Lipschitz, se tiene

$$H_0^k(\Omega) = \left\{ v \in H^k(\Omega) : D^\alpha v \equiv 0 \text{ sobre } \partial\Omega, \text{ si } 0 \leq |\alpha| \leq k - 1 \right\}.$$

El conjunto  $C^k(\Omega)$  es el espacio de funciones continuas en  $\Omega$  cuyas derivadas de órdenes menores o iguales que  $k$  existen y son también continuas, y ponemos  $C(\Omega) = C^0(\Omega)$ .

Una triangulación  $\mathcal{T}$  de un dominio  $\Omega \subset \mathbb{R}^n$  es una colección de un número finito de subconjuntos  $K \subset \bar{\Omega}$ , llamados elementos finitos, de tal manera que las siguientes propiedades se verifiquen: (i)  $\bar{\Omega} = \cup_{K \in \mathcal{T}} K$ , (ii) Para cada  $K \in \mathcal{T}$ , el conjunto  $K$  es cerrado y el interior  $K^\circ$  es no vacío, (iii) Para cada par de elementos distintos  $K_1, K_2 \in \mathcal{T}$ , se tiene  $K_1^\circ \cap K_2^\circ = \emptyset$ , (iv) Para cada  $K \in \mathcal{T}$ , el borde  $\partial K$  es Lipschitz continuo.

El espacio de polinomios de grado menor o igual que  $k$  es denotado por  $\mathcal{P}_k$ . Además, denotamos por  $\mathcal{Q}_1$  el espacio de funciones bilineales (en el caso bidimensional) o trilineales (en el caso tridimensional).

En toda la Tesis la letra  $C$  indicará una constante, pero no necesariamente la misma en las distintas apariciones. La constante  $C$  será siempre independiente del parámetro de perturbación y de las mallas (es decir, independiente de  $\varepsilon$  y  $h$ ).

# Introduction

In this Thesis we develop and analyze finite element methods for stationary singularly perturbed problems such as reaction-diffusion or convection-diffusion problems. It is known that standard discretization techniques do not give good approximations to the solution of this kind of problems when the perturbation parameter is small because of the presence of boundary or internal layers. The results of this Thesis are related with robust methods that work for all the values of the singular perturbation parameter. Basically we study two finite elements techniques in order to obtain this kind of methods. One of them is based on suitable mesh refinement in accordance with *a priori* estimates for the solution. Therefore, for this approach we need some knowledge of the solution and the exact location of their singularities. The other one deals with a stabilization technique known as Exponential Fitting. In this case we do not need any precise knowledge of the solution properties.

The first technique leads us to treat with highly non uniform meshes such that the mesh size is much smaller near the singularities or layers than far from them. In the case of boundary layers these meshes contain very narrow or anisotropic elements.

We obtain new error estimates for  $\mathcal{Q}_1$  (piecewise bilinear in 2d or trilinear in 3d) approximations on meshes containing anisotropic rectangular elements, i. e., rectangles with sides of different orders. The classic error analysis is based on the so called regularity assumption which excludes this kind of elements (see for example [12, 17]). However, it is now well known that this assumption is not needed. Indeed, many papers have been written to prove error estimates under more general conditions. In particular, for rectangular elements we refer to [1, 20, 38] and their references.

We will prove the error estimates for a mean average interpolation. There are two reasons to work with this kind of approximation instead of the Lagrange interpolation. The first one is to approximate non smooth functions for which the Lagrange interpolation is not even defined, in fact this is the reason that motivated the introduction of average

interpolations (see [18]). On the other hand, it has already been observed that, in the three dimensional case, average interpolations have better approximation properties than the Lagrange interpolation even for smooth functions when narrow elements are used (see [1, 20]).

In order to motivate the principal results of this Thesis, here we consider the following example. We refer to the end of this Introduction for general notations and definitions. In what follows, the letter  $C$  will denote a constant which may be not the same in different occurrences, but it will be always independent of the function  $u$  and of the meshes. Let  $\Omega \subset \mathbb{R}^2$  be a polygonal domain and  $u \in H^2(\Omega)$  be the solution of a general second order elliptic equation, whose weak formulation is given by: Find  $u \in V$  such that

$$a(u, v) = (F, v) \quad \forall v \in V$$

where  $V$  is a subspace of  $H^1(\Omega)$ ,  $F \in L^2(\Omega)$  and  $a$  is a continuous and coercive bilinear form. Given a family of triangular (or rectangular) meshes  $\mathcal{T}_h$ , with  $h = \max\{\text{diam } K, K \in \mathcal{T}_h\}$ , let  $u_h$  be the conforming piecewise linear (or bilinear) finite element approximation of  $u$  associated with  $\mathcal{T}_h$ . By Cea's lemma we know that

$$|u - u_h|_{1,\Omega} \leq C|u - u_I|_{1,\Omega}$$

where  $C$  depends only on the bilinear form  $a$ , and  $u_I \in V$  is some continuous piecewise linear (or bilinear) approximation of  $u$ , for example an interpolation. Then, in order to prove the convergence of the method when  $h \rightarrow 0$  we should bound  $|u - u_I|_{1,\Omega}$ . If the family of meshes  $\mathcal{T}_h$  is regular, that is

$$\frac{h_K}{\rho_K} \leq C \quad \forall K \in \mathcal{T}_h, h > 0. \quad (1)$$

where  $h_K$  and  $\rho_K$  are the outer and inner diameters of the element  $K$ , and if  $u_I$  is the piecewise linear (or bilinear) Lagrange interpolation of  $u$ , then it holds (see for example [17])

$$|u - u_I|_{1,\Omega} \leq Ch|u|_{2,\Omega}$$

which implies the convergence. In the case of rectangular meshes, the condition (1) implies that there exist constants  $C_1$  and  $C_2$  such that

$$C_1 \leq \frac{h_{1,K}}{h_{2,K}} \leq C_2 \quad \forall K \in \mathcal{T}_h, h > 0. \quad (2)$$

( $h_{i,K}$  is the length of  $K$  on the  $i$  direction). But, it is known that the regularity assumption (1) is not needed in order to have the error estimates. That have been studied, for example, by Apel and Dobrowolski [4], Babuska and Aziz [9], Jamet [25], Krížek [28], in the case of triangular meshes, and by Jamet [25] and Al Shenk [38] in the case of rectangular meshes. In the first case, the regularity assumption can be replaced by a “maximum angle condition” (angles bounded away from  $\pi$ ). For rectangular elements we can proceed as follows. Let  $\hat{K}$  be a reference rectangular element and consider a function  $f \in H^2(\hat{K})$ . Let  $p \in \mathcal{P}_1$  be such that

$$\left\| \frac{\partial}{\partial x_1}(f - p) \right\|_{0,\hat{K}} \leq C \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0,\hat{K}}$$

( $p$  can be taken as the averaged Taylor polynomial of degree 1, see for example [11]). And now, let  $f_I \in \mathcal{Q}_1$  be the piecewise bilinear Lagrange interpolation of  $f$  on  $\hat{K}$ . Then we have

$$\left\| \frac{\partial}{\partial x_1}(f - f_I) \right\|_{0,\hat{K}} \leq \left\| \frac{\partial}{\partial x_1}(f - p) \right\|_{0,\hat{K}} + \left\| \frac{\partial}{\partial x_1}(p - f_I) \right\|_{0,\hat{K}}.$$

So, it is enough to estimate  $\left\| \frac{\partial}{\partial x_1}(p - f_I) \right\|_{0,\hat{K}}$ . Set  $v = p - f_I \in \mathcal{Q}_1$ , it is easy to see that (see Figure 3)

$$\left\| \frac{\partial v}{\partial x_1} \right\|_{0,\hat{K}} \sim |v(\mathbf{B}) - v(\mathbf{A})|^2 + |v(\mathbf{D}) - v(\mathbf{C})|^2.$$

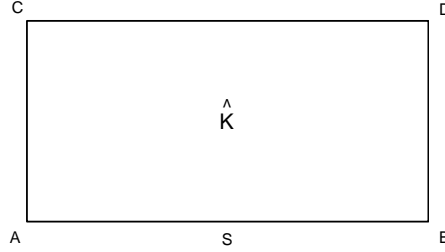


Figure 3: A reference element.

We have

$$\begin{aligned} |v(\mathbf{B}) - v(\mathbf{A})| &= |(p(\mathbf{B}) - f(\mathbf{B})) - (p(\mathbf{A}) - f(\mathbf{A}))| \\ &\leq \left| \int_s \frac{\partial}{\partial x_1}(p - f) ds \right| \\ &\leq C \left\{ \left\| \frac{\partial}{\partial x_1}(p - f) \right\|_{0,\hat{K}} + \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0,\hat{K}} \right\} \end{aligned}$$



where we have used the trace inequality

$$\|w\|_{0,S} \leq C \|w\|_{1,\hat{K}} \quad \forall w \in H^1(\hat{K}). \quad (3)$$

Analogously we bound  $|v(D) - v(C)|$ , and so we obtain

$$\left\| \frac{\partial}{\partial x_1} (f - f_I) \right\|_{0,\hat{K}} \leq C \left\| \nabla \frac{\partial f}{\partial x_1} \right\|_{0,\hat{K}} \quad (4)$$

Note that the derivative  $\frac{\partial^2 f}{\partial x_2^2}$  does not appear in the right hand side. This fact is important, because in this case, if  $K$  is a rectangle with lengths  $h_1$  and  $h_2$  (see Figure 4), by a change of variables we easily obtain from (4) for the solution  $u$  of the problem that we are considering

$$\left\| \frac{\partial}{\partial x_1} (u - u_I) \right\|_{0,K} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,K} + h_2 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,K} \right\} \quad (5)$$

( $u_I$  is now the bilinear interpolant of  $u$  on  $K$ ). Clearly, we can also obtain

$$\left\| \frac{\partial}{\partial x_2} (u - u_I) \right\|_{0,K} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,K} + h_2 \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,K} \right\}. \quad (6)$$

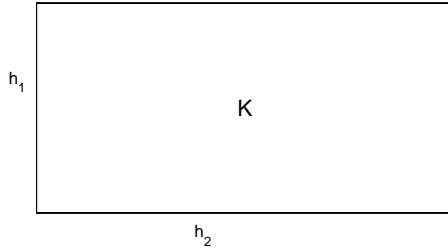


Figure 4: A generic element.

The previous two inequalities imply the convergence of the finite element method, even if the meshes  $\mathcal{T}_h$  contain strongly narrow elements, that is, when the condition (2) is not verified.

The situation is different if  $\Omega$  is a domain in  $\mathbb{R}^3$ . Now the mesh  $\mathcal{T}_h$  is made of 3d-rectangles. We can not repeat the argument because the trace inequality (3) is not true

if  $\hat{K}$  is a parallelepiped in  $\mathbb{R}^3$  and  $S$  is one of its edges. Indeed, let  $\hat{K} = [0, 1]^3$  and  $S = [0, 1] \times \{0\} \times \{0\}$  be an edge of  $\hat{K}$ , and for each  $\varepsilon > 0$  consider the function

$$w_\varepsilon(x) = \min \left( 1, \varepsilon \log \left| \log \frac{r}{e} \right| \right) \in H^1(\hat{K}),$$

with  $r = \sqrt{x_2^2 + x_3^2}$ . It follows that  $w_\varepsilon|_S \equiv 1$ , and then  $\|w_\varepsilon\|_{0,S} = 1$ , but, on the other hand, since  $w_\varepsilon \searrow 0$  a.e. for  $\varepsilon \rightarrow 0$  we have  $\|w_\varepsilon\|_{0,\hat{K}} \rightarrow 0$ , and

$$\int_{\hat{K}} |\nabla w_\varepsilon|^2 dx \leq C\varepsilon^2 \int_0^2 r^{-1} \left| \log \frac{r}{e} \right|^{-2} dr \rightarrow 0,$$

that is,  $\|w_\varepsilon\|_{1,\hat{K}} \rightarrow 0$ , showing that the trace inequality (3) does not hold for  $\hat{K} \subset \mathbb{R}^3$ . This example is a trivial modification of the one given by Apel and Dobrowolski [4] to show that, in fact, the corresponding inequality analogous to (4) does not hold true if  $\hat{K}$  is replaced by the unit 3-simplex  $\Lambda_1$ . As we now show, the same example works for  $\hat{K} = [0, 1]^3$ . Consider for each  $\varepsilon > 0$  the function

$$u_\varepsilon(x) = \left( 1 - \min \left( 1, \varepsilon \log \left| \log \frac{r}{e} \right| \right) \right) x_1 \in H^2(\hat{K}).$$

Then  $u_{\varepsilon,I}$  is the bilinear function defined by  $u_{\varepsilon,I}(0, 0, 0) = u_{\varepsilon,I}(0, 1, 0) = u_{\varepsilon,I}(0, 0, 1) = u_{\varepsilon,I}(0, 1, 1) = u_{\varepsilon,I}(1, 0, 0) = 0$ ,  $u_{\varepsilon,I}(1, 1, 0) = u_{\varepsilon,I}(1, 0, 1) = 1$  and  $u_{\varepsilon,I}(1, 1, 1) = 1 - \varepsilon \log \left| \log \frac{\sqrt{2}}{e} \right|$  for  $\varepsilon$  small enough. Note that  $u_{\varepsilon,I}(1, 1, 1) \rightarrow 1$  if  $\varepsilon \rightarrow 0$ . Also,  $\frac{\partial u_\varepsilon}{\partial x_1} \nearrow 1$  a.e. for  $\varepsilon \rightarrow 0$ . Then, it follows easily that

$$\int_{\hat{K}} \left| \frac{\partial}{\partial x_1} (u_\varepsilon - u_{\varepsilon,I}) \right|^2 dx \rightarrow 0$$

while

$$\int_{\hat{K}} \left| \nabla \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 dx = \int_{\hat{K}} |\nabla w_\varepsilon|^2 dx \rightarrow 0$$

as we wanted.

In view of these results, a natural question is whether or not optimal order estimates like (5) and (6) are valid uniformly for narrow elements for other interpolations and more singular functions. In this direction, R. Durán [20] has introduced an average interpolation operator for which he has obtained error estimates for narrow elements. This average interpolation is defined even for non-continuous functions, for example, for functions in  $H^1(\Omega)$  with a two or three dimensional domain  $\Omega$ . In Chapter 2 we will prove error estimates for an average interpolation analogous to the one defined by Durán. Our estimates extend previously known results in several aspects:

First, our assumptions include more general meshes than those allowed in the previous papers. Indeed, in [20] it was required that the meshes were quasiuniform in each direction. This requirement was relaxed in [1] but not enough to include the meshes that arise naturally in the approximation of boundary layers, which will be included under our assumptions. To prove our error estimates we require only that neighboring elements are of comparable size and so, our results are valid for a rather general family of anisotropic meshes.

Second, we consider the approximation of functions vanishing on the boundary by finite element functions with the same property. This is a non trivial point that was not considered in the above mentioned references.

Finally, we generalize the error estimates allowing weaker norms on the right hand side. These norms are weighted Sobolev norms where the weights are related with the distance to the boundary.

The use of weighted norms to design appropriate meshes in finite element approximations of singular problems is a well known procedure. In particular, error estimates for functions in weighted Sobolev spaces have been obtained in several works (see for example [3, 6, 10, 23]). In those works, the weights considered are related with the distance to a point or an edge (in the 3d case), instead here we consider weights related with the distance to the boundary.

Our interest of working with these norms arises in the approximation of boundary layers. Indeed, for many singularly perturbed problems it is possible to prove that the solution has first and second derivatives which are bounded, uniformly in the perturbation parameter, in appropriate weighted Sobolev norms. For example, let  $u \in H_0^1([0, 1])$  be the solution of the equation

$$-\varepsilon u'' + b(x)u' + c(x)u = f \quad \text{in } (0, 1) \quad (7)$$

where  $b$ ,  $c$ , and  $f$  are regular functions defined on  $[0, 1]$ , and  $\varepsilon \in (0, 1]$  is a small positive parameter. If there exist a constant  $b_0$  such that  $b(x) < b_0 < 0$  then  $u$  present in general an exponential boundary layer near  $x = 0$ . It can be proved (see Section 1.1) that

$$\varepsilon^\beta \|x^\alpha u'\|_{0,[0,1]} \leq C \quad \text{if } \alpha + \beta \geq \frac{1}{2}, \quad \alpha, \beta \geq 0$$

and

$$\varepsilon^\beta \|x^\alpha u''\|_{0,[0,1]} \leq C \quad \text{if } \alpha + \beta \geq \frac{3}{2}, \quad \alpha, \beta \geq 0$$

with the constant  $C$  independent of  $\varepsilon$ . In Chapter 1 we use these estimates to prove the following result. Let  $h, \delta < 1$  be positive parameters, and consider the grid  $\{x_i\}$  defined by

$x_0 = 0$ ,  $x_1 = h^{\frac{1}{1-\delta}}$ ,  $x_{i+1} = x_i + hx_i^\delta$  for  $i = 1, \dots, N-1$  where  $N-1$  is the first number such that  $x_{N-1} + hx_{N-1}^\delta \geq 1$ , and finally redefine  $x_N = 1$ . Let  $\Pi u$  be the piecewise linear Lagrange interpolation of  $u$  associated with the grid just defined. Then if  $\delta = 1 - \frac{1}{\log \frac{1}{\varepsilon}}$  we have the following interpolation error estimates (see Theorem 1.3.1)

$$\|u - \Pi u\|_{0,[0,1]} \leq Ch \log \frac{1}{\varepsilon} \quad \text{and} \quad \varepsilon^{\frac{1}{2}} \|(u - \Pi u)'\|_{0,[0,1]} \leq Ch$$

where  $C$  is a constant depending on the data  $b, c$  and  $f$ , but independent of  $h$  and  $\varepsilon$ . Since it can be proved that

$$h \leq C \frac{1}{N} \log N \log \frac{1}{\varepsilon}$$

we obtain the almost optimal (with respect to the number of nodes) interpolation estimate

$$\|u - \Pi u\|_{\varepsilon,[0,1]} \leq C \frac{1}{N} \log N \log^2 \frac{1}{\varepsilon} \quad (8)$$

in the norm  $\|\cdot\|_\varepsilon$  defined by  $\|v\|_\varepsilon^2 = \|v\|_0^2 + \varepsilon \|v'\|_0^2$  (this norm is related with the energy). We say that the above estimate is almost optimal because for a regular problem, for example when  $\varepsilon = 1$ , and for a quasi-uniform grid with  $N$  nodes it can be proved that

$$\|u - \Pi u\|_{1,[0,1]} \leq C \frac{1}{N} |u|_{2,[0,1]}.$$

It must be remarked that in our case, in general, we have  $|u|_{2,[0,1]} \sim \varepsilon^{-\frac{3}{2}}$ , and then, using this fact, the following interpolation error estimate for the energy norm follows (see Section 1.1)

$$\|u - \Pi u\|_\varepsilon \leq C \frac{1}{N} \left( \frac{1}{N} \varepsilon^{-\frac{3}{2}} + \varepsilon^{-1} \right)$$

which, in practice, implies that a large  $N$  will be needed to obtain an acceptable approximation when  $\varepsilon$  is small.

For problems in more space dimensions we need to use another approximation, a mean average interpolation, in order to obtain error estimates in weighted Sobolev spaces. Furthermore, that interpolation operator will be defined even for functions for which the Lagrange interpolation is not defined.

For the equation considered here we are not able obtain an estimate like (8) with  $\Pi u$  replaced by the continuous piecewise linear finite element solution  $u_h$ . However, some numerical experiments move us to think that such an estimate could hold true. By the other hand, for a reaction-diffusion problem we obtain the desired approximation error estimate.

As we remark at the beginning of this Introduction, we need know *a priori* estimates for the solution, and the location of its singularities in order to design the family of meshes just described. However, as it is now well known, there exist many methods for singularly perturbed problems that remain stable even if meshes that are not locally refined are used. These techniques are known as stabilized methods. We propose one of them in this Thesis to solve a problem of interest in the modelling of semiconductors. More precisely, given a continuous function  $\psi$  defined on a polygonal domain  $\Omega \subset \mathbb{R}^2$  we consider the approximation of the solution of the following problem

$$\begin{aligned} -\operatorname{div}(\varepsilon \nabla u - \nabla \psi u) &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma_D \\ (\varepsilon \nabla u - \nabla \psi u) \cdot n &= 0 && \text{on } \Gamma_N. \end{aligned}$$

For  $\varepsilon \ll 1$  the solution can exhibit internal and boundary layers. Introducing the variable

$$\rho = u e^{-\frac{\psi}{\varepsilon}} \tag{9}$$

we can rewrite the problem as

$$\begin{aligned} -\operatorname{div}(\varepsilon e^{\frac{\psi}{\varepsilon}} \nabla \rho) &= f && \text{in } \Omega \\ \rho &= \chi && \text{on } \Gamma_D \\ \varepsilon e^{\frac{\psi}{\varepsilon}} \frac{\partial \rho}{\partial n} &= 0 && \text{on } \Gamma_N \end{aligned} \tag{10}$$

with  $\chi = g e^{-\frac{\psi}{\varepsilon}}$ . This is a convenient formulation of the problem in order to approximate it by a finite element method. Doing that, we arrive at a linear system for the approximation  $\rho_h$  of  $\rho$  like

$$\widetilde{M} \rho_h = b.$$

Notice that the coefficient  $\varepsilon e^{\frac{\psi}{\varepsilon}}$  in the left hand side of the first line of (10), may be a source of numerical problems when the matrix  $\widetilde{M}$  is constructed. For this reason, and since we are interested in obtaining an approximation of  $u$ , we will perform, at the matrix level, a discrete inverse transformation  $u_h = \mathcal{R}(\rho_h)$ , obtaining the desirable form

$$M u_h = b.$$

This procedure is known as Exponential Fitting Stabilization and we refer to [13] for an extensive study on the subject. In that article, the authors analyze different finite element

approximations of (10) like mixed finite elements and finite volume methods. In this Thesis we will consider a Discontinuous Galerkin method, more precisely, a modification of the Interior Penalty method introduced in [7]. As follows by our discussion in Section 4.2 the Interior Penalty method is not suited for exponential fitting stabilization in the mixed formulation, and then we need to introduce some modifications. We present some numerical experiments (see Section 4.4) where it can be observed the effectiveness of the stabilized scheme in capturing internal and boundary layers without any spurious oscillations.

In Chapter 1 we deal with the interpolation error estimates for the solution of singularly perturbed one dimensional convection diffusion problems. We work there with the Lagrange interpolation, and we obtain quasi-optimal error estimates on suitably graded grids, like those mentioned above.

In Chapter 2 we introduce a mean average interpolation operator defined on functions that are not too smooth and we prove error estimates in weighted Sobolev norms. We also consider the approximation of functions vanishing on the boundary by functions with the same property.

In Chapter 3, as an application of the results obtained in the previous Chapter, we present some examples and numerical tests of different singularly perturbed equations. For a two dimensional reaction-diffusion equation we prove quasi-optimal approximation error for the finite element solution if the mesh is suitably graded. We also study a two dimensional convection-diffusion problem, and we present some numerical examples which allows us to conjecture that adequate approximations by the standard finite element method (without any stabilization) can be obtained if graded meshes are used. Finally we consider a fourth order singularly perturbed problem, for which a robust finite element method is proposed in [33]. We propose here a different finite element based on the Adini's rectangle and we prove that if quasi-uniform meshes are used, the convergence order obtained in [33] is recovered. We show, only by numerical experiments, that better approximations are obtained using that finite element method on graded meshes.

In Chapter 4, we consider again the approximation of singularly perturbed problems, but we follow a different approach. We seek finite element methods that remain stable even using meshes that are not locally refined. We introduce a Discontinuous Galerkin method with stabilization of Exponential Fitting type. We present some numerical examples, and finally, following the techniques introduced in [8] we analyze the convergence properties of

the DG method (without stabilization) considered.

We use the following general notations and definitions. Given a domain  $\Omega \subset \mathbb{R}^n$  and a positive integer  $k$ ,  $H^k(\Omega)$  denotes the Sobolev space of functions in  $L^2(\Omega)$  whose derivatives of order less than or equal  $k$  belong also to  $L^2(\Omega)$ . We denote by  $\|\cdot\|_{0,\Omega}$  the  $L^2$ -norm, and the norm of a function  $v \in H^k(\Omega)$  is given by

$$\|v\|_{k,\Omega}^2 = \sum_{|\alpha| \leq k} \left\| \frac{\partial^{|\alpha|} v}{\partial x^\alpha} \right\|_{0,\Omega}^2,$$

where for a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  we have  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . In  $H^k(\Omega)$  we also consider the seminorm

$$|v|_{k,\Omega}^2 = \sum_{|\alpha|=k} \left\| \frac{\partial^{|\alpha|} v}{\partial x^\alpha} \right\|_{0,\Omega}^2.$$

Sometimes we also use the notation  $D^\alpha$  for the derivative  $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ . If there is no confusion, we omit the subindex  $\Omega$  in the above notations. Also  $H_0^k(\Omega)$  is the closure in  $H^k(\Omega)$  of the subspace  $C_0^\infty(\Omega)$  of all infinitely differentiable functions with compact support contained in  $\Omega$ . It follows that if  $\Omega$  is a Lipschitz domain then

$$H_0^k(\Omega) = \left\{ v \in H^k(\Omega) : D^\alpha v \equiv 0 \text{ on } \partial\Omega, \text{ if } 0 \leq |\alpha| \leq k-1 \right\}.$$

The set  $C^k(\Omega)$  is the space of continuous functions on  $\Omega$  whose derivatives of order less than or equal  $k$  exist and they are also continuous, and we set  $C(\Omega) = C^0(\Omega)$ .

A triangulation  $\mathcal{T}$  of a domain  $\Omega \subset \mathbb{R}^n$  is a collection of a finite number of subsets  $K \subset \overline{\Omega}$ , called finite elements, in such a way that the following properties are satisfied: (i)  $\overline{\Omega} = \cup_{K \in \mathcal{T}} K$ , (ii) for each  $K \in \mathcal{T}$ , the set  $K$  is closed and the interior  $K^\circ$  is non empty, (iii) for each distinct  $K_1, K_2 \in \mathcal{T}$  one has  $K_1^\circ \cap K_2^\circ = \emptyset$ , (iv) for each  $K \in \mathcal{T}$ , the boundary  $\partial K$  is Lipschitz continuous.

The space of polynomials of degree less than or equal  $k$  is denoted by  $\mathcal{P}_k$ . Also we denote by  $\mathcal{Q}_1$  either the space of the bilinear functions (in the two dimensional case) or the space of the trilinear functions (in the three dimensional case).

In all the Thesis letter  $C$  will denote a constant, but not necessarily the same in different occurrences. The constant  $C$  will be always independent of the perturbation parameter and of the meshes (i.e., independent of  $\varepsilon$  and  $h$ ).

# Chapter 1

## Weighted Error Estimates for the Lagrange Interpolation

### 1.1 Introduction

In this Chapter we discuss the numerical approximation of the boundary value problem of convection-diffusion type

$$\begin{aligned} -\varepsilon u'' + b(x)u' + c(x)u &= f \quad \text{in } (0, 1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{1.1.1}$$

where  $b$ ,  $c$ , and  $f$  are smooth functions defined on  $[0, 1]$ , and  $\varepsilon$  is a small positive parameter. We also assume that there exist constants  $b_0$  and  $\gamma_0$  such that

$$b(x) \leq b_0 < 0, \quad \text{and} \quad c - \frac{b'}{2} \geq \gamma_0 > 0.$$

Problem (1.1.1) has a unique solution. When  $\varepsilon \ll 1$  that equation becomes singularly perturbed, indeed, the solution  $u$  shows a boundary layer near the endpoint  $x = 0$ .

The following *a priori* estimates are known (see Lemma 1.6 in Chapter 1 of [36]).

**Lemma 1.1.1.** *The solution  $u$  of Problem (1.1.1) verifies*

$$|u^{(i)}(x)| \leq C \left( 1 + \frac{1}{\varepsilon^i} \exp\left(b_0 \frac{x}{\varepsilon}\right) \right) \tag{1.1.2}$$

for  $i = 1, 2, \dots$ .

Let  $\hat{I} = [0, 1]$  and  $V = H_0^1(\hat{I})$ . We define the bilinear form  $a_{cd} : V \times V \rightarrow \mathbb{R}$  by

$$a_{cd}(v, w) = \int_0^1 (\varepsilon v' w' + b v' w + c v w) dx.$$



Then the variational formulation of Problem (1.1.1) is: Find  $u \in V$  such that

$$a_{cd}(u, v) = \int_0^1 f v \, dx \quad \forall v \in V.$$

In  $V$  we will consider the norm  $\|\cdot\|_\varepsilon$  defined by

$$\|v\|_\varepsilon^2 = \varepsilon \|v'\|_{0,\hat{I}}^2 + \|v\|_{0,\hat{I}}^2.$$

The bilinear form  $a_{cd}$  is  $V$ -coercive uniformly in  $\varepsilon$  (with the energy norm), i. e.  $a_{cd}(v, v) \geq C\|v\|_\varepsilon^2$  for all  $v \in V$ , with  $C$  independent of  $\varepsilon$ . Unfortunately, it is not continuous uniformly in  $\varepsilon$ , because we only have  $|a_{cd}(v, w)| \leq C\varepsilon^{-\frac{1}{2}}\|v\|_\varepsilon\|w\|_\varepsilon$ .

Given a grid  $0 = x_0 < x_1 < \dots < x_N = 1$ , let  $\Pi$  be its associated Lagrange interpolation operator on the piecewise linear functions (see Section 1.3). If the grid is quasi-uniform, that is, there exist positive constants  $c_1 \leq c_2$  such that

$$c_1 \frac{1}{N} \leq x_i - x_{i-1} \leq c_2 \frac{1}{N},$$

it follows from standard error estimates [17] that

$$\|v - \Pi v\|_{0,\hat{I}} \leq C \frac{1}{N^2} |v|_{2,\hat{I}}, \quad \|(v - \Pi v)'\|_{0,\hat{I}} \leq C \frac{1}{N} |v|_{2,\hat{I}}$$

for the interpolation error. If  $u$  is the solution of Problem (1.1.1), it results from estimates (1.1.2) that  $|u|_{2,\hat{I}} \leq C\varepsilon^{-\frac{3}{2}}$ , and then we have

$$\|u - \Pi u\|_\varepsilon \leq C \frac{1}{N} \left( \frac{1}{N} \varepsilon^{-\frac{3}{2}} + \varepsilon^{-1} \right). \quad (1.1.3)$$

As we can see, this estimate is not good when  $\varepsilon \searrow 0$ . We are interested in obtaining inequalities like (1.1.3), but with a better dependence on  $\varepsilon$  in the right hand side. Indeed, we will design a family of grids on which we have the estimate

$$\|u - \Pi u\|_\varepsilon \leq C \frac{1}{N} \log \frac{1}{\varepsilon}.$$

We will do that using the following weighted *a priori* estimates for the solution  $u$  which are immediate consequences of the Lemma 1.1.1.

**Proposition 1.1.2.** *Let  $\alpha_0 > 0$ . If (H1) holds and if  $\alpha$  and  $\beta$  are non negative numbers with  $\alpha \leq \alpha_0$  and*

$$\alpha + \beta \geq i - \frac{1}{2}$$

for the positive integer  $i$ , then the solution  $u$  of Problem (1.1.1) satisfies

$$\varepsilon^\beta \|x^\alpha u^{(i)}\|_0 \leq C \quad (1.1.4)$$

with the constant  $C$  independent of  $\alpha, \beta$  and  $\varepsilon$ .

*Proof.* Using the inequalities (1.1.2), we have

$$\begin{aligned} \varepsilon^{2\beta} \|x^\alpha u^{(i)}\|_0^2 &\leq C \varepsilon^{2\beta} \int_0^1 x^{2\alpha} \left(1 + \frac{1}{\varepsilon^i} \exp\left(b_0 \frac{x}{\varepsilon}\right)\right)^2 dx \\ &\leq C \varepsilon^{2\beta} \int_0^1 x^{2\alpha} dx + C \varepsilon^{2\beta} \int_0^1 x^{2\alpha} \frac{1}{\varepsilon^{2i}} \exp\left(2b_0 \frac{x}{\varepsilon}\right) dx \\ &= C \frac{\varepsilon^{2\beta}}{2\alpha+1} + C \varepsilon^{2(\beta+\alpha-i)} \int_0^1 \left(\frac{x}{\varepsilon}\right)^{2\alpha} \exp\left(2b_0 \frac{x}{\varepsilon}\right) dx \\ &= C \frac{\varepsilon^{2\beta}}{2\alpha+1} + C \varepsilon^{2(\beta+\alpha-i)+1} \int_0^{\frac{1}{\varepsilon}} y^{2\alpha} \exp(2b_0 y) dy \\ &\leq C \frac{\varepsilon^{2\beta}}{2\alpha+1} + \varepsilon^{2(\beta+\alpha-i)+1} \int_0^\infty y^{2\alpha} \exp(2b_0 y) dy \\ &=: C \frac{\varepsilon^{2\beta}}{2\alpha+1} + C \varepsilon^{2(\beta+\alpha-i)+1} A(\alpha) \\ &\leq 1 + A(\alpha) \end{aligned}$$

whenever  $\alpha \geq 0$ ,  $\beta \geq 0$  y  $2(\beta + \alpha - i) + 1 \geq 0$ . Therefore, we obtain the inequality (1.1.4) with  $C = 1 + \max_{0 \leq \alpha \leq \alpha_0} A(\alpha) < \infty$ .  $\square$

The following particular cases of the last Proposition will be used later on:

$$\varepsilon^\beta \|x^\alpha u'\|_0 \leq C \quad \text{if } 0 \leq \alpha \leq 1, \beta \geq 0, \alpha + \beta \geq \frac{1}{2} \quad (1.1.5)$$

$$\varepsilon^\beta \|x^\alpha u''\|_0 \leq C \quad \text{if } 0 \leq \alpha \leq 1, \beta \geq 0, \alpha + \beta \geq \frac{3}{2}. \quad (1.1.6)$$

## 1.2 Weighted inequalities for the local interpolation error

Let  $I = [a, b]$  be an interval of the real line. For each  $u \in H^1(I)$  we define the linear interpolant  $u_I \in \mathcal{P}_1(I)$  which verifies  $u_I(a) = u(a)$  and  $u_I(b) = u(b)$ .

**Proposition 1.2.1.** For  $\alpha < \frac{1}{2}$  we have

$$\|u - u_I\|_{0,I} \leq \frac{C}{1-2\alpha} |I|^{1-\alpha} \|(x-a)^\alpha u'\|_{0,I}. \quad (1.2.1)$$

where  $C$  is independent of the function  $u$  and of the interval  $I$ .

*Proof.* First we consider the particular case  $I = \hat{I} := [0, 1]$ . We have  $(u - u_{\hat{I}})(0) = (u - u_{\hat{I}})(1) = 0$  and then it follows

$$(u - u_{\hat{I}})(x) = \int_0^x (u - u_{\hat{I}})'(y) dy.$$

Having in mind that  $\alpha < \frac{1}{2}$ , and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} |(u - u_{\hat{I}})(x)| &\leq \int_0^1 |(u - u_{\hat{I}})'(y)| dy \\ &= \int_0^1 y^{-\alpha} y^{\alpha} |(u - u_{\hat{I}})'(y)| dy \\ &\leq \left( \int_0^1 y^{-2\alpha} dy \right)^{\frac{1}{2}} \left( \int_0^1 y^{2\alpha} |(u - u_{\hat{I}})'(y)|^2 dy \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{1 - 2\alpha} \right)^{\frac{1}{2}} \|y^{\alpha} (u - u_{\hat{I}})'\|_{0, \hat{I}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|u - u_{\hat{I}}\|_{0, \hat{I}} &\leq \left( \frac{1}{1 - 2\alpha} \right)^{\frac{1}{2}} \|x^{\alpha} (u - u_{\hat{I}})'\|_{0, \hat{I}} \\ &\leq \left( \frac{1}{1 - 2\alpha} \right)^{\frac{1}{2}} \left( \|x^{\alpha} u'\|_{0, \hat{I}} + \|x^{\alpha} u'_{\hat{I}}\|_{0, \hat{I}} \right). \end{aligned} \quad (1.2.2)$$

Taking into account that

$$u_{\hat{I}}(x) = u(0) + x(u(1) - u(0))$$

we obtain

$$\begin{aligned} u'_{\hat{I}}(x) &= u(1) - u(0) \\ &= \int_0^1 u'(y) dy. \end{aligned}$$

Then

$$\|x^{\alpha} u'_{\hat{I}}\|_{0, \hat{I}}^2 = \frac{1}{2\alpha + 1} \left( \int_0^1 u'(y) dy \right)^2. \quad (1.2.3)$$

Now, using again the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \int_0^1 u'(y) dy \right| &\leq \int_0^1 x^{-\alpha} x^{\alpha} |u'(y)| dy \\ &\leq \left( \int_0^1 x^{-2\alpha} \right)^{\frac{1}{2}} \left( \int_0^1 x^{2\alpha} |u'(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{1 - 2\alpha} \right)^{\frac{1}{2}} \|x^{\alpha} u'\|_{0, \hat{I}} \end{aligned}$$

whenever  $\alpha < \frac{1}{2}$ . Then, from equation (1.2.3) we arrive at

$$\|x^\alpha u'_I\|_{0,I} \leq \left(\frac{1}{1-2\alpha}\right)^{\frac{1}{2}} \|x^\alpha u'\|_{0,I}. \quad (1.2.4)$$

Inserting (1.2.4) in (1.2.2) it follows that

$$\begin{aligned} \|u - u_I\|_{0,I} &\leq \left(\frac{1}{1-2\alpha}\right)^{\frac{1}{2}} \left( \|x^\alpha u'\|_{0,I} + \left(\frac{1}{1-2\alpha}\right)^{\frac{1}{2}} \|x^\alpha u'\|_{0,I} \right) \\ &= \frac{(1-2\alpha)^{\frac{1}{2}} + 1}{(1-2\alpha)} \|x^\alpha u'\|_{0,I} \\ &\leq \frac{C}{1-2\alpha} \|x^\alpha u'\|_{0,I} \end{aligned} \quad (1.2.5)$$

as we wanted to prove. In the general case, let  $I = [a, b]$ ,  $u \in H^1(I)$  and  $u_I$  be the interpolant of  $u$ . Consider the map  $\Phi : \hat{I} \rightarrow I$ , given by  $\Phi(\hat{x}) = a + \hat{x}(b-a)$ , and, for a function  $v$  define  $\hat{v} = v \circ \Phi$ . Then  $(u_I)^\wedge = \hat{u}_I$  and then the following identities hold

$$\int_0^1 ((\hat{u} - (u_I)^\wedge)(\hat{x}))^2 d\hat{x} = \frac{1}{|I|} \int_a^b ((u - u_I)(x))^2 dx \quad (1.2.6)$$

$$\int_0^1 \hat{x}^{2\alpha} |\hat{u}'(\hat{x})|^2 d\hat{x} = |I|^{1-2\alpha} \int_a^b (x-a)^{2\alpha} |u'(x)|^2 dx. \quad (1.2.7)$$

Inserting (1.2.6) and (1.2.7) in (1.2.5) we obtain the inequality (1.2.1)  $\square$

*Remark 1.2.1.* The following example shows that an estimate as (1.2.1) is not true for  $\alpha > \frac{1}{2}$ : Consider the sequence of functions

$$u_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Then

$$\|u_n - u_{n,I}\|_{0,I} \longrightarrow \left( \int_0^1 (1-x)^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}.$$

while

$$\|x^\alpha u'_n\|_{0,I} = \int_0^{\frac{1}{n}} n^2 x^{2\alpha} dx = \frac{1}{2\alpha+1} n^{1-2\alpha} \longrightarrow 0$$

for  $\alpha > \frac{1}{2}$ .

In the proof of the next Lemma we will make use of the following inequality which is known as “Hardy inequality” (see, for example, [31]): There exists a constant  $C$  such that

$$\left\| \frac{v(x)}{x(x-1)} \right\|_{0,[0,1]} \leq C \|v'\|_{0,[0,1]} \quad (1.2.8)$$

for all  $v \in H_0^1([0, 1])$ .

**Lemma 1.2.2.** *Let  $0 \leq \alpha \leq 1$ . If  $u \in H^1(\hat{I})$  and  $\int_{\hat{I}} u = 0$  then it holds*

$$\|u\|_{0,\hat{I}} \leq C \|x^\alpha u'\|_{0,\hat{I}}. \quad (1.2.9)$$

*The constant  $C$  is independent of  $\alpha$  and  $u$ .*

*Proof.* Let  $v : \hat{I} \rightarrow \mathbb{R}$  defined by

$$v(x) = \int_0^x u(y) dy.$$

Then  $v$  solves the problem

$$v' = u, \quad \text{in } (0, 1), \quad v(0) = v(1) = 0,$$

besides

$$\|v\|_{H^1(\hat{I})} \leq C \|u\|_{0,\hat{I}}.$$

Therefore

$$\begin{aligned} \|u\|_{0,\hat{I}}^2 &= \int_0^1 u(x)^2 dx \\ &= \int_0^1 u(x)v'(x) dx \\ &= - \int_0^1 u'(x)v(x) dx \\ &= \int_0^1 \frac{-v(x)}{x(x-1)} x(x-1)u'(x) dx \\ &\leq \left\| \frac{-v}{x(x-1)} \right\|_{0,\hat{I}} \|x(x-1)u'\|_{0,\hat{I}} \\ &\leq C \|u\|_{0,\hat{I}} \|xu'\|_{0,\hat{I}} \end{aligned}$$

where we have used the Hardy inequality, which holds since  $v(0) = v(1) = 0$ , to obtain

$$\left\| \frac{v(x)}{x(x-1)} \right\|_{0,\hat{I}} \leq C \|v'\|_{0,\hat{I}} = C \|u\|_{0,\hat{I}}.$$

Thus we have proved

$$\|u\|_{0,\hat{I}} \leq C\|xu'\|_{0,\hat{I}} \leq \|x^\alpha u'\|_{0,\hat{I}}$$

for any  $0 \leq \alpha \leq 1$ . □

If  $v \in H^2(\hat{I})$ , then  $(v - v_{\hat{I}})' \in H^1(\hat{I})$  and we have  $\int_{\hat{I}}(v - v_{\hat{I}})' = 0$ . Then we obtain the following

**Corollary 1.2.3.** *Let  $0 \leq \alpha \leq 1$ . If  $v \in H^2(\hat{I})$  then*

$$\|(v - v_{\hat{I}})'\|_{0,\hat{I}} \leq C\|x^\alpha v''\|_{0,\hat{I}}. \quad (1.2.10)$$

*The constant  $C$  is independent of  $\alpha$  and  $v$ .*

*Remark 1.2.2.* The classical proofs of estimates like (1.2.10) use compactness arguments. For example, we can prove the inequality

$$\|(v - v_{\hat{I}})'\|_{0,\hat{I}} \leq C\|v''\|_{0,\hat{I}}$$

as follows [17, 35]: The linear operator  $\hat{\Pi} : H^2(\hat{I}) \rightarrow H^1(\hat{I})$ ,  $\hat{\Pi}v = v_{\hat{I}}$ , is continuous (since  $H^1(\hat{I}) \subset C^0(\hat{I})$ ), and it verifies  $\hat{\Pi}(p) = p$  for all  $p \in \mathcal{P}_1(\hat{I})$ . Then we have

$$\|(v - \hat{\Pi}v)'\|_{0,\hat{I}} \leq C\|v + p\|_{2,\hat{I}} \quad \forall p \in \mathcal{P}_1(\hat{I}).$$

Now, using that  $H^1(\hat{I})$  is compactly embedded in  $L^2(\hat{I})$  (and so  $H^2(\hat{I}) \subset H^1(\hat{I})$  compactly) it follows that

$$\inf_{p \in \mathcal{P}_1(\hat{I})} \|v + p\|_{H^2(\hat{I})} \leq C\|v''\|_{0,\hat{I}} \quad \forall v \in H^2(\hat{I})$$

and then we obtain the desired inequality. This argument can not be used to prove (1.2.10), at least for  $\alpha = 1$ , because the inclusion

$$H^{1,d}(\hat{I}) \subset L^2(\hat{I})$$

is not compact. Here  $H^{1,d}(\hat{I})$  is defined by

$$H^{1,d}(\hat{I}) = \left\{ v \in L^2(\hat{I}) : \|xv'\|_{0,\hat{I}} < \infty \right\}, \quad \|v\|_{H^{1,d}(\hat{I})}^2 = \|v\|_{0,\hat{I}}^2 + \|xv'\|_{0,\hat{I}}^2.$$

Indeed, for  $n \in \mathbb{N}$  consider the functions

$$u_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$$

and let  $w_n = \sqrt{n}u_n$ . Then  $\|w_n\|_{H^{1,d}(\hat{I})}^2 = \frac{10}{3}$ . If  $H^{1,d}(\hat{I}) \subset L^2(\hat{I})$  is compact, then there exists a subsequence  $w_{n_k}$ , and  $w \in L^2(\hat{I})$  such that  $w_{n_k} \rightarrow w$  in  $L^2(\hat{I})$  if  $k \rightarrow \infty$ . Since  $w_n(x) \rightarrow 0$  for all  $x \in \hat{I}$  we have  $w \equiv 0$ . But  $\|w_n\|_{0,\hat{I}}^2 = \frac{2}{3}$ , and then we arrive at a contradiction. In the following Section, we will use the estimates (1.2.10) for  $\alpha < 1$ , but it will be important to have a constant in the right hand side independent of  $\alpha$ .

Now, we obtain an estimate of the  $|\cdot|_1$  seminorm of the interpolation error as a simple consequence of the previous Corollary.

**Proposition 1.2.4.** *Let  $0 \leq \alpha \leq 1$  and  $I = [a, b]$ . If  $u \in H^2(I)$  then*

$$\|(u - u_I)'\|_{0,I} \leq C|I|^{1-\alpha} \|(x - a)^\alpha u''\|_{0,I}.$$

*Proof.* As in the proof of Proposition 1.2.1 let  $\Phi : \hat{I} \rightarrow I$  be the map defined by  $\Phi(\hat{x}) = a + \hat{x}(b - a)$ , and, for a function  $v$ , consider  $\hat{v} = v \circ \Phi$ . Then  $(u_I)^\wedge = \hat{u}_I$  and the following identities hold

$$\int_0^1 ((\hat{u} - (u_I)^\wedge)'(\hat{x}))^2 d\hat{x} = |I| \int_a^b ((u - u_I)'(x))^2 dx \quad (1.2.11)$$

$$\int_0^1 \hat{x}^{2\alpha} |\hat{u}''(\hat{x})|^2 d\hat{x} = |I|^{3-2\alpha} \int_a^b (x - a)^{2\alpha} |u''(x)|^2 dx. \quad (1.2.12)$$

Now, the Proposition follows from Corollary 1.2.3 and equations (1.2.11) and (1.2.12).  $\square$

### 1.3 Estimates for the global interpolation error

Let  $x_0 = 0 < x_1 < \dots < x_N = 1$  be a mesh on  $\hat{I}$ , and define  $I_i = [x_{i-1}, x_i], i = 1, \dots, N$ . Then for each function  $u \in H^2(\hat{I})$  we can define the piecewise linear function  $\Pi u$  such that  $\Pi u(x_i) = u(x_i), i = 0, \dots, N$ .

Now, we want to obtain an uniform (or almost uniform) estimate, in the parameter  $\varepsilon$ , of the interpolation error  $\|u - \Pi u\|_\varepsilon$ , being  $u$  the solution of Problem (1.1.1).

We will do it for a particular class of meshes, which will be designed taking into account the local interpolation error estimates established in the previous Section. We begin with the  $L^2$  part of the error. Let  $\alpha < \frac{1}{2}$  and  $\beta$  be nonnegative real numbers. Then by the

Proposition 1.2.1 we have

$$\begin{aligned} \|u - \Pi u\|_{0,\hat{I}}^2 &= \sum_{i=1}^N \|u - \Pi u\|_{0,I_i}^2 \\ &\leq \varepsilon^{-2\beta} \frac{C}{(1-2\alpha)^2} |I_1|^{2-2\alpha} \varepsilon^{2\beta} \|x^\alpha u'\|_{0,I_1}^2 + C \sum_{i=2}^N |I_i|^2 \|u'\|_{0,I_i}^2. \end{aligned} \quad (1.3.1)$$

Given a positive parameter  $h$  we impose the following restrictions on the selection of the points  $x_i$  which define the mesh:

$$x_1 \leq \sigma h^{\frac{1}{1-\alpha}} \quad (1.3.2)$$

$$x_{i+1} \leq x_i + \sigma h x_i^\alpha, \quad i = 1, \dots, N-1 \quad (1.3.3)$$

with  $\sigma > 0$  fixed. Under these conditions we obtain from inequality (1.3.1)

$$\begin{aligned} \|u - \Pi u\|_{0,\hat{I}}^2 &\leq \varepsilon^{-2\beta} \frac{C}{(1-2\alpha)^2} x_1^{2-2\alpha} \varepsilon^{2\beta} \|x^\alpha u'\|_{0,I_1}^2 + Ch^2 \sum_{i=2}^N \|x^\alpha u'\|_{0,I_i}^2 \\ &\leq \varepsilon^{-2\beta} \frac{C}{(1-2\alpha)^2} h^2 \varepsilon^{2\beta} \|x^\alpha u'\|_{0,I_1}^2 + C\varepsilon^{-2\beta} h^2 \sum_{i=2}^N \varepsilon^{2\beta} \|x^\alpha u'\|_{0,I_i}^2 \\ &\leq C\varepsilon^{-2\beta} \max \left\{ 1, \frac{1}{(1-2\alpha)^2} \right\} h^2 \varepsilon^{2\beta} \|x^\alpha u'\|_{0,\hat{I}}^2. \end{aligned} \quad (1.3.4)$$

Now, we take

$$\beta = \frac{1}{2} - \alpha = \frac{1}{\log \frac{1}{\varepsilon}},$$

so

$$\varepsilon^{-\beta} = e, \quad \text{and} \quad \frac{1}{1-2\alpha} = \frac{1}{2} \log \frac{1}{\varepsilon}.$$

From (1.3.4) it follows that

$$\|u - \Pi u\|_{0,\hat{I}}^2 \leq C \max \left\{ 1, \frac{1}{2} \log^2 \frac{1}{\varepsilon} \right\} h^2 \varepsilon^{2\beta} \|x^\alpha u'\|_{0,\hat{I}}^2.$$

Since  $u$  is the solution of Problem (1.1.1), the inequality (1.1.5) holds if  $\varepsilon$  is small enough because, in this case,  $\alpha > 0$  and  $\alpha + \beta = \frac{1}{2}$ ; so we have proved that

$$\|u - \Pi u\|_{0,\hat{I}} \leq Ch \log \frac{1}{\varepsilon}. \quad (1.3.5)$$



Now, we deal with the remain part of the error,  $\varepsilon^{\frac{1}{2}}\|(u - \Pi u)'\|_{0,\hat{I}}$ . The procedure is similar. By Proposition 1.2.4 we have for positive numbers  $\gamma$  and  $\delta \leq 1$

$$\begin{aligned} \varepsilon\|(u - \Pi u)'\|_{0,\hat{I}}^2 &= \varepsilon \sum_{i=1}^N \|(u - \Pi u)'\|_{0,I_i}^2 \\ &\leq \varepsilon^{-2\gamma} C |I_1|^{2-2\delta} \varepsilon^{1+2\gamma} \|x^\delta u''\|_{0,I_1}^2 + \\ &\quad + C \varepsilon^{-2\gamma} \sum_{i=2}^N |I_i|^2 \varepsilon^{1+2\gamma} \|u''\|_{0,I_i}^2 \end{aligned} \quad (1.3.6)$$

Let

$$\gamma = 1 - \delta = \frac{1}{\log \frac{1}{\varepsilon}}$$

and so,

$$\varepsilon^{-\gamma} = e \quad \text{and} \quad \frac{1}{1 - \delta} = \log \frac{1}{\varepsilon}.$$

Now, we add other restrictions to the mesh. We suppose that the points  $\{x_i\}$  satisfy (1.3.2), (1.3.3) and

$$x_1 \leq \sigma h^{\frac{1}{1-\delta}} \quad (1.3.7)$$

$$x_{i+1} \leq x_i + \sigma h x_i^\delta, \quad i = 1, \dots, N-1. \quad (1.3.8)$$

Then, it follow from inequality (1.3.6) that

$$\begin{aligned} \varepsilon\|(u - \Pi u)'\|_{0,\hat{I}}^2 &\leq Ch^2 \varepsilon^{1+2\gamma} \|x^\delta u''\|_{0,I_1}^2 + Ch^2 \sum_{i=2}^N \varepsilon^{1+2\gamma} \|x^\delta u''\|_{0,I_i}^2 \\ &\leq Ch^2 \varepsilon^{1+2\gamma} \|x^\delta u''\|_{0,\hat{I}}^2. \end{aligned} \quad (1.3.9)$$

Again, for  $\varepsilon$  small enough  $\gamma > 0$  and  $\frac{1}{2} + \gamma + \delta = \frac{3}{2}$ , and then equation (1.1.6) holds. Then we arrive at

$$\varepsilon^{\frac{1}{2}}\|(u - \Pi u)'\|_{0,\hat{I}} \leq Ch. \quad (1.3.10)$$

Note that for small  $\varepsilon$

$$\alpha = \frac{1}{2} - \frac{1}{\log \frac{1}{\varepsilon}} \sim \frac{1}{2}, \quad \frac{1}{1 - \alpha} \sim 2,$$

$$\delta = 1 - \frac{1}{\log \frac{1}{\varepsilon}} \sim 1, \quad \frac{1}{1 - \delta} = \log \frac{1}{\varepsilon},$$

and so, the restrictions (1.3.7) and (1.3.8) imply (1.3.2) and (1.3.3).

Then we have proved the following Theorem.

**Theorem 1.3.1.** *Let  $h > 0$  be fixed. If  $\delta = 1 - \frac{1}{\log(\frac{1}{\varepsilon})}$  and the finite sequency  $\{x_i\}_{i=0}^N$  satisfies the conditions (1.3.7) and (1.3.8), then for  $\varepsilon$  small enough we have the following estimates for the interpolation error*

$$\|u - \Pi u\|_{0,\hat{I}} \leq Ch \log \frac{1}{\varepsilon} \quad \text{and} \quad \varepsilon^{\frac{1}{2}} \|(u - \Pi u)'\|_{0,\hat{I}} \leq Ch.$$

## 1.4 Concluding remarks

Given a parameter  $h > 0$ , if  $\{x_i\}_{i=0}^N$  is the graded grid considered in Theorem 1.3.1, we have the estimate

$$\|u - \Pi u\|_{\varepsilon} \leq Ch \log \frac{1}{\varepsilon}. \quad (1.4.1)$$

Such a grid can be constructed with a number  $N$  of nodes, in such a way that

$$h \leq C \frac{1}{N} \log N \log \frac{1}{\varepsilon}$$

with the constant  $C$  independent of  $h$  and  $\varepsilon$  (see the proof of Corollary 3.2.5 and the comments after the Theorem 3.3.4), and then we see that the estimate (1.4.1) is almost optimal with respect to the number of nodes.

We introduce the finite element space

$$V_h = \{v \in V : v|_{[x_{i-1}, x_i]} \in \mathcal{P}_1([x_{i-1}, x_i]), i = 1, \dots, N\}$$

and let  $u_h$  be the finite element approximation of problem (1.1.1), that is,  $u_h \in V_h$  such that

$$a_{cd}(u_h, v) = \int_0^1 f v \, dx \quad \forall v \in V_h.$$

This problem has a unique solution  $u_h$  in view of the coerciveness and continuity of the bilinear form  $a_{cd}$ . Unfortunately we are not able to obtain an error estimate as (1.4.1) with  $u_h$  instead of  $\Pi u$ . Indeed, using the Cea's Lemma, we have

$$\|u - u_h\|_{\varepsilon} \leq C \varepsilon^{-\frac{1}{2}} \|u - \Pi u\|_{\varepsilon} \leq C \varepsilon^{-\frac{1}{2}} h \log \frac{1}{\varepsilon}$$

if we use the graded grid. The factor  $\varepsilon^{-\frac{1}{2}}$  that appear in the right hand side is a consequence of the non-uniform continuity (with respect to the parameter  $\varepsilon$ ) of the bilinear form  $a_{cd}$ . By some numerical experiments (see Chapter 3) we can conjecture that the factor  $\varepsilon^{-\frac{1}{2}}$  could be removed from the estimate.

In Chapter 3 we will consider the 2-dimensional form of the following reaction-diffusion problem

$$\begin{aligned} -\varepsilon^2 u'' + c(x)u &= f & \text{in } (0, 1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{1.4.2}$$

The solution of this problem verifies the *a priori* estimates

$$|u^{(i)}(x)| \leq C \left( 1 + \frac{1}{\varepsilon^i} \exp\left(b_0 \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon^i} \exp\left(b_0 \frac{1-x}{\varepsilon}\right) \right) \tag{1.4.3}$$

for  $i = 1, 2, \dots$ . The bilinear form associated with this problem is

$$a_{rd}(v, w) = \int_0^1 (\varepsilon^2 v' w' + cvw) dx,$$

and the energy norm is now defined by  $\|v\|_\varepsilon^2 = \varepsilon^2 \|v'\|_{0,\hat{I}}^2 + \|v\|_{0,\hat{I}}^2$ . So, for the energy norm, the form  $a_{rd}$  is  $V$ -coercive and continuous uniformly in the parameter  $\varepsilon$ . By the same methods used in the previous sections, we can obtain the interpolation error estimate (1.4.1), if a graded grid, like those introduced in the previous Section, is used, and then, by Cea's Lemma it follows the following almost optimal inequality

$$\|u - u_h\|_\varepsilon \leq Ch \log \frac{1}{\varepsilon}$$

for the finite element solution  $u_h$ . But, in Chapter 3, we will consider another interpolation operator, which will allow us to obtain the above estimate on a graded grid performed independently of  $\varepsilon$ .

## Chapter 2

# Error Estimates for an Average Interpolation

### 2.1 Introduction

In the finite element approximation of functions which have singularities or boundary layers it is necessary to use highly non uniform meshes such that the mesh size is much smaller near the singularities than far from them. In the case of boundary layers these meshes contain very narrow or anisotropic elements.

The goal of this Chapter is to obtain new error estimates for  $\mathcal{Q}_1$  (piecewise bilinear in 2d or trilinear in 3d) approximations on meshes containing anisotropic rectangular elements, i. e., rectangles with sides of different orders. The classic error analysis excludes this kind of elements because it is based on the so called regularity assumption (see for example [12, 17]). However, it is now well known that this assumption is not needed. Indeed, many papers have been written to prove error estimates under more general conditions. In particular, for rectangular elements we refer to [1, 20, 38] and their references.

We will prove the error estimates for a mean average interpolation. There are two reasons to work with this kind of approximation instead of the Lagrange interpolation. The first one is to approximate non smooth functions for which the Lagrange interpolation is not even defined. On the other hand, it has already been observed that, in the three dimensional case, average interpolations have better approximation properties than the Lagrange interpolation even for smooth functions when narrow elements are used (see [1, 20]).

Our estimates extend previous known results in several aspects:

First, our assumptions include more general meshes than those allowed in the previous papers. Indeed, in [20] it was required that the meshes were quasiuniform in each direction. This requirement was relaxed in [1] but not enough to include the meshes that arise naturally in the approximation of boundary layers, which will be included under our assumptions. To prove our error estimates we require only that neighboring elements are of comparable size and so, our results are valid for a rather general family of anisotropic meshes.

Second, we generalize the error estimates allowing weaker norms on the right hand side. These norms are weighted Sobolev norms where the weights are related with the distance to the boundary. The interest of working with these norms arise in the approximation of boundary layers. Indeed, for many singular perturbed problems it is possible to prove that the solution has first and second derivatives which are bounded, uniformly in the perturbation parameter, in appropriate weighted Sobolev norms.

Finally, we consider the approximation of functions vanishing on the boundary by finite element functions with the same property. This is a non trivial point that was not considered in the above mentioned references.

Our mean average interpolation is similar to that introduced in [20] but the difference is that we define it directly on the given mesh instead of using reference elements. This is important in order to relax the regularity assumptions on the elements.

We will prove our estimates for the domain  $\Omega = [0, 1]^d$ ,  $d = 2, 3$ . It will be clear that the interior estimates derived in Section 2.4 are valid for any domain which can be decomposed in  $d$ -rectangles. However, the extension of our results of Section 2.5 for interpolations satisfying Dirichlet boundary conditions to other domains is not straightforward and would require a further analysis.

To prove the weighted estimates we will use a result of Boas and Straube [11] which, as we show, can be derived from the classic Hardy inequality in higher dimensions.

In Sections 2.2 and 2.3 we construct the mean average interpolation and prove lemmas that will be useful later on. In Section 2.4, error estimates for interior elements are proved. Section 2.5 deals with the approximation on boundary elements. Since the proofs of this section are rather technical we give them in the two dimensional case. However, it is not difficult (although very tedious!) to see that our arguments apply also in three dimensions.

## 2.2 An average interpolation. Definitions

In this section we define a piecewise  $\mathcal{Q}_1$  mean average interpolation. The approximation introduced here is a variant of that considered in [20]. The difference is that we define it directly in the given mesh instead of using a reference one. Working in this way we are able to remove the restrictions used in [1, 20]. In particular, our results apply for the anisotropic meshes arising in the approximation of boundary layers.

Let  $\mathcal{T}$  be a partition into rectangular elements of  $\Omega = (0, 1)^d$ ,  $d = 2, 3$ . We call  $\mathcal{N}$  the set of nodes of  $\mathcal{T}$  and  $\mathcal{N}_{in}$  the set of interior nodes.

Given an element  $R \in \mathcal{T}$ , let  $h_{R,i}$  be the length of the side of  $R$  in the direction  $x_i$ .

We assume that there exists a constant  $\sigma$  such that, for  $R, S \in \mathcal{T}$  neighboring elements,

$$\frac{h_{R,i}}{h_{S,i}} \leq \sigma \quad 1 \leq i \leq d. \quad (2.2.1)$$

For each  $\mathbf{v} \in \mathcal{N}$  we define

$$h_{\mathbf{v},i} = \min\{h_{R,i} : \mathbf{v} \text{ is a vertex of } R\}, \quad 1 \leq i \leq d.$$

and  $h_{\mathbf{v}} = (h_{\mathbf{v},1}, h_{\mathbf{v},2})$  if  $d = 2$  or  $h_{\mathbf{v}} = (h_{\mathbf{v},1}, h_{\mathbf{v},2}, h_{\mathbf{v},3})$  if  $d = 3$ . If  $p, q \in \mathbb{R}^d$  we denote by  $p : q$  the vector  $(p_1q_1, p_2q_2)$  if  $d = 2$  or  $(p_1q_1, p_2q_2, p_3q_3)$  if  $d = 3$ . Take  $\psi \in C^\infty(\mathbb{R}^d)$  with support in a ball centered at the origin and radius  $r \leq 1/\sigma$  and such that  $\int \psi = 1$ , and for  $\mathbf{v} \in \mathcal{N}_{in}$  let

$$\psi_{\mathbf{v}}(x) = \frac{1}{h_{\mathbf{v},1}h_{\mathbf{v},2}} \psi \left( \frac{\mathbf{v}_1 - x_1}{h_{\mathbf{v},1}}, \frac{\mathbf{v}_2 - x_2}{h_{\mathbf{v},2}} \right)$$

if  $d = 2$  or

$$\psi_{\mathbf{v}}(x) = \frac{1}{h_{\mathbf{v},1}h_{\mathbf{v},2}h_{\mathbf{v},3}} \psi \left( \frac{\mathbf{v}_1 - x_1}{h_{\mathbf{v},1}}, \frac{\mathbf{v}_2 - x_2}{h_{\mathbf{v},2}}, \frac{\mathbf{v}_3 - x_3}{h_{\mathbf{v},3}} \right)$$

if  $d = 3$ . Given a function  $u$  we call  $P(x, y)$  its Taylor polynomial of degree 1 at the point  $x$ , namely,

$$P(x, y) = u(x) + \nabla u(x) \cdot (y - x).$$

Then, for  $\mathbf{v} \in \mathcal{N}_{in}$  we introduce the regularized average

$$u_{\mathbf{v}}(y) = \int P(x, y) \psi_{\mathbf{v}}(x) dx. \quad (2.2.2)$$

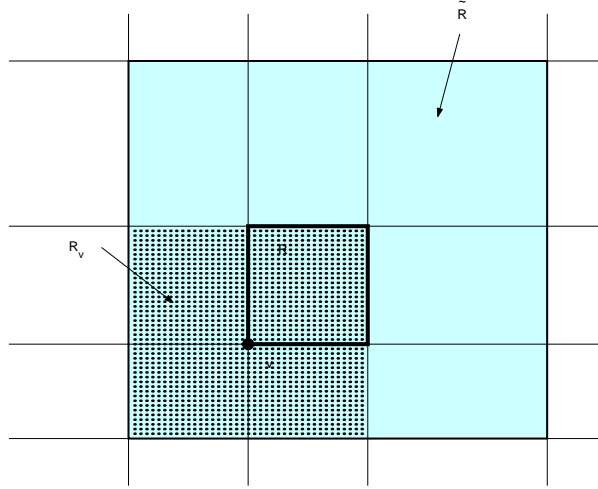


Figure 2.1: Notation.

Now, given  $u \in H_0^1(\Omega)$  we define  $\Pi u$  as the unique piecewise (with respect to  $\mathcal{T}$ )  $Q_1$  function such that, for  $\mathbf{v} \in \mathcal{N}_{in}$ ,  $\Pi u(\mathbf{v}) = u_{\mathbf{v}}(\mathbf{v})$  while  $\Pi u(\mathbf{v}) = 0$  for boundary nodes  $\mathbf{v}$ .

Introducing the standard basis functions  $\lambda_{\mathbf{v}}$  associated with the nodes  $\mathbf{v}$  we can write

$$\Pi u(x) = \sum_{\mathbf{v} \in \mathcal{N}_{in}} u_{\mathbf{v}}(\mathbf{v}) \lambda_{\mathbf{v}}(x).$$

For  $R \in \mathcal{T}$  and  $\mathbf{v} \in \mathcal{N}$  we define (see Figure 2.1 for the 2d case)

$$\tilde{R} = \bigcup \{S \in \mathcal{T} : S \text{ is a neighboring element of } R\}$$

and

$$R_{\mathbf{v}} = \bigcup \{S \in \mathcal{T} : \mathbf{v} \text{ is a vertex of } S\}.$$

In our analysis we will also make use of the regularized average of  $u$ , namely,

$$Q_{\mathbf{v}}(u) = \int u(x) \psi_{\mathbf{v}}(x) dx$$

for  $\mathbf{v} \in \mathcal{N}_{in}$ .

We remark that, since  $r \leq 1/\sigma$ , it follows from our assumption (2.2.1) that the support of  $\psi_{\mathbf{v}}(x)$  is contained in  $R_{\mathbf{v}}$ .

## 2.3 Preliminary lemmas

Now we prove some weighted estimates which will be useful for our error analysis. For any set  $D$  we call  $d_D(x)$  the distance of  $x$  to the boundary of  $D$ . For a  $d$ -rectangle  $R = \Pi_{i=1}^d(a_i, b_i)$  we have  $d_R(x) = \min\{x_i - a_i, b_i - x_i : 1 \leq i \leq d\}$ . For such  $R$  we will also consider the following function

$$\delta_R(x) := \min \left\{ \frac{x_i - a_i}{h_{R,i}}, \frac{b_i - x_i}{h_{R,i}} : 1 \leq i \leq d \right\}.$$

In what follows we will make use of the Hardy Inequality (1.2.8). We will also need the following generalization to higher dimensions: If  $D$  is a convex domain and  $u \in H_0^1(D)$  then

$$\left\| \frac{u}{d_D} \right\|_{L^2(D)} \leq 2 \|\nabla u\|_{L^2(D)} \quad (2.3.1)$$

(see for example [31]).

The following lemma gives an ‘‘anisotropic’’ version of (2.3.1). It can be proved by standard scaling arguments.

**Lemma 2.3.1.** *Let  $R = \Pi_{i=1}^d(a_i, b_i)$  be a  $d$ -rectangle and  $h_i = b_i - a_i, 1 \leq i \leq d$ . For all  $u \in H_0^1(R)$*

$$\left\| \frac{u}{\delta_R} \right\|_{L^2(R)} \leq 2 \sum_{i=1}^d h_i \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(R)}. \quad (2.3.2)$$

Another consequence of (2.3.1) is the inequality that we prove in the following lemma. This inequality was proved for Lipschitz domains by Boas and Straube in [11]. We give a different proof here because we are interested in the dependence of the constant on the domain, which is not stated in [11] because the proof given there is based on compactness arguments.

**Lemma 2.3.2.** *Let  $R$  be a  $d$ -rectangle with sides of lengths  $h_i, 1 \leq i \leq d$ , such that  $\frac{1}{\delta} \leq h_i \leq \delta$ , and let  $\psi \in C_0(R)$  be a function such that  $\int_R \psi = 1$ . Then, there exists a constant  $C$  depending only on  $\delta$  and  $\psi$ , such that, for all  $u \in H^1(R)$  with  $\int_R u \psi = 0$ ,*

$$\|u\|_{L^2(R)} \leq C \|d_R \nabla u\|_{L^2(R)}. \quad (2.3.3)$$



*Proof.* Since  $v := u - (\int_R u)\psi$  has vanishing mean value, there exists  $F \in H_0^1(R)^d$  such that

$$-\operatorname{div} F = v \quad (2.3.4)$$

and such that

$$\|F\|_{H_0^1(R)^2} \leq C\|v\|_{L^2(R)}. \quad (2.3.5)$$

Moreover, from the explicit bound for the constant given in [21] it follows that  $C$  can be taken depending only on  $\delta$ .

Now, since  $\int_R u\psi = 0$ , we have from (2.3.4)

$$\|u\|_{L^2(R)}^2 = \int_R uv = - \int_R u \operatorname{div} F$$

and therefore, integrating by parts and using (2.3.1) for each component of  $F$ , we obtain

$$\|u\|_{L^2(R)}^2 = \int_R \nabla u \cdot F \leq \|d_R \nabla u\|_{L^2(R)} \left\| \frac{F}{d_R} \right\|_{L^2(R)} \leq 2 \|d_R \nabla u\|_{L^2(R)} \|\nabla F\|_{L^2(R)}$$

but,

$$\|v\|_{L^2(R)}^2 \leq (1 + |R| \|\psi\|_{L^2(R)}^2) \|u\|_{L^2(R)}^2$$

and so, the proof concludes by using (2.3.5) and the fact that the constant in that estimate depends only on  $\delta$ .  $\square$

As a consequence of the previous lemma we obtain the following weighted estimates.

**Lemma 2.3.3.** *For  $\mathbf{v} \in \mathcal{N}_{in}$  there exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that, for all  $u \in H^1(R_{\mathbf{v}})$ ,*

$$\|u - Q_{\mathbf{v}}(u)\|_{L^2(R_{\mathbf{v}})} \leq C \sum_{i=1}^d h_{\mathbf{v},i} \left\| \delta_{R_{\mathbf{v}}} \frac{\partial u}{\partial x_i} \right\|_{L^2(R_{\mathbf{v}})} \quad (2.3.6)$$

and, for all  $u \in H^2(R_{\mathbf{v}})$ ,

$$\left\| \frac{\partial(u - u_{\mathbf{v}})}{\partial x_j} \right\|_{L^2(R_{\mathbf{v}})} \leq C \sum_{i=1}^d h_{\mathbf{v},i} \left\| \delta_{R_{\mathbf{v}}} \frac{\partial^2 u}{\partial x_j \partial x_i} \right\|_{L^2(R_{\mathbf{v}})}. \quad (2.3.7)$$

*Proof.* Let  $K_{\mathbf{v}}$  be the image of  $R_{\mathbf{v}}$  by the map  $x \rightarrow \bar{x}$  with

$$\bar{x}_i = \frac{\mathbf{v}_i - x_i}{h_{\mathbf{v},i}} \quad 1 \leq i \leq d$$

and, for  $\bar{x} \in K_{\mathbf{v}}$ , define  $\bar{u}$  by  $\bar{u}(\bar{x}) = u(x)$ . Then,  $Q_{\mathbf{v}}(u) = \bar{Q}(\bar{u})$  where

$$\bar{Q}(\bar{u}) = \int \bar{u}(\bar{x})\psi(\bar{x})d\bar{x}.$$

Now, in view of our assumption (2.2.1), the  $d$ -rectangle  $K_{\mathbf{v}}$  satisfies the hypothesis of Lemma 2.3.2 with  $\delta = 2\sigma$ . Moreover, since  $r \leq \frac{1}{\sigma}$ , the support of  $\psi$  is contained in  $K_{\mathbf{v}}$ . Therefore, since  $\int (\bar{u} - \bar{Q}(\bar{u}))\psi = 0$ , it follows from Lemma 2.3.2 that there exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that

$$\|\bar{u} - \bar{Q}(\bar{u})\|_{L^2(K_{\mathbf{v}})} \leq C \|d_{K_{\mathbf{v}}}\nabla\bar{u}\|_{L^2(K_{\mathbf{v}})}$$

and (2.3.6) follows by going back to the variable  $x$ .

To prove (2.3.7), observe that  $u_{\mathbf{v}}(y) = \bar{u}_0(\bar{y})$  where

$$\bar{u}_0(\bar{y}) = \int (\bar{u}(\bar{x}) + \nabla(\bar{u})(\bar{x}) \cdot (\bar{y} - \bar{x}))\psi(\bar{x})d\bar{x}$$

and so, since

$$\int \frac{\partial(\bar{u} - \bar{u}_0)}{\partial\bar{x}_i}\psi = 0,$$

we obtain from Lemma 2.3.2 that there exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that

$$\left\| \frac{\partial(\bar{u} - \bar{u}_0)}{\partial\bar{x}_i} \right\|_{L^2(K_{\mathbf{v}})} \leq C \left\| d_{K_{\mathbf{v}}}\nabla \frac{\partial\bar{u}}{\partial\bar{x}_i} \right\|_{L^2(K_{\mathbf{v}})}$$

and the proof concludes going back to the variable  $x$ . □

## 2.4 Error estimates for interior elements

In the rest of this Chapter we prove error estimates for the average interpolation introduced in the previous sections for functions in some weighted Sobolev spaces. The weights

considered are powers of the distance to the boundary. As we show in Chapter 1 for one-dimensional problems, this kind of weights arise naturally in problems with boundary layers. In this Section we estimate the approximation error for interior elements.

We start with the  $L^2$  norm. From now on  $C$  will be a generic constant which depends only on  $\sigma$  and  $\psi$ . In view of our hypothesis (2.2.1),  $h_{\mathbf{v},i}$  and  $h_{R,i}$  are equivalent up to a constant depending on  $\sigma$  whenever  $\mathbf{v}$  is a vertex of  $R$ . We will use this fact repeatedly without making it explicitly.

**Theorem 2.4.1.** *There exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that*

(i) *For all  $R \in \mathcal{T}$  and  $u \in H^1(\tilde{R})$  we have*

$$\|\Pi u\|_{L^2(R)} \leq C \|u\|_{L^2(\tilde{R})}. \quad (2.4.1)$$

(ii) *for all  $R \in \mathcal{T}$  such that  $R$  is not a boundary element and  $u \in H^1(\tilde{R})$  we have*

$$\|u - \Pi u\|_{L^2(R)} \leq C \sum_{i=1}^d h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial u}{\partial x_i} \right\|_{L^2(\tilde{R})}. \quad (2.4.2)$$

*Proof.* To prove (i) we write

$$(\Pi u)|_R = \sum_{j=1}^{n_R} u_{\mathbf{v}_j}(\mathbf{v}_j) \lambda_{\mathbf{v}_j}$$

where  $\{\mathbf{v}_j\}_1^{n_R}$  are the interior nodes of  $R$ . Then,

$$\|\Pi u\|_{L^2(R)} \leq C \left( \prod_{i=1}^d h_{R,i} \right)^{\frac{1}{2}} \sum_{j=1}^{n_R} \|u_{\mathbf{v}_j}\|_{L^\infty(R)} \quad (2.4.3)$$

and we have to estimate  $\|u_{\mathbf{v}_j}\|_{L^\infty(R)}$  for each  $j$ . To simplify notation we write  $\mathbf{v} = \mathbf{v}_j$  (and so the subindexes denote now the components of  $\mathbf{v}$ ). We have

$$\left| \int u(x) \psi_{\mathbf{v}}(x) dx \right| \leq C \left( \prod_{i=1}^d h_{R,i} \right)^{-\frac{1}{2}} \|u\|_{L^2(\tilde{R})}. \quad (2.4.4)$$

On the other hand, since  $\psi_{\mathbf{v}} = 0$  on  $\partial \tilde{R}$ , integration by parts gives

$$\begin{aligned}
\left| \int \frac{\partial u}{\partial x_i}(x)(y_i - x_i)\psi_{\mathbf{v}}(x)dx \right| &= \left| \int u(x)\psi_{\mathbf{v}}(x)dx - \int u(x)(y_i - x_i)\frac{\partial \psi_{\mathbf{v}}}{\partial x_i}(x)dx \right| \\
&\leq C \left( \prod_{i=1}^d h_{R,i} \right)^{-\frac{1}{2}} \|u\|_{L^2(\tilde{R})}
\end{aligned} \tag{2.4.5}$$

where we have used that  $|y_i - x_i| \leq Ch_{\mathbf{v},i}$ . Thus, (2.4.1) follows from (2.4.3), (2.4.4), (2.4.5) and the definition of  $u_{\mathbf{v}}$  given in (2.2.2).

To prove (ii), choose a node of  $R$ , say  $\mathbf{v}_1$ . Since  $Q_{\mathbf{v}_1}(u)$  is a constant function and  $R$  is not a boundary element, we have  $\Pi Q_{\mathbf{v}_1}(u) = Q_{\mathbf{v}_1}(u)$  on  $R$  and so

$$\begin{aligned}
\|u - \Pi u\|_{L^2(R)} &\leq \|u - Q_{\mathbf{v}_1}(u)\|_{L^2(R)} + \|\Pi(Q_{\mathbf{v}_1}(u) - u)\|_{L^2(R)} \\
&\leq C \|u - Q_{\mathbf{v}_1}(u)\|_{L^2(\tilde{R})}
\end{aligned} \tag{2.4.6}$$

where we have used (2.4.1). Now, estimate (2.4.2) follows from (2.4.6) and (2.3.6) (observe that inequality (2.3.6) remain valid if  $R_{\mathbf{v}}$  is replaced by  $\tilde{R}$ ).  $\square$

In what follows, we estimate the approximation error for the first derivatives for interior elements. We will use the

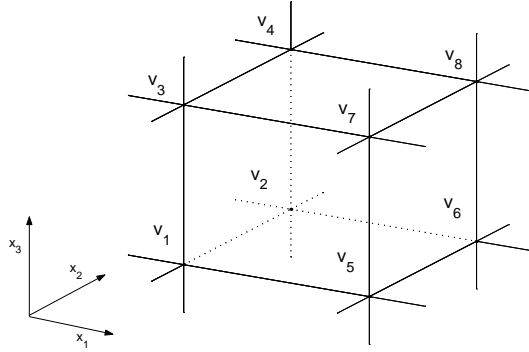


Figure 2.2: Notation for Theorem 2.4.2.

**Theorem 2.4.2.** *There exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that, if  $R \in \mathcal{T}$  is not a boundary element then, for all  $u \in H^2(\tilde{R})$  we have*

$$\left\| \frac{\partial}{\partial x_j}(u - \Pi u) \right\|_{L^2(R)} \leq C \sum_{i=1}^d h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\tilde{R})} \quad 1 \leq j \leq d. \tag{2.4.7}$$

*Proof.* We will consider the case  $d = 3, j = 1$ . Clearly, the other cases are analogous. We have

$$u - \Pi u = (u - u_{\mathbf{v}_1}) + (u_{\mathbf{v}_1} - \Pi u)$$

and from (2.3.7) we know that  $\left\| \frac{\partial(u - u_{\mathbf{v}_1})}{\partial x_1} \right\|_{L^2(R)}$  is bounded by the right hand side of (2.4.7). Therefore, we have to estimate  $\left\| \frac{\partial(u_{\mathbf{v}_1} - \Pi u)}{\partial x_1} \right\|_{L^2(R)}$ . Since  $w := u_{\mathbf{v}_1} - \Pi u \in \mathcal{Q}_1$  we have (see for example [38])

$$\frac{\partial w}{\partial x_1} = \sum_{i=1}^4 (w(\mathbf{v}_i) - w(\mathbf{v}_{i+4})) \frac{\partial \lambda_{\mathbf{v}_i}}{\partial x_1}$$

then,

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{L^2(R)} \leq \sum_{i=1}^4 |w(\mathbf{v}_i) - w(\mathbf{v}_{i+4})| \left\| \frac{\partial \lambda_{\mathbf{v}_i}}{\partial x_1} \right\|_{L^2(R)}. \quad (2.4.8)$$

But, it is easy to see that

$$\left\| \frac{\partial \lambda_{\mathbf{v}_i}}{\partial x_1} \right\|_{L^2(R)} \leq C \left( \frac{h_{\mathbf{v}_i,2} h_{\mathbf{v}_i,3}}{h_{\mathbf{v}_i,1}} \right)^{\frac{1}{2}}. \quad (2.4.9)$$

So, we have to estimate  $|w(\mathbf{v}_i) - w(\mathbf{v}_{i+4})|$  for  $1 \leq i \leq 4$ . We have

$$\begin{aligned} w(\mathbf{v}_1) - w(\mathbf{v}_5) &= u_{\mathbf{v}_5}(\mathbf{v}_5) - u_{\mathbf{v}_1}(\mathbf{v}_5) \\ &= \int P(x, \mathbf{v}_5) \psi_{\mathbf{v}_5}(x) dx - \int P(x, \mathbf{v}_5) \psi_{\mathbf{v}_1}(x) dx. \end{aligned} \quad (2.4.10)$$

So, changing variables we obtain

$$w(\mathbf{v}_1) - w(\mathbf{v}_5) = \int [P(\mathbf{v}_5 - h_{\mathbf{v}_5} : y, \mathbf{v}_5) - P(\mathbf{v}_1 - h_{\mathbf{v}_1} : y, \mathbf{v}_5)] \psi(y) dy. \quad (2.4.11)$$

We introduce the notation  $\mathbf{v}_i = (\mathbf{v}_i^1, \mathbf{v}_i^2, \mathbf{v}_i^3)$ . Define now

$$\theta = (\theta_1, 0, 0) := (\mathbf{v}_5^1 - \mathbf{v}_1^1 + (h_{\mathbf{v}_1,1} - h_{\mathbf{v}_5,1})y_1, 0, 0)$$

and

$$F_y(t) := P(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + t\theta, \mathbf{v}_5).$$

Then, since  $h_{\mathbf{v}_1,2} = h_{\mathbf{v}_5,2}$ ,  $h_{\mathbf{v}_1,3} = h_{\mathbf{v}_5,3}$  and  $\mathbf{v}_1^2 = \mathbf{v}_5^2$ ,  $\mathbf{v}_1^3 = \mathbf{v}_5^3$ , we have

$$P(\mathbf{v}_5 - h_{\mathbf{v}_5} : y, \mathbf{v}_5) - P(\mathbf{v}_1 - h_{\mathbf{v}_1} : y, \mathbf{v}_5) = F_y(1) - F_y(0)$$

and replacing in (2.4.11) we obtain

$$w(\mathbf{v}_1) - w(\mathbf{v}_5) = \int \int_0^1 F'_y(t) \psi(y) dt dy = \int_0^1 \left\{ \int F'_y(t) \psi(y) dy \right\} dt$$

and therefore it is enough to estimate

$$I(t) := \int F'_y(t) \psi(y) dy$$

for  $0 \leq t \leq 1$ . But, from the definition of  $F_y$  and  $P$ , we have

$$\begin{aligned} |I(t)| \leq \int \left\{ \left| \frac{\partial^2 u}{\partial x_1^2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + t\theta) \right| \times |\mathbf{v}_5^1 - \mathbf{v}_1^1 + h_{\mathbf{v}_1,1} y_1 - t\theta_1| \right. \\ \left. + \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + t\theta) \right| \times |\mathbf{v}_5^2 - \mathbf{v}_1^2 + h_{\mathbf{v}_1,2} y_2| \right. \\ \left. + \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + t\theta) \right| \times |\mathbf{v}_5^3 - \mathbf{v}_1^3 + h_{\mathbf{v}_1,3} y_3| \right\} |\theta_1| \psi(y) dy. \end{aligned}$$

Now, for  $|y| \leq 1$  and  $0 \leq t \leq 1$ , we have

$$|\theta| = |\theta_1| \leq Ch_{\mathbf{v}_1,1}, \quad |\mathbf{v}_5^i - \mathbf{v}_1^i + h_{\mathbf{v}_1,i} y_i - \theta_1 t| \leq Ch_{\mathbf{v}_1,i},$$

and therefore, since  $\text{supp}(\psi) \subset B(0, 1)$ , we have

$$\begin{aligned} |I(t)| \leq C \int \left\{ \left| \frac{\partial^2 u}{\partial x_1^2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t) \right| (h_{\mathbf{v}_1,1})^2 + \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t) \right| h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,2} \right. \\ \left. + \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t) \right| h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,3} \right\} \psi(y) dy. \end{aligned}$$

Now, making the change of variables  $z = \mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t$  and calling

$$\phi(z) = \psi \left( -\frac{z_1 - [(1-t)\mathbf{v}_1^1 + t\mathbf{v}_2^1]}{(1-t)h_{\mathbf{v}_1,1} + th_{\mathbf{v}_5,1}}, -\frac{z_2 - \mathbf{v}_1^2}{h_{\mathbf{v}_1,2}}, -\frac{z_3 - \mathbf{v}_1^3}{h_{\mathbf{v}_1,3}} \right)$$

we obtain

$$|I(t)| \leq C \frac{1}{h_{\mathbf{v}_1,2} h_{\mathbf{v}_1,3}} \sum_{i=1}^3 h_{\mathbf{v}_1,i} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_i}(z) \right| \phi(z) dz$$

where we have used that  $h_{\mathbf{v}_1,1} \geq C((1-t)h_{\mathbf{v}_1,1} + th_{\mathbf{v}_5,1})$ . But, since  $\text{supp } \psi \subset B(0, \frac{1}{\sigma})$ , it follows that  $\text{supp } \phi \subset \tilde{R}$ . Then, using the Schwarz inequality we obtain

$$|I(t)| \leq C \frac{1}{h_{\mathbf{v}_1,2} h_{\mathbf{v}_1,3}} \sum_{i=1}^3 h_{\mathbf{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\| \left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|$$

and from Lemma 2.3.1 we know that

$$\left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \leq C(h_{\mathbf{v}_1,1}h_{\mathbf{v}_1,2}h_{\mathbf{v}_1,3})^{\frac{1}{2}}.$$

Finally, using (2.4.9) we obtain

$$|w(\mathbf{v}_1) - w(\mathbf{v}_5)| \left\| \frac{\partial \lambda_{\mathbf{v}_1}}{\partial x_1} \right\|_{L^2(\tilde{R})} \leq C \sum_{i=1}^3 h_{\mathbf{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\|_{L^2(\tilde{R})}. \quad (2.4.12)$$

Now, to estimate  $|w(\mathbf{v}_2) - w(\mathbf{v}_6)|$  we write

$$\begin{aligned} w(\mathbf{v}_2) - w(\mathbf{v}_6) &= (u_{\mathbf{v}_1}(\mathbf{v}_2) - u_{\mathbf{v}_2}(\mathbf{v}_2)) - (u_{\mathbf{v}_1}(\mathbf{v}_6) - u_{\mathbf{v}_6}(\mathbf{v}_6)) \\ &= (u_{\mathbf{v}_1}(\mathbf{v}_2) - u_{\mathbf{v}_1}(\mathbf{v}_6)) - (u_{\mathbf{v}_2}(\mathbf{v}_2) - u_{\mathbf{v}_2}(\mathbf{v}_6)) - (u_{\mathbf{v}_2}(\mathbf{v}_6) - u_{\mathbf{v}_6}(\mathbf{v}_6)) \quad (2.4.13) \\ &=: I - II - III. \end{aligned}$$

Now we estimate  $I - II$ . We have

$$I = \int \frac{\partial u}{\partial x_1}(x)(\mathbf{v}_2^1 - \mathbf{v}_6^1)\psi_{\mathbf{v}_1}(x)dx \quad \text{and} \quad II = \int \frac{\partial u}{\partial x_1}(x)(\mathbf{v}_2^1 - \mathbf{v}_6^1)\psi_{\mathbf{v}_2}(x)dx$$

where we have used that  $\mathbf{v}_2 - \mathbf{v}_6 = (\mathbf{v}_2^1 - \mathbf{v}_6^1, 0, 0)$ . After a change of variables in both integrals we obtain

$$I - II = \int \left[ \frac{\partial u}{\partial x_1}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y) - \frac{\partial u}{\partial x_1}(\mathbf{v}_2 - h_{\mathbf{v}_2} : y) \right] (\mathbf{v}_2^1 - \mathbf{v}_6^1)\psi(y)dy$$

and so, defining  $\theta = (0, \theta_2, 0) := (0, \mathbf{v}_2^2 - \mathbf{v}_1^2 - (h_{\mathbf{v}_2,2} - h_{\mathbf{v}_1,2})y_2, 0)$  and

$$F_y(t) = \frac{\partial u}{\partial x_1}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t)$$

and taking into account that  $h_{\mathbf{v}_1,1} = h_{\mathbf{v}_2,1}$  and  $h_{\mathbf{v}_1,3} = h_{\mathbf{v}_2,3}$  we have

$$\begin{aligned} I - II &= - \int_0^1 \int_0^1 F'_y(t)(\mathbf{v}_2^1 - \mathbf{v}_6^1)\psi(y)dt dy \\ &= - \int_0^1 \left\{ \int_0^1 F'_y(t)(\mathbf{v}_2^1 - \mathbf{v}_6^1)\psi(y)dy \right\} dt \\ &=: \int_0^1 I(t)dt. \end{aligned}$$

Since

$$F'_y(t) = \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t)\theta_2$$

and for  $y \in \text{supp } \psi$ ,  $|y| \leq 1$ , we have

$$\begin{aligned} |I(t)| &\leq \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t) \right| |\theta_2| |\mathbf{v}_2^1 - \mathbf{v}_6^1| |\psi(y)| dy \\ &\leq Ch_{\mathbf{v}_2,1} h_{\mathbf{v}_2,2} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t) \right| |\psi(y)| dy. \end{aligned}$$

Change now to the variable  $z = \mathbf{v}_1 - h_{\mathbf{v}_1} : y + \theta t$  and define

$$\phi(z) = \psi \left( -\frac{z_1 - \mathbf{v}_1^1}{h_{\mathbf{v}_1,1}}, -\frac{z_2 - [(1-t)\mathbf{v}_1^2 + t\mathbf{v}_2^2]}{(1-t)h_{\mathbf{v}_1,2} + th_{\mathbf{v}_2,2}}, -\frac{z_3 - \mathbf{v}_1^3}{h_{\mathbf{v}_1,3}} \right).$$

Then, since  $\text{supp } \phi \subset \tilde{R}$  (because  $\text{supp } \psi \subset B(0, \frac{1}{\sigma})$ ), we can use Lemma 2.3.1 to obtain

$$\begin{aligned} |I(t)| &\leq C \frac{1}{h_{\mathbf{v}_1,3}} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(z) \right| \phi(z) dz \\ &\leq C \frac{1}{h_{\mathbf{v}_1,3}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \left\| \frac{\phi(z)}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \\ &\leq C \left( \frac{h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,2}}{h_{\mathbf{v}_1,3}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \end{aligned}$$

Therefore,

$$|I - II| \leq C \left( \frac{h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,2}}{h_{\mathbf{v}_1,3}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}.$$

The term  $III$  in equation (2.4.13) can be bounded by the same arguments used to obtain (2.4.12). Therefore we obtain

$$|w(\mathbf{v}_2) - w(\mathbf{v}_6)| \left\| \frac{\partial \lambda_{\mathbf{v}_1}}{\partial x_1} \right\|_{L^2(R)} \leq C \sum_{i=1}^d h_{\mathbf{v}_1,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_i} \right\|_{L^2(\tilde{R})}. \quad (2.4.14)$$

The estimate of  $w(\mathbf{v}_3) - w(\mathbf{v}_7)$  follows by the same arguments used to estimate  $w(\mathbf{v}_2) - w(\mathbf{v}_6)$ . Then, it remains to estimate  $w(\mathbf{v}_4) - w(\mathbf{v}_8)$ . We have

$$\begin{aligned} w(\mathbf{v}_4) - w(\mathbf{v}_8) &= (u_{\mathbf{v}_1}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_4)) - (u_{\mathbf{v}_1}(\mathbf{v}_8) - u_{\mathbf{v}_8}(\mathbf{v}_8)) \\ &= [(u_{\mathbf{v}_1}(\mathbf{v}_4) - u_{\mathbf{v}_1}(\mathbf{v}_8)) - (u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_3}(\mathbf{v}_8))] \\ &\quad + [(u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_3}(\mathbf{v}_8)) - (u_{\mathbf{v}_4}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_8))] + [u_{\mathbf{v}_8}(\mathbf{v}_8) - u_{\mathbf{v}_4}(\mathbf{v}_8)] \\ &=: I + II + III. \end{aligned}$$



Now we deal with the term  $I$ . One can check that

$$I = \int \left[ \frac{\partial u}{\partial x_1}(\mathbf{v}_1 - h_{\mathbf{v}_1} : y) - \frac{\partial u}{\partial x_1}(\mathbf{v}_3 - h_{\mathbf{v}_3} : y) \right] (\mathbf{v}_4^1 - \mathbf{v}_8^1) \psi(y) dy.$$

Defining now

$$F_y(t) := \frac{\partial u}{\partial x_1}(\mathbf{v}_3 - h_{\mathbf{v}_3} : y + t\theta)$$

where  $\theta = (0, 0, \theta_3) := (0, 0, \mathbf{v}_1^3 - \mathbf{v}_3^3 - (h_{\mathbf{v}_1,3} - h_{\mathbf{v}_3,3})y_3)$  we have

$$\begin{aligned} I &= \int \int_0^1 F'_y(t) (\mathbf{v}_4^1 - \mathbf{v}_8^1) \psi(y) dy dt \\ &= \int_0^1 \int F'_y(t) (\mathbf{v}_4^1 - \mathbf{v}_8^1) \psi(y) dt dy =: \int_0^1 I(t) dt. \end{aligned}$$

Since

$$F'_y(t) = \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathbf{v}_3 - h_{\mathbf{v}_3} : y + t\theta) \theta_3$$

and  $|\theta_3| \leq Ch_{\mathbf{v}_1,3}$  if  $|y| \leq 1$  it follows that

$$|I(t)| \leq h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,3} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(\mathbf{v}_3 - h_{\mathbf{v}_3} : y + t\theta) \right| \psi(y) dy$$

and so, changing variables and setting

$$\phi(z) = \psi \left( -\frac{z_1 - \mathbf{v}_3^1}{h_{\mathbf{v}_3,1}}, -\frac{z_2 - \mathbf{v}_3^2}{h_{\mathbf{v}_3,2}}, -\frac{z_3 - [(1-t)\mathbf{v}_3^3 + t\mathbf{v}_1^3]}{(1-t)h_{\mathbf{v}_3,3} + th_{\mathbf{v}_1,3}} \right),$$

we obtain

$$|I(t)| \leq C \frac{1}{h_{\mathbf{v}_1,2}} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_3}(z) \right| \phi(z) dz.$$

Now, taking into account that  $\phi = 0$  on  $\partial\tilde{R}$ , it follows by the Schwarz inequality and Lemma 2.3.1 that

$$\begin{aligned} |I(t)| &\leq C \frac{1}{h_{\mathbf{v}_1,2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})} \left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \\ &\leq \left( \frac{h_{\mathbf{v}_1,1} h_{\mathbf{v}_1,3}}{h_{\mathbf{v}_1,2}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})}, \end{aligned}$$

and therefore,

$$|I| \left\| \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1} \right\|_{L^2(\tilde{R})} \leq h_{\mathbf{v}_1,3} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right\|_{L^2(\tilde{R})}. \quad (2.4.15)$$

Finally, estimates for the terms *II* and *III* can be obtained with the arguments used for  $(u_{v_1}(v_2) - u_{v_1}(v_6)) - (u_{v_2}(v_2) - u_{v_2}(v_6))$  in (2.4.13) and  $u_{v_5}(v_5) - u_{v_1}(v_5)$  in (2.4.10) respectively. These estimates together with the inequalities (2.4.12), (2.4.14) and (2.4.15) conclude the proof.  $\square$

## 2.5 Error estimates for boundary elements

In this section we deal with the interpolation error on boundary elements for functions satisfying a homogeneous Dirichlet condition. For the sake of simplicity and because the proof is rather technical, we state and prove the main Theorem in the two dimensional case. However, analogous results can be obtained in three dimensions by using similar arguments.

We will use the notation of the previous section. Further, if  $R = (a_1, b_1) \times (a_2, b_2)$  is a rectangle in  $\mathcal{T}$ , we set  $R_{1i} = a_i$  and  $l_{R,i} = (a_i, b_i)$ . Also we define the function  $\delta_{-,R}$  by

$$\delta_{-,R}(x) = \min \left\{ \frac{x_1 - a_1}{h_{R,1}}, \frac{x_2 - a_2}{h_{R,2}} \right\}.$$

We have  $\delta_R(x) \leq \delta_{-,R}(x)$  for all  $x \in R$ .

To estimate the error on a boundary element  $R$  we need to consider different cases according to the position of  $R$ . So, we decompose  $\Omega$  into four regions (see Figure 2.3):

$$\Omega_1 = \bigcup \{R \in \mathcal{T} : R \cap \partial\Omega = \emptyset\}$$

$$\Omega_2 = \bigcup \{R \in \mathcal{T} : R \cap \{x : x_1 = 0\} = \emptyset \text{ and } R \cap \{x : x_2 = 0\} \neq \emptyset\}$$

$$\Omega_3 = \bigcup \{R \in \mathcal{T} : R \cap \{x : x_1 = 0\} \neq \emptyset \text{ and } R \cap \{x : x_2 = 0\} = \emptyset\}$$

$$\Omega_4 = R \in \mathcal{T} \text{ such that } (0,0) \in R.$$

**Theorem 2.5.1.** *There exists a constant  $C$  depending only on  $\sigma$  and  $\psi$  such that if  $R \in \mathcal{T}$  for all  $u \in H^2(\tilde{R})$  the following estimates hold,*

(i) *If  $R \subset \Omega_2$  and  $u \equiv 0$  on  $\{x : x_2 = 0\}$*

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,1} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\} \quad (2.5.1)$$

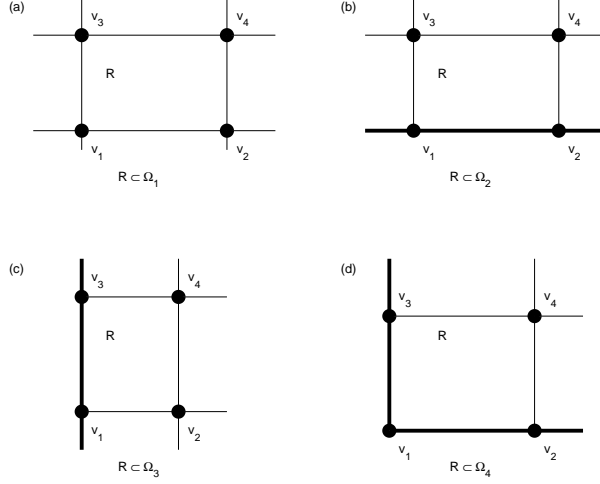


Figure 2.3: Relative positions of the rectangle  $R$ . The bold face line is the boundary of  $\Omega$ .

and

$$\left\| \frac{\partial}{\partial x_2} (u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,1} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} \right\}. \quad (2.5.2)$$

(ii) If  $R \subset \Omega_3$  and  $u \equiv 0$  on  $\{x : x_1 = 0\}$

$$\left\| \frac{\partial}{\partial x_1} (u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,1} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\}. \quad (2.5.3)$$

and

$$\left\| \frac{\partial}{\partial x_2} (u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,1} \left\| \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} \right\}. \quad (2.5.4)$$

(iii) If  $R \subset \Omega_4$  and  $u \equiv 0$  on  $\{x : x_1 = 0 \text{ or } x_2 = 0\}$

$$\left\| \frac{\partial}{\partial x_1} (u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,1} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \left\{ \frac{x_1}{h_{R,1}} + \frac{x_2}{h_{R,2}} \right\} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\}. \quad (2.5.5)$$

and

$$\left\| \frac{\partial}{\partial x_2}(u - \Pi u) \right\|_{L^2(R)} \leq C \left\{ h_{R,2} \left\| \delta_{-\tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} + h_{R,1} \left\| \left\{ \frac{x_1}{h_{R,1}} + \frac{x_2}{h_{R,2}} \right\} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\}. \quad (2.5.6)$$

*Proof of Part (i).* We now use the notation of Figure 2.3(b). We have

$$\Pi u|_R = u_{\mathbf{v}_3}(\mathbf{v}_3)\lambda_{\mathbf{v}_3} + u_{\mathbf{v}_4}(\mathbf{v}_4)\lambda_{\mathbf{v}_4}.$$

From (2.3.7) we know that  $\left\| \frac{\partial}{\partial x_1}(u - u_{\mathbf{v}_3}) \right\|_{L^2(R)}$  is bounded by the right hand side of (2.5.1).

So, to prove (2.5.1), it is enough to estimate  $\left\| \frac{\partial}{\partial x_1}(u_{\mathbf{v}_3} - \Pi u) \right\|_{L^2(R)}$ .

Since  $(u_{\mathbf{v}_3} - \Pi u)|_R \in \mathcal{Q}_1$  we have (see for example [38])

$$\begin{aligned} \frac{\partial}{\partial x_1}(u_{\mathbf{v}_3} - \Pi u) &= ((u_{\mathbf{v}_3} - \Pi u)(\mathbf{v}_2) - (u_{\mathbf{v}_3} - \Pi u)(\mathbf{v}_1)) \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1} \\ &\quad + ((u_{\mathbf{v}_3} - \Pi u)(\mathbf{v}_4) - (u_{\mathbf{v}_3} - \Pi u)(\mathbf{v}_3)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1} \\ &= (u_{\mathbf{v}_3}(\mathbf{v}_2) - u_{\mathbf{v}_3}(\mathbf{v}_1)) \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1} + (u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_4)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1}. \end{aligned} \quad (2.5.7)$$

Taking into account that  $\frac{\partial u}{\partial x_1} \equiv 0$  on  $(x_1, 0)$  it is easy to see that

$$u_{\mathbf{v}_3}(\mathbf{v}_2) - u_{\mathbf{v}_3}(\mathbf{v}_1) = (\mathbf{v}_2^1 - \mathbf{v}_1^1) \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \int_0^{x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \psi_{\mathbf{v}_3}(x) dt dx_1 dx_2$$

and then,

$$\begin{aligned} |u_{\mathbf{v}_3}(\mathbf{v}_2) - u_{\mathbf{v}_3}(\mathbf{v}_1)| &\leq C h_{\mathbf{v}_3,1} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \right| \psi_{\mathbf{v}_3}(x) dt dx_1 dx_2 \\ &\leq C h_{\mathbf{v}_3,1} \int_{l_{\tilde{R},2}} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \left\| \psi_{\mathbf{v}_3}(x) \frac{h_{\mathbf{v}_3,1}}{x_1 - \tilde{R}_{11}} \right\|_{L^2(\tilde{R})} dx_2. \end{aligned}$$

Using the one dimensional Hardy inequality (1.2.8) we have

$$\begin{aligned} \int_{l_{\tilde{R},1}} \left| \frac{\psi_{\mathbf{v}_3}(x)}{x_1 - \tilde{R}_{11}} \right|^2 dx_1 &\leq \frac{C}{h_{\mathbf{v}_3,1}^4 h_{\mathbf{v}_3,2}^2} \int_{l_{\tilde{R},1}} \left| \frac{\partial \psi}{\partial x_1} \left( \frac{\mathbf{v}_3^1 - x_1}{h_{\mathbf{v}_3,1}}, \frac{\mathbf{v}_1^2 - x_2}{h_{\mathbf{v}_3,2}} \right) \right|^2 dx_1 \\ &\leq C \frac{1}{h_{\mathbf{v}_3,1}^3 h_{\mathbf{v}_3,2}^2} \end{aligned} \quad (2.5.8)$$

and then it follows that

$$|u_{\mathbf{v}_3}(\mathbf{v}_2) - u_{\mathbf{v}_3}(\mathbf{v}_1)| \leq C(h_{\mathbf{v}_3,1}h_{\mathbf{v}_3,2})^{\frac{1}{2}} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}$$

and so

$$|u_{\mathbf{v}_3}(\mathbf{v}_2) - u_{\mathbf{v}_3}(\mathbf{v}_1)| \left\| \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1} \right\|_{L^2(\tilde{R})} \leq Ch_{\mathbf{v}_3,2} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \quad (2.5.9)$$

On the other hand, with the same argument that we have used to obtain (2.4.12) in the proof of Theorem 2.4.2 we can show that

$$|(u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_4))| \left\| \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1} \right\|_{L^2(R)} \leq C \sum_{i=1}^2 \left( h_{\mathbf{v}_3,i} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\tilde{R})} \right)$$

which together with (2.5.7) and (2.5.9) concludes the proof of (2.5.1).

Now, to prove (2.5.2), using again Lemma 2.3.3, we have to estimate  $\left\| \frac{\partial}{\partial x_2} (u_{\mathbf{v}_3} - \Pi u) \right\|_{L^2(R)}$ . Using again the expression for the derivative of a  $\mathcal{Q}_1$  function we have

$$\begin{aligned} \frac{\partial}{\partial x_2} (u_{\mathbf{v}_3} - \Pi u) &= -u_{\mathbf{v}_3}(\mathbf{v}_1) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} + (u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_4)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} - u_{\mathbf{v}_3}(\mathbf{v}_2) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} \\ &= -u_{\mathbf{v}_3}(\mathbf{v}_1) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} + (u_{\mathbf{v}_3}(\mathbf{v}_4) - u_{\mathbf{v}_3}(\mathbf{v}_2)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} - \\ &\quad (u_{\mathbf{v}_4}(\mathbf{v}_4) - u_{\mathbf{v}_4}(\mathbf{v}_2)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} - u_{\mathbf{v}_4}(\mathbf{v}_2) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} \\ &=: -u_{\mathbf{v}_3}(\mathbf{v}_1) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} + (I - II) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2} - u_{\mathbf{v}_4}(\mathbf{v}_2) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2}. \end{aligned} \quad (2.5.10)$$

Defining now

$$\theta = (\theta_1, 0) := (\mathbf{v}_4^1 - \mathbf{v}_3^1 - (h_{\mathbf{v}_4,1} - h_{\mathbf{v}_3,1})y_1, 0)$$

and

$$F_y(t) = \frac{\partial u}{\partial x_2}(\mathbf{v}_3 - h_{\mathbf{v}_3} : y + \theta t)$$

we have

$$\begin{aligned}
I - II &= (v_4^2 - v_2^2) \int \left[ \frac{\partial u}{\partial x_2}(v_3 - h_{v_3} : y) - \frac{\partial u}{\partial x_2}(v_4 - h_{v_4} : y) \right] \psi(y) dy \\
&= (v_4^2 - v_2^2) \int (F_y(0) - F_y(1)) \psi(y) dy \\
&= -(v_4^2 - v_2^2) \int \int_0^1 F'_y(t) dt \psi(y) dy
\end{aligned}$$

but,

$$F'_y(t) = \frac{\partial^2 u}{\partial x_1 \partial x_2}(v_3 - h_{v_3} : y + \theta t) \theta_1$$

and so,

$$\begin{aligned}
I - II &= -(v_4^2 - v_2^2) \int_0^1 \int \frac{\partial^2 u}{\partial x_1 \partial x_2}(v_3 - h_{v_3} : y + \theta t) \theta_1 \psi(y) dy dt \\
&=: -(v_4^2 - v_2^2) \int_0^1 I(t) dt.
\end{aligned}$$

We will estimate  $I(t)$ . Since  $\text{supp } \psi \subset B(0, 1)$  we have

$$|I(t)| \leq Ch_{v_3,1} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(v_3 - h_{v_3} : y + \theta t) \right| \psi(y) dy.$$

Now, setting  $z = v_3 - h_{v_3} : y + \theta t$ , taking into account that  $Ch_{v_3,1} \leq (1-t)h_{v_3,1} + th_{v_4,1}$  ( $0 \leq t \leq 1$ ), and defining

$$\phi(z) = \psi \left( \frac{(1-t)v_3^1 + tv_4^1 - z_1}{(1-t)h_{v_3,1} + th_{v_4,1}}, \frac{v_3^2 - z_2}{h_{v_3,2}} \right)$$

we obtain

$$|I(t)| \leq C \frac{1}{h_{v_3,2}} \int \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(z) \right| \phi(z) dz$$

and, since  $\phi \equiv 0$  on  $\partial \tilde{R}$  we can use Lemma 2.3.1 to obtain

$$\begin{aligned}
|I(t)| &\leq C \frac{1}{h_{\mathbf{v}_3,2}} \left\| \frac{\phi}{\delta_{\tilde{R}}} \right\|_{L^2(\tilde{R})} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \\
&\leq C \frac{1}{h_{\mathbf{v}_3,2}} \left( h_{R,1} \left\| \frac{\partial \phi}{\partial z_1} \right\|_{L^2(\tilde{R})} + h_{R,2} \left\| \frac{\partial \phi}{\partial z_2} \right\|_{L^2(\tilde{R})} \right) \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \\
&\leq C \left( \frac{h_{R,1}}{h_{R,2}} \right)^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}.
\end{aligned}$$

Therefore,

$$|I - II| \leq C(h_{R,1}h_{R,2})^{\frac{1}{2}} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})},$$

so

$$\left\| (I - II) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} \right\|_{L^2(R)} \leq Ch_{R,1} \left\| \delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \quad (2.5.11)$$

Now, to estimate the first term of formula (2.5.10),  $u_{\mathbf{v}_3}(\mathbf{v}_1) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2}$ , we observe that, since  $u(x_1, 0) \equiv 0$  then one can check that

$$\begin{aligned}
u_{\mathbf{v}_3}(\mathbf{v}_1) &= - \int \int_{t=0}^{x_2} \frac{\partial^2 u}{\partial x_2^2}(x_1, t) (t - \mathbf{v}_1^2) \psi_{\mathbf{v}_3}(x) dt dx + \int (\mathbf{v}_1^1 - x_1) \frac{\partial u}{\partial x_1}(x) \psi_{\mathbf{v}_3}(x) dx \\
&=: A + B.
\end{aligned} \quad (2.5.12)$$

We will estimate  $A$  and  $B$ . Since  $\mathbf{v}_1^2 = 0$  we have

$$\begin{aligned}
|A| &\leq Ch_{\mathbf{v}_3,2} \int \int_{t=0}^{x_2} \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{t}{h_{\mathbf{v}_3,2}} \left| \frac{\partial^2 u}{\partial x_2^2}(x_1, t) \right| \psi_{\mathbf{v}_3}(x) \frac{h_{\mathbf{v}_3,1}}{x_1 - \tilde{R}_{11}} dt dx \\
&\leq Ch_{\mathbf{v}_3,2} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \delta_{-, \tilde{R}}(x_1, t) \left| \frac{\partial^2 u}{\partial x_2^2}(x_1, t) \right| \psi_{\mathbf{v}_3}(x) \frac{h_{\mathbf{v}_3,1}}{x_1 - \tilde{R}_{11}} dt dx_1 dx_2.
\end{aligned}$$

Therefore, using the Schwarz inequality and (2.5.8) we obtain

$$|A| \leq C \frac{(h_{\mathbf{v}_3,2})^{\frac{3}{2}}}{(h_{\mathbf{v}_3,1})^{\frac{1}{2}}} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})}$$

and then,

$$|A| \left\| \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} \right\|_{L^2(R)} \leq Ch_{\mathbf{v}_3,2} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})}. \quad (2.5.13)$$

In order to estimate  $B$  we note that, since  $\frac{\partial u}{\partial x_1}(x_1, 0) \equiv 0$  then,

$$\begin{aligned} B &= \int_{l_{\tilde{R},1}} (\mathbf{v}_1^1 - x_1) \int_{l_{\tilde{R},2}} \frac{\partial u}{\partial x_1}(x) \psi_{\mathbf{v}_3}(x) dx_2 dx_1 \\ &= \int_{l_{\tilde{R},1}} (\mathbf{v}_1^1 - x_1) \int_{l_{\tilde{R},2}} \int_{t=0}^{x_2} \frac{\partial^2 u}{\partial x_2 \partial x_1}(x_1, t) \psi_{\mathbf{v}_3}(x) dt dx_2 dx_1. \end{aligned}$$

Then,

$$\begin{aligned} |B| &\leq Ch_{\mathbf{v}_3,1} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \right| \psi_{\mathbf{v}_3}(x) dt dx_1 dx_2 \\ &\leq C(h_{\mathbf{v}_3,1} h_{\mathbf{v}_3,2})^{\frac{1}{2}} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \end{aligned}$$

where we have used the Schwarz inequality and the same argument used to obtain (2.5.9).

Consequently we obtain

$$|B| \left\| \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} \right\|_{L^2(R)} \leq Ch_{\mathbf{v}_3,1} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}$$

which together with (2.5.12) and (2.5.13) implies

$$\left\| u_{\mathbf{v}_3}(\mathbf{v}_1) \frac{\partial \lambda_{\mathbf{v}_3}}{\partial x_2} \right\|_{L^2(R)} \leq C \left\{ h_{\mathbf{v}_3,2} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{R})} + h_{\mathbf{v}_3,1} \left\| \frac{x_1 - \tilde{R}_{11}}{h_{\mathbf{v}_3,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})} \right\}.$$

Clearly an analogous estimate follows for  $\|u_{\mathbf{v}_4}(\mathbf{v}_2) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_2}\|_{L^2(R)}$ , and then, in view of (2.5.10) and (2.5.11) we conclude the proof of inequality (2.5.2).

The proof of Part (ii) is, of course, analogous to that of Part (i).

Proof of Part (iii): We will use the notation of the Figure 2.3(d). Then

$$\Pi u|_R = u_{\mathbf{v}_4}(\mathbf{v}_4) \lambda_{\mathbf{v}_4}.$$

In this case the error can be split as

$$(u - \Pi u)|_R = (u - u_{\mathbf{v}_4}) + (u_{\mathbf{v}_4} - \Pi u)$$



and it is enough to bound  $u_{\mathbf{v}_4} - \Pi u$ , which is piecewise  $\mathcal{Q}_1$ . Then we have

$$\begin{aligned} \frac{\partial}{\partial x_1}(u_{\mathbf{v}_4} - \Pi u) &= ((u_{\mathbf{v}_4} - \Pi u)(\mathbf{v}_4) - (u_{\mathbf{v}_4} - \Pi u)(\mathbf{v}_3)) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1} \\ &\quad + ((u_{\mathbf{v}_4} - \Pi u)(\mathbf{v}_2) - (u_{\mathbf{v}_4} - \Pi u)(\mathbf{v}_1)) \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1} \\ &= -u_{\mathbf{v}_4}(\mathbf{v}_3) \frac{\partial \lambda_{\mathbf{v}_4}}{\partial x_1} + (u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1)) \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1}. \end{aligned} \quad (2.5.14)$$

First we estimate  $|u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1)|$ . Using that  $\frac{\partial u}{\partial x_1}(x_1, 0) \equiv 0$  we have

$$\begin{aligned} u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1) &= \int (P(x, \mathbf{v}_2) - P(x, \mathbf{v}_1)) \psi_{\mathbf{v}_4}(x) dx \\ &= (\mathbf{v}_2^1 - \mathbf{v}_1^1) \int \frac{\partial u}{\partial x_1}(x) \psi_{\mathbf{v}_4}(x) dx \\ &= (\mathbf{v}_2^1 - \mathbf{v}_1^1) \int \int_{t=0}^{x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \psi_{\mathbf{v}_4}(x) dt dx. \end{aligned}$$

It follows that

$$\begin{aligned} |u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1)| &\leq Ch_{\mathbf{v}_4,1} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \right| \psi_{\mathbf{v}_4}(x) dt dx_1 dx_2 \\ &\leq Ch_{\mathbf{v}_4,1} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \frac{x_1}{h_{\mathbf{v}_4,1}} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, t) \right| \psi_{\mathbf{v}_4}(x) \frac{h_{\mathbf{v}_4,1}}{x_1} dx_1 dt dx_2, \end{aligned}$$

and an argument similar to that used to obtain (2.5.9) gives

$$|u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1)| \leq C(h_{\mathbf{v}_4,1} h_{\mathbf{v}_4,2})^{\frac{1}{2}} \left\| \frac{x_1}{h_{\mathbf{v}_4,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}.$$

Therefore,

$$|u_{\mathbf{v}_4}(\mathbf{v}_2) - u_{\mathbf{v}_4}(\mathbf{v}_1)| \left\| \frac{\partial \lambda_{\mathbf{v}_2}}{\partial x_1} \right\| \leq Ch_{\mathbf{v}_4,2} \left\| \frac{x_1}{h_{\mathbf{v}_4,1}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \quad (2.5.15)$$

Now we consider the other term in (2.5.14). We have to estimate  $|u_{\mathbf{v}_4}(\mathbf{v}_3)|$ . Using that  $u(0, x_2) \equiv 0$  and  $\mathbf{v}_3 = (0, \mathbf{v}_3^2)$  we obtain

$$\begin{aligned} u_{\mathbf{v}_4}(\mathbf{v}_3) &= - \int \int_{t=0}^{x_1} t \frac{\partial^2 u}{\partial x_1^2}(t, x_2) \psi_{\mathbf{v}_4}(x) dt dx + \int (\mathbf{v}_3^2 - x_2) \frac{\partial u}{\partial x_2}(x) \psi_{\mathbf{v}_4}(x) dx \\ &=: A + B \end{aligned}$$

and we have to estimate  $A$  and  $B$ . We have

$$\begin{aligned} |A| &\leq h_{v_4,1} \int \int_{t=0}^{x_1} \frac{t}{h_{v_4,1}} \frac{x_2}{h_{v_4,2}} \left| \frac{\partial^2 u}{\partial x_1^2}(t, x_2) \right| \psi_{v_4}(x) \frac{h_{v_4,2}}{x_2} dt dx \\ &\leq C h_{v_4,1} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \delta_{-, \tilde{R}}(t, x_2) \left| \frac{\partial^2 u}{\partial x_1^2}(t, x_2) \right| \psi_{v_4}(x) \frac{h_{v_4,2}}{x_2} dt dx_2 dx_1. \end{aligned}$$

But again, by an argument similar to that used in the proof of (2.5.9), we obtain

$$|A| \leq C \frac{(h_{v_4,1})^{\frac{3}{2}}}{(h_{v_4,2})^{\frac{1}{2}}} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})}.$$

Therefore,

$$|A| \left\| \frac{\partial \lambda_{v_4}}{\partial x_1} \right\|_{L^2(R)} \leq C h_{v_4,1} \left\| \delta_{-, \tilde{R}} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})}. \quad (2.5.16)$$

On the other hand, using now that  $\frac{\partial u}{\partial x_2}(0, x_2) \equiv 0$ , we have

$$\begin{aligned} B &= \int (v_3^2 - x_2) \frac{\partial u}{\partial x_2}(x) \psi_{v_4}(x) dx \\ &= \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} (v_3^2 - x_2) \int_{t=0}^{x_1} \frac{\partial^2 u}{\partial x_1 \partial x_2}(t, x_2) \psi_{v_4}(x) dt dx_2 dx_1 \end{aligned}$$

and then

$$|B| \leq C h_{v_4,2} \int_{l_{\tilde{R},1}} \int_{l_{\tilde{R},2}} \int_{l_{\tilde{R},1}} \frac{x_2}{h_{v_4,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2}(t, x_2) \psi_{v_4}(x) \frac{h_{v_4,2}}{x_2} dt dx_2 dx_1.$$

Hence

$$|B| \left\| \frac{\partial \lambda_{v_4}}{\partial x_1} \right\|_{L^2(R)} \leq C h_{v_4,2} \left\| \frac{x_2}{h_{v_4,2}} \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}. \quad (2.5.17)$$

Now, inequality (2.5.5) follows from (2.5.14), (2.5.15), (2.5.16) and (2.5.17).

Since (2.5.6) is analogous to (2.5.5) the proof is concluded.  $\square$

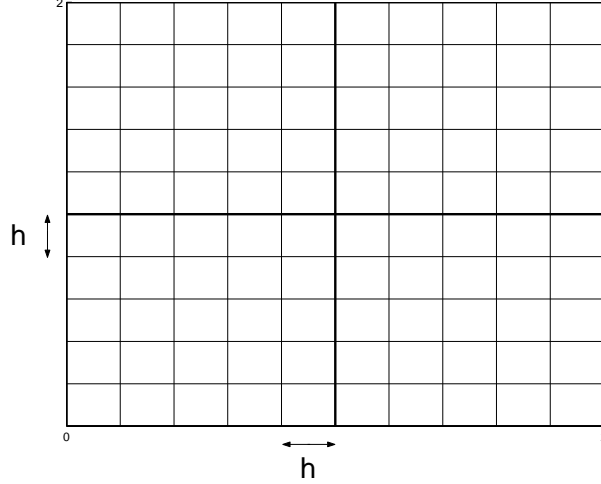


Figure 2.4: Uniform mesh on  $[0, 2]^2$ .

## 2.6 Applications

Let  $D = (0, 2)^2$ , and consider on  $D$  an uniform mesh  $\mathcal{T}_h$  made of squares of sides of length  $h$ , as is shown in Figure 2.4. Consider  $0 < \alpha \leq 1$ . We continue with the notation previously introduced.

If  $R \in \mathcal{T}_h$  is not a boundary element of  $D$ , we have for all  $(x_1, x_2) \in R$

$$\begin{aligned} \delta_{\tilde{R}}(x_1, x_2) &\leq C \left( \frac{d_{\tilde{R}}(x_1, x_2)}{h} \right)^\alpha \\ &\leq Ch^{-\alpha} d_D(x_1, x_2)^\alpha \end{aligned}$$

Then, summing up the estimates given by part (ii) of Theorem 2.4.1 for all the interior elements  $R$  in  $\mathcal{T}_h$ , we obtain

$$\|u - \Pi u\|_{0, D^\circ} \leq Ch^{1-\alpha} \left( \left\| d_D^\alpha \frac{\partial u}{\partial x_1} \right\|_{0, D} + \left\| d_D^\alpha \frac{\partial u}{\partial x_2} \right\|_{0, D} \right)$$

where  $D^\circ$  is the union of all the interior elements of  $\mathcal{T}_h$ . If  $D_\partial$  denotes the union of the boundary elements of  $\mathcal{T}_h$ , using part (i) of Theorem 2.4.1, we obtain

$$\begin{aligned} \|u - \Pi u\|_{0, D_\partial} &\leq \sum_{R \subset D_\partial} C \|u\|_{0, \tilde{R}} \\ &\leq C \|u\|_{0, \tilde{D}_\partial} \end{aligned}$$

where  $\widetilde{D}_\partial$  denotes the union of the elements intersecting  $D_\partial$ . So, we have the estimate

$$\|u - \Pi u\|_{0,D} \leq C \|u\|_{0,\widetilde{D}_\partial} + Ch^{1-\alpha} \left( \left\| d_D^\alpha \frac{\partial u}{\partial x_1} \right\|_{0,D} + \left\| d_D^\alpha \frac{\partial u}{\partial x_2} \right\|_{0,D} \right). \quad (2.6.1)$$

Now we consider the derivatives of the interpolation error. Let  $\Omega = (0,1)^2$ , and  $R \in \mathcal{T}_h$  with  $R \subset \bar{\Omega}$ . Then we have the estimates

$$\begin{aligned} \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} &\leq C \left( \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \right)^\alpha \leq Ch^{-\alpha} d_{D,1}(x_1, x_2)^\alpha, \\ \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} &\leq C \left( \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \right)^\alpha \leq Ch^{-\alpha} d_{D,2}(x_1, x_2)^\alpha, \end{aligned}$$

and

$$\delta_{-, \tilde{R}} \leq C \min \left\{ \left( \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \right)^\alpha, \left( \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \right)^\alpha \right\} \leq Ch^{-\alpha} d_D(x_1, x_2)^\alpha,$$

for all  $(x_1, x_2) \in R$ , where  $d_{D,i}(x_1, x_2) = \min\{x_i, 2 - x_i\}$ . If we consider the estimates for  $\|\frac{\partial}{\partial x_1}(u - \Pi u)\|_{0,R}$  given by Theorems 2.4.2 and 2.5.1, for each  $R \subset \bar{\Omega}$ , and sum up them, we can easily obtain (for the particular mesh considered here)

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{0,\Omega} \leq Ch^{1-\alpha} \left( \left\| d_D^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,\tilde{\Omega}} + \left\| (d_{1,D}^\alpha + d_{2,D}^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,\tilde{\Omega}} \right)$$

where  $\tilde{\Omega}$  is the union of the elements intersecting  $\Omega$ . Clearly, the corresponding estimates for the rest of the domain  $D$  can be obtained similarly. We can proceed analogously with  $\|\frac{\partial}{\partial x_2}(u - \Pi u)\|_{0,R}$ . Then we have the following weighted estimates

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{0,D} \leq Ch^{1-\alpha} \left( \left\| d_D^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,D} + \left\| (d_{1,D}^\alpha + d_{2,D}^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,D} \right) \quad (2.6.2)$$

$$\left\| \frac{\partial}{\partial x_2}(u - \Pi u) \right\|_{0,D} \leq Ch^{1-\alpha} \left( \left\| (d_{1,D}^\alpha + d_{2,D}^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,D} + \left\| d_D^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,D} \right). \quad (2.6.3)$$

Notice, that if we know *a priori* estimates for the function  $u$  involving

$$d_D^\alpha \frac{\partial u}{\partial x_1}, \quad d_D^\alpha \frac{\partial u}{\partial x_2}, \quad d_D^\alpha \frac{\partial^2 u}{\partial x_1^2}, \quad d_D^\alpha \frac{\partial^2 u}{\partial x_2^2}, \quad d_{D,1}^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2}, \quad d_{D,2}^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

and the  $L^2$  norm of  $u$  on some “small” subset of  $D$ , then we are able to bound the interpolation error. We will use this idea in the next chapter, combining it with a suitable design of the mesh, which will allow us to recover the optimal order  $h$  in the estimate of the interpolation and the approximation errors for a particular singularly perturbed reaction diffusion problem.

## Chapter 3

# Applications and Numerical Examples

### 3.1 Introduction

As an application of the results obtained in Chapter 2, we consider the finite element approximation of three singularly perturbed model problems: a reaction-diffusion equation in Section 3.2, a convection-diffusion equation in Section 3.3, and a fourth order problem in Section 3.4.

In Theorem 3.2.4 we obtain average interpolation estimates analogous to those of inequalities (2.6.1), (2.6.2) and (2.6.3) on suitable graded meshes in order to recover the optimal order. As a consequence of that theorem we prove that, for the reaction diffusion case, the finite element method using continuous piecewise bilinears on such classes of meshes achieve quasi-optimal approximation order, uniform in the perturbation parameter. In this case the meshes can be performed independently of the perturbation parameter. For the convection diffusion case, the same theorem, permit us to prove uniform quasi-optimal interpolation error estimates on graded meshes, that in this case must be constructed taking into account the perturbation parameter. However, we are not able to prove uniform error estimates for the finite element approximation error, we can only conjecture them by numerical experiments.

For the fourth order problem, we consider a nonconforming finite element method based on Adini's rectangle. We prove that on quasi-uniform meshes, a sub-optimal convergence order can be obtained. The proof presented here is exactly the same to the one given in [33] for a modification of Morley's finite elements, but since that our finite element has as nodal

variables some evaluations of derivatives, we need to reestablish some technical ingredients of that proof. In particular, we need to use an interpolation operator introduced in [22]. We also present numerical examples that show that if our method is used on graded meshes the optimal order seem to be recovered.

### 3.2 A reaction-diffusion model equation

As an example of application of our results we consider in this section the singular perturbation model problem

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= f && \text{in } (0, 2) \times (0, 2) \\ u &= 0 && \text{on } \partial\{(0, 2) \times (0, 2)\}. \end{aligned} \tag{3.2.1}$$

We assume that  $f \in C^2([0, 2] \times [0, 2])$  and the following compatibility conditions

$$f(0, 0) = f(2, 0) = f(2, 2) = f(0, 2) = 0$$

which ensure that the solution  $u$  of (3.2.1) belong to  $C^4((0, 2) \times (0, 2)) \cap C^2([0, 2] \times [0, 2])$  (see, for example, [24, 29, 30, 36]). Such compatibility conditions are necessary for the pointwise derivative estimates of the solution.

As we will show, appropriate graded anisotropic meshes can be defined in order to obtain almost optimal order error estimates in the energy norm valid uniformly in the parameter  $\varepsilon$ . These estimates follow from the results of Sections 2.4 and 2.5 of Chapter 2.

The meshes that we construct are very different from the Shishkin type meshes that have been used in other papers for this problem (see for example [5, 30]). In particular, our almost optimal error estimate in the energy norm is obtained with meshes independent of  $\varepsilon$ .

Given a partition  $\mathcal{T}_h$  of  $(0, 2) \times (0, 2)$  into rectangles, we call  $u_h$  the  $\mathcal{Q}_1$  finite element approximation of the solution of problem (3.2.1). Since  $u_h$  is the orthogonal projection in the scalar product associated with the energy norm

$$\|v\|_\varepsilon = \left\{ \varepsilon^2 \|\nabla v\|_{L^2((0,2)^2)}^2 + \|v\|_{L^2((0,2)^2)}^2 \right\}^{\frac{1}{2}}$$

we know that, for any  $v_h$  in the finite element space,

$$\|u - u_h\|_\varepsilon \leq \|u - v_h\|_\varepsilon.$$

In particular, if  $\Pi$  is the average interpolation operator associated with the partition  $\mathcal{T}_h$  introduced in Section 2.2, we have

$$\|u - u_h\|_\varepsilon \leq \|u - \Pi u\|_\varepsilon. \quad (3.2.2)$$

Therefore, we will construct the meshes in order to have a good estimate for the right hand side of (3.2.2).

We will obtain our estimates in  $\Omega = (0, 1) \times (0, 1)$ . Clearly, analogous arguments can be applied for the rest of the domain. The constant  $C$  will be always independent of  $\varepsilon$ .

We will make use of the fact that the solution of (3.2.1) satisfies some weighted a priori estimates which are valid uniformly in the parameter  $\varepsilon$ . We state these a priori estimates in the next three lemmas but postpone the proofs until the end of the section.

**Lemma 3.2.1.** *There exists a constant  $C$  such that if  $\alpha \geq \frac{1}{2}$  then*

$$\left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C \quad \text{and} \quad \left\| x_2^\alpha \frac{\partial u}{\partial x_2} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C. \quad (3.2.3)$$

**Lemma 3.2.2.** *There exists a constant  $C$  such that if  $\alpha \geq \frac{1}{2}$  then*

$$\varepsilon \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C, \quad \varepsilon \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C, \quad (3.2.4)$$

$$\varepsilon \left\| x_1^\alpha \frac{\partial u}{\partial x_1 \partial x_2} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C \quad \text{and} \quad \varepsilon \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2((0, \frac{3}{2}) \times (0, \frac{3}{2}))} \leq C. \quad (3.2.5)$$

To estimate the error in the  $L^2$  norm we will use a priori estimates in the following norms. For  $v : R \rightarrow \mathbb{R}$ , where  $R$  is the rectangle  $R = l_1 \times l_2$ , define

$$\|v\|_{\infty \times \underline{1}, R} := \left\| \|v(x_1, \cdot)\|_{L^1(l_2)} \right\|_{L^\infty(l_1)} \quad \text{and} \quad \|v\|_{\underline{1} \times \infty, R} := \left\| \|v(\cdot, x_2)\|_{L^1(l_1)} \right\|_{L^\infty(l_2)}. \quad (3.2.6)$$

Then we have the following lemma, which also will be proved at the end of the section.

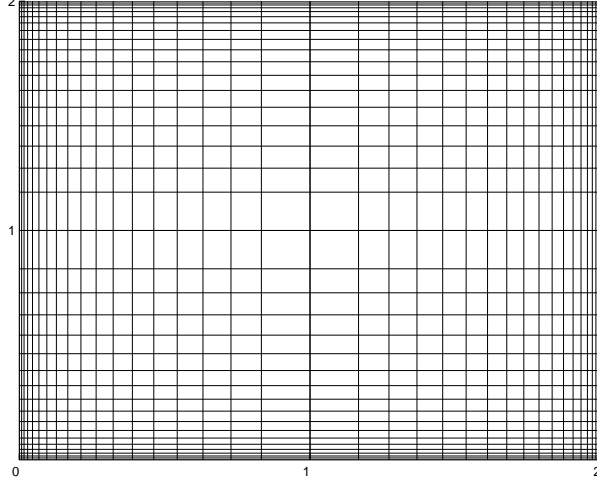


Figure 3.1: An example of graded mesh

**Lemma 3.2.3.** *There exists a constant  $C$  such that*

$$\left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1} \times \infty, (0, \frac{3}{2}) \times (0, \frac{3}{2})} \leq C \quad \text{and} \quad \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, (0, \frac{3}{2}) \times (0, \frac{3}{2})} \leq C.$$

Let us now define the graded meshes. Given a parameter  $h > 0$  and  $\alpha \in (0, 1)$  we introduce the partition  $\{\xi_i\}_{i=0}^N$  of the interval  $[0, 1]$  given by  $\xi_0 = 0$ ,  $\xi_1 = h^{\frac{1}{1-\alpha}}$ ,  $\xi_{i+1} = \xi_i + h\xi_i^\alpha$  for  $i = 1, \dots, N-2$ , where  $N$  is such that  $\xi_{N-1} < 1$  and  $\xi_{N-1} + h\xi_{N-1}^\alpha \geq 1$ , and  $\xi_N = 1$ . We assume that the last interval  $(\xi_{N-1}, 1)$  is not too small in comparison with the previous one  $(\xi_{N-2}, \xi_{N-1})$  (if this is not the case we just eliminate the node  $\xi_{N-1}$ ).

We define the partitions  $\mathcal{T}_{h,\alpha}$  such that they are symmetric with respect to the lines  $x_1 = 1$  and  $x_2 = 1$  (see Figure 3.1) and in the subdomain  $\Omega = (0, 1) \times (0, 1)$  are given by

$$\{R \subset \bar{\Omega} : R = (\xi_{i-1}, \xi_i) \times (\xi_{j-1}, \xi_j) \text{ for } 1 \leq i, j \leq N\}.$$

Observe that the family of meshes  $\mathcal{T}_{h,\alpha}$  satisfies our local regularity condition (2.2.1) with  $\sigma = 2^\alpha$ , that is, if  $S, T \in \mathcal{T}_{h,\alpha}$  are neighboring elements then

$$\frac{h_{T,i}}{h_{S,i}} \leq 2^\alpha.$$

For these meshes we have the following error estimates. We set  $\tilde{\Omega} = \cup\{\tilde{R} : R \subset \Omega\}$  where we are using the notations of the previous Chapter.



**Theorem 3.2.4.** *If  $u \in H^2(\Omega)$  and  $u \equiv 0$  on  $\{x : x_1 = 0 \text{ or } x_2 = 0\}$  then there exists a constant  $C$  such that*

$$\begin{aligned} \|u - \Pi u\|_{L^2(\Omega)} &\leq Ch \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{L^2(\tilde{\Omega})} + \left\| x_2^\alpha \frac{\partial u}{\partial x_2} \right\|_{L^2(\tilde{\Omega})} \right\} \\ &\quad + Ch^{\frac{1}{2-2\alpha}} \left\{ \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, \tilde{\Omega}} + \left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1} \times \infty, \tilde{\Omega}} \right\}, \end{aligned} \quad (3.2.7)$$

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_1} \right\|_{L^2(\Omega)} \leq Ch \left( \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{\Omega})} + \left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{\Omega})} \right), \quad (3.2.8)$$

and

$$\left\| \frac{\partial(u - \Pi u)}{\partial x_2} \right\|_{L^2(\Omega)} \leq Ch \left( \left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{\Omega})} + \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\tilde{\Omega})} \right). \quad (3.2.9)$$

*Proof.* We will estimate the error on each element according to its position. So, we decompose the domain  $\Omega$  into four parts,  $\Omega_i$ ,  $i = 1, \dots, 4$  defined as

$$\begin{aligned} \Omega_1 &= [\xi_1, \xi_N]^2 & \Omega_2 &= [\xi_1, \xi_N] \times [0, \xi_1] \\ \Omega_3 &= [0, \xi_1] \times [\xi_1, \xi_N] & \Omega_4 &= [0, \xi_1]^2, \end{aligned}$$

and we set  $\tilde{\Omega}_i = \cup\{\tilde{R} : R \subset \Omega_i\}$ ,  $i = 1, \dots, 4$ .

In order to prove (3.2.7) we split the error as follows

$$\|u - \Pi u\|_{L^2(\Omega)}^2 = \sum_{i=1}^4 \|u - \Pi u\|_{L^2(\Omega_i)}^2 =: S_1 + S_2 + S_3 + S_4. \quad (3.2.10)$$

First we estimate  $S_1$ . If  $\tilde{R} \cap \{x : x_1 = 0 \text{ or } x_2 = 0\} = \emptyset$  we have that, for each  $S \subset \tilde{R}$ ,  $h_{S,1} \leq hx_1^\alpha$  and  $h_{S,2} \leq hx_2^\alpha$  for all  $(x_1, x_2) \in S$  and then, Theorem 2.4.1 gives

$$\begin{aligned} \|u - \Pi u\|_{L^2(R)}^2 &\leq C \left\{ h_{R,1}^2 \int_{\tilde{R}} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\leq C \sum_{S \subset \tilde{R}} \left\{ h_{S,1}^2 \int_S \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h_{S,2}^2 \int_S \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\leq C \sum_{S \subset \tilde{S}} \left\{ h^2 \int_S x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_S x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &= C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\}. \end{aligned}$$

Now, suppose that  $R \subset \Omega_1$ ,  $\tilde{R} \cap \{x : x_2 = 0\} \neq \emptyset$  and  $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$ . Then  $\tilde{R}_{12} = 0$  and, if  $S \subset \tilde{R}$ , we have  $h_{S,1} \leq hx_1^\alpha$  for  $(x_1, x_2) \in S$  and  $h_{S,2} \leq Ch^{\frac{1}{1-\alpha}}$ . Therefore, using Theorem 2.4.1 we obtain

$$\begin{aligned} \|u - \Pi u\|_{L^2(R)}^2 &\leq Ch_{R,1}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch_{R,2}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial u}{\partial x_2} \right|^2 dx \\ &\leq C \sum_{S \subset \tilde{R}} \left\{ h_{S,1}^2 \int_S \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch_{S,2}^{2-2\alpha} \int_S x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\leq C \sum_{S \subset \tilde{R}} \left\{ h^2 \int_S x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch^2 \int_S x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &= C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\}. \end{aligned}$$

Now, if  $0 \in \tilde{R}$ , that is  $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$  and  $\tilde{R} \cap \{x : x_2 = 0\} \neq \emptyset$ , then,  $\tilde{R}_{11} = \tilde{R}_{12} = 0$  and  $h_{R,1} \leq Ch^{\frac{1}{1-\alpha}}$ ,  $h_{R,2} \leq Ch^{\frac{1}{1-\alpha}}$ . Then, from Theorem 2.4.1 we have

$$\begin{aligned} \|u - \Pi u\|_{L^2(R)}^2 &\leq Ch_{R,1}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch_{R,2}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial u}{\partial x_2} \right|^2 dx \\ &\leq Ch_{R,1}^{2-2\alpha} \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch_{R,2}^{2-2\alpha} \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \\ &\leq C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\}. \end{aligned}$$

A similar estimate can be obtained for  $\|u - \Pi u\|_{L^2(R)}$  when  $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$  and  $\tilde{R} \cap \{x : x_2 = 0\} = \emptyset$ . Therefore, we have

$$\begin{aligned} S_1 &\leq C \sum_{R \subset \Omega_1} \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \right\} \\ &\leq Ch^2 \int_{\tilde{\Omega}} x_1^{2\alpha} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + Ch^2 \int_{\tilde{\Omega}} x_2^{2\alpha} \left| \frac{\partial u}{\partial x_2} \right|^2 dx. \end{aligned} \tag{3.2.11}$$

Now, we estimate  $S_2$ . From Theorem 2.4.1 we know that  $\|\Pi u\|_{L^2(R)} \leq C\|u\|_{L^2(\tilde{R})}$  for all  $R \in \mathcal{T}_{h,\alpha}$  and therefore

$$S_2 = \sum_{R \subset \Omega_2} \|u - \Pi u\|_{L^2(R)}^2 \leq C \sum_{R \subset \Omega_2} \|u\|_{L^2(\tilde{R})}^2 \leq C \|u\|_{L^2(\tilde{\Omega}_2)}^2. \tag{3.2.12}$$

So, we have to estimate  $\|u\|_{L^2(\tilde{\Omega}_2)}$ . We have  $\tilde{\Omega}_2 = l_{\tilde{\Omega}_2,1} \times l_{\tilde{\Omega}_2,2}$  with  $|l_{\tilde{\Omega}_2,1}| \leq C$  and  $|l_{\tilde{\Omega}_2,2}| \leq Ch^{\frac{1}{1-\alpha}}$ . Using that  $u(x_1, 0) \equiv 0$  we have

$$\begin{aligned}
\|u\|_{L^2(\tilde{\Omega}_2)}^2 &= \int_{l_{\tilde{\Omega}_2,1}} \int_{l_{\tilde{\Omega}_2,2}} u^2(x) dx \\
&= \int_{l_{\tilde{\Omega}_2,1}} \int_{l_{\tilde{\Omega}_2,2}} \left\{ \int_0^{x_2} \frac{\partial u}{\partial x_2}(x_1, t) dt \right\}^2 dx_2 dx_1 \\
&\leq C \int_{l_{\tilde{\Omega}_2,2}} \left\| \left\| \frac{\partial u}{\partial x_2}(x_1, \cdot) \right\|_{L^1(l_{\tilde{\Omega}_2,2})} \right\|_{L^\infty(l_{\tilde{\Omega}_2,1})}^2 dx_2 \\
&\leq Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, \tilde{\Omega}_2}^2
\end{aligned} \tag{3.2.13}$$

and so, it follows from (3.2.12) and (3.2.13) that

$$S_2 \leq Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, \tilde{\Omega}_2}^2. \tag{3.2.14}$$

Analogously we can prove that

$$S_3 \leq Ch^{\frac{1}{1-\alpha}} \left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1} \times \infty, \tilde{\Omega}_3}^2, \tag{3.2.15}$$

$$S_4 \leq Ch^{\frac{2}{1-\alpha}} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, \tilde{\Omega}_4}^2 \quad \text{and} \quad S_4 \leq Ch^{\frac{2}{1-\alpha}} \left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1} \times \infty, \tilde{\Omega}_4}^2 \tag{3.2.16}$$

and inserting inequalities (4.2.4), (4.2.5), (4.3.13) and (3.2.16) in (3.2.10) we obtain (3.2.7) (note that  $\tilde{\Omega}_4 \subset \tilde{\Omega}_2$  and  $\tilde{\Omega}_4 \subset \tilde{\Omega}_3$ ).

Let us now prove (3.2.8). Inequality (3.2.9) follows in a similar way. Again we use the decomposition of  $\Omega$  into the four subsets  $\Omega_i, i = 1, \dots, 4$  defined above. Then we have

$$\left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(\Omega)}^2 = \sum_{i=1}^4 \left\| \frac{\partial}{\partial x_1}(u - \Pi u) \right\|_{L^2(\Omega_i)}^2 =: S_1 + S_2 + S_3 + S_4 \tag{3.2.17}$$

and we have to estimate  $S_i, i = 1, \dots, 4$ .

For  $S_1$ , Theorem 2.4.2 gives

$$\begin{aligned}
S_1 &= \sum_{R \subset \Omega_1} \left\| \frac{\partial}{\partial x_1} (u - \Pi u) \right\|_{L^2(R)}^2 \\
&\leq \sum_{R \subset \Omega_1} \left\{ h_{R,1}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial^2 u}{\partial x_1^2}(x) \right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \delta_{\tilde{R}}^{2\alpha}(x) \left| \frac{\partial^2 u}{\partial x_1 \partial x_2}(x) \right|^2 dx \right\} \\
&=: \sum_{R \subset \Omega_1} I_R.
\end{aligned}$$

Now, if  $\tilde{R} \cap \{x : x_1 = 0 \text{ or } x_2 = 0\} = \emptyset$  we have

$$|I_R| \leq C \sum_{T \subset \tilde{R}} \left\{ h_{T,1}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{T,2}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}$$

but, for  $T \subset \Omega_1$ , we have that

$$h_{T,1} \leq Chx_1^\alpha, \quad h_{T,2} \leq Chx_2^\alpha \quad \forall (x_1, x_2) \in T, \quad (3.2.18)$$

and therefore,

$$|I_R| \leq C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{R}} x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}.$$

On the other hand, if  $\tilde{R} \cap \{x : x_2 = 0\} \neq \emptyset$  and  $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$ , there are some elements  $T \subset \tilde{R}$  that not verify condition (3.2.18). For a such elements  $T$  we have  $h_{T,2} \leq h^{\frac{1}{1-\alpha}}$  while the condition on  $h_{T,1}$  in (3.2.18) remains valid. So we obtain

$$\begin{aligned}
|I_R| &\leq C \sum_{T \subset \tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{T,2}^{2-2\alpha} \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\
&\leq C \sum_{T \subset \tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}.
\end{aligned}$$

Now, if  $(0, 0) \in \tilde{R}$  we have  $h_{T,1} \leq Ch^{\frac{1}{1-\alpha}}$  and  $h_{T,2} \leq Ch^{\frac{1}{1-\alpha}}$  for all  $T \subset \tilde{R}$  and therefore,

$$\begin{aligned}
|I_R| &\leq C \sum_{T \subset \tilde{R}} \left\{ h_{T,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{T,2}^{2-2\alpha} \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\
&\leq C \sum_{T \subset \tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}.
\end{aligned}$$

If  $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$  and  $\tilde{R} \cap \{x : x_2 = 0\} = \emptyset$  we can estimate  $I_R$  analogously and so we obtain

$$S_1 \leq C \left\{ h^2 \int_{\tilde{\Omega}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{\Omega}} x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \quad (3.2.19)$$

Let us now estimate  $S_2$ . From Theorem 2.5.1(i) we have

$$\begin{aligned} S_2 &\leq \sum_{R \in \Omega_2} \left\{ h_{R,1}^2 \int_{\tilde{R}} \delta_{-, \tilde{R}}^{2\alpha}(x) \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left( \frac{x_1 - \tilde{R}_{11}}{h_{R,1}} \right)^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\ &=: \sum_{R \in \Omega_2} I_R. \end{aligned} \quad (3.2.20)$$

Now, if  $R \subset \Omega_2$  is such that  $\tilde{R} \cap \{x : x_1 = 0\} = \emptyset$  then, we have

$$|I_R| \leq C \left\{ \sum_{T \subset \tilde{R}} h_{R,1}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + \sum_{T \subset \tilde{R}} h_{R,2}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}$$

but, in this case, for  $T \subset \tilde{R}$ ,

$$h_{T,2} \leq Ch_{T,1} \leq Chx_1^\alpha \quad \forall x = (x_1, x_2) \in T$$

and therefore,

$$|I_R| \leq C \left\{ h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \quad (3.2.21)$$

On the other hand, if  $R \subset \Omega_2$  is such that  $\tilde{R} \cap \{x : x_1 = 0\} \neq \emptyset$ ,  $\tilde{R}_{11} = 0$  and so, it follows from (3.2.20) that (note that  $h_{R,2} \leq h_{R,1}$ )

$$|I_R| \leq C \left\{ \sum_{T \subset \tilde{R}} h_{R,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + \sum_{T \subset \tilde{R}} h_{R,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}$$

but in this case, for  $T \subset \tilde{R}$ ,  $h_{T,1} \leq Ch^{\frac{1}{1-\alpha}}$  and then

$$|I_R| \leq h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{R}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx. \quad (3.2.22)$$

Therefore, inserting inequalities (3.2.21) and (3.2.22) in (3.2.20) we obtain

$$S_2 \leq C \left\{ h^2 \int_{\tilde{\Omega}_2} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{\Omega}_2} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \quad (3.2.23)$$

Let us now estimate  $S_3$ . Using Theorem 2.5.1(ii) we have

$$\begin{aligned} S_3 &\leq C \sum_{R \in \Omega_3} \left\{ h_{R,1}^2 \int_{\tilde{R}} \delta_{-, \tilde{R}}^{2\alpha}(x) \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{R,2}^2 \int_{\tilde{R}} \left( \frac{x_2 - \tilde{R}_{12}}{h_{R,2}} \right)^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\ &=: \sum_{R \in \Omega_3} I_R. \end{aligned}$$

If  $R \subset \Omega_3$  is such that  $\tilde{R} \cap \{x : x_2 = 0\} = \emptyset$  then, for  $T \subset \tilde{R}$ ,

$$h_{T,1} \leq Ch^{\frac{1}{1-\alpha}} \quad h_{T,2} \leq Chx_2^\alpha \quad \forall (x_1, x_2) \in T,$$

and so

$$\begin{aligned} |I_R| &\leq C \sum_{T \subset \tilde{R}} \left\{ h_{T,1}^{2-2\alpha} \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h_{T,2}^2 \int_T \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\} \\ &\leq \sum_{T \subset \tilde{R}} \left\{ h^2 \int_T x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_T x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \end{aligned} \quad (3.2.24)$$

If  $\tilde{R} \cap \{x : x_2 = 0\} \neq \emptyset$  then  $\tilde{R}_{12} = 0$  and so (3.2.24) can be obtained also for this case using similar arguments. Therefore, we have

$$S_3 \leq C \left\{ h^2 \int_{\tilde{\Omega}_3} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 dx + h^2 \int_{\tilde{\Omega}_3} x_2^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \right\}. \quad (3.2.25)$$

Finally, to estimate  $S_4$ , note that  $\Omega_4$  contains only one element  $R$ . Now, using Theorem 2.5.1(iii) and the fact that for this element  $h_{R,1} = h_{R,2} = h^{\frac{1}{1-\alpha}}$  we obtain

$$S_4 \leq Ch^2 \left\{ \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{R})}^2 + \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}^2 + \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{R})}^2 \right\}. \quad (3.2.26)$$

Collecting the inequalities (3.2.19), (3.2.23), (3.2.25) and (3.2.26) we obtain (3.2.8) concluding the proof.  $\square$

As a consequence of Theorem 3.2.4 and the a priori estimates for the solution of problem (3.2.1) we obtain the following error estimates for the finite element approximations obtained using the family of meshes  $\mathcal{T}_{h,\alpha}$ . To simplify notation we omit the subscript  $\alpha$  in the approximate solution.

**Corollary 3.2.5.** *Let  $u$  be the solution of (3.2.1) and  $u_h$  its  $\mathcal{Q}_1$  finite element approximation obtained using the mesh  $\mathcal{T}_{h,\alpha}$  with  $\frac{1}{2} \leq \alpha < 1$ . If  $N$  is the number of nodes of  $\mathcal{T}_{h,\alpha}$  then, there exists a constant  $C$  independent of  $\varepsilon$  and  $N$  such that*

$$\|u - u_h\|_\varepsilon \leq C \frac{1}{1-\alpha} \frac{1}{\sqrt{N}} \log N. \quad (3.2.27)$$

*Proof.* From (3.2.2), Lemmas 3.2.1, 3.2.2 and 3.2.3, and Theorem 3.2.4 (and its extension to the rest of  $(0, 2) \times (0, 2)$ ) it follows that if  $h$  is small enough ( $h < \frac{1}{2}$  is sufficient) and  $\alpha \geq \frac{1}{2}$ , then

$$\|u - u_h\|_\varepsilon \leq Ch.$$

So we have to estimate  $h$  in terms of  $N$ . If we denote with  $M$  the number of nodes in each direction in the subdomain  $\Omega$ , we have  $N \sim M^2$  and we will estimate  $M$ . Let  $f(\xi) = \xi + h\xi^\alpha$ . Then,  $\xi_0 = 0$ ,  $\xi_1 = h^{\frac{1}{1-\alpha}}$  and  $\xi_{i+1} = f(\xi_i)$ ,  $i = 1, \dots, M_f - 1$  where  $M_f (= M)$  is the first number  $i$  such that  $\xi_i \geq 1$ . Since  $\alpha < 1$  we have that

$$f(\xi) > \xi + h\xi =: g(\xi), \quad \forall \xi \in (0, 1).$$

Now, consider the sequence  $\{\eta_i\}_{i=0}^{M_g}$  given by  $\eta_1 = \xi_1$ , and  $\eta_{i+1} = g(\eta_i)$ ,  $i = 2, \dots, M_g$  where  $M_g$  is defined analogously to  $M_f$ . Then, it is easy to see that  $M_f < M_g$  and therefore, it is enough to estimate  $M_g$ . But,  $M_g = [m]$  where  $m$  solves  $(1+h)^{m-1}\xi_1 = 1$ . Since  $\xi_1 = h^{\frac{1}{1-\alpha}}$ , for  $0 \leq h \leq 1$ , we obtain

$$\frac{1}{1-\alpha} \frac{1}{h} \log \frac{1}{h} \leq m - 1 \leq C \frac{1}{1-\alpha} \frac{1}{h} \log \frac{1}{h}. \quad (3.2.28)$$

Now, from inequalities (3.2.28) we easily arrive at

$$h \leq C \frac{1}{1-\alpha} \frac{1}{M} \log M \quad (3.2.29)$$

for all  $h$  small enough. □

*Remark 3.2.1.* If we use a uniform mesh with elements of side  $h$ , we can use inequalities (2.6.1), (2.6.2) and (2.6.3) to estimate the interpolation error. It is easy to see that for  $\alpha \geq \frac{1}{2}$  we obtain

$$\|u - \Pi u\|_{\varepsilon, D} \leq Ch^{1-\alpha}$$

where  $D = [0, 2]^2$ , and  $C$  is a constant independent of  $\varepsilon$  and  $\alpha$ . We observe that in our case we have  $|\tilde{D}_\partial| \leq 8h$  and that  $\|u\|_\infty \leq C$ , and so  $\|u\|_{0, \tilde{D}_\partial}^2 \leq Ch$ . In this way we obtain

$$\|u - u_h\|_{\varepsilon, D} \leq Ch^{1-\alpha}.$$

It results that the method converges with order  $\frac{1}{2}$  with respect to the mesh size  $h$ .

The Lemmas 3.2.1, 3.2.2 and 3.2.3 are straightforward consequences of the following estimates

$$\left| \frac{\partial^k u}{\partial x_1^k}(x_1, x_2) \right| \leq C \left\{ 1 + \varepsilon^{-k} e^{-\frac{x_1}{\varepsilon}} + \varepsilon^{-k} e^{-\frac{2-x_1}{\varepsilon}} \right\} \quad (3.2.30)$$

$$\left| \frac{\partial^k u}{\partial x_2^k}(x_1, x_2) \right| \leq C \left\{ 1 + \varepsilon^{-k} e^{-\frac{x_2}{\varepsilon}} + \varepsilon^{-k} e^{-\frac{2-x_2}{\varepsilon}} \right\} \quad (3.2.31)$$

provided that  $0 \leq k \leq 4$  and  $(x_1, x_2) \in [0, 2] \times [0, 2]$ , which are proved in [30]. As an example we prove the first inequality in (3.3.4). Observe that, for  $r = 0, 1, 2$ ,  $\frac{\partial^r u}{\partial x_i^r}(x_1, x_2) \equiv 0$  when  $x_2 = 0$  or  $x_2 = 2$  for  $i = 1$  and when  $x_1 = 0$  or  $x_1 = 2$  for  $i = 2$ . Then we have

$$\begin{aligned} & \int_0^2 \int_0^{\frac{3}{2}} x_1^{2\alpha} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx_1 dx_2 = - \int_0^{\frac{3}{2}} \int_0^2 x_1^{2\alpha} \frac{\partial u}{\partial x_1} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} dx_2 dx_1 \\ & = - \int_0^2 \left\{ x_1^{2\alpha} \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_2^2} \Big|_{x_1=0}^{\frac{3}{2}} - \int_0^{\frac{3}{2}} \frac{\partial}{\partial x_1} \left( x_1^{2\alpha} \frac{\partial u}{\partial x_1} \right) \frac{\partial^2 u}{\partial x_2^2} dx_1 \right\} dx_2 \\ & = - \left( \frac{3}{2} \right)^{2\alpha} \int_0^2 \frac{\partial u}{\partial x_1} \left( \frac{3}{2}, x_2 \right) \frac{\partial^2 u}{\partial x_2^2} \left( \frac{3}{2}, x_2 \right) dx_2 + \int_0^2 \int_0^{\frac{3}{2}} 2\alpha x_1^{2\alpha-1} \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_2^2} dx_1 dx_2 \\ & \quad + \int_0^2 \int_0^{\frac{3}{2}} x_1^{2\alpha} \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} dx_1 dx_2 \\ & =: I + II + III. \end{aligned} \quad (3.2.32)$$

Now, since



$$\begin{aligned}
\left| \frac{\partial u}{\partial x_1} \left( \frac{3}{2}, x_2 \right) \right| &\leq C \\
\left| \frac{\partial^2 u}{\partial x_2^2} \left( \frac{3}{2}, x_2 \right) \right| &\leq C(1 + \varepsilon^{-2}) \\
\left| \frac{\partial u}{\partial x_1} (x_1, x_2) \right| &\leq C(1 + \varepsilon^{-1} e^{-\frac{x_1}{\varepsilon}}) \quad (0 \leq x_1 \leq 3/2) \\
\left| \frac{\partial^2 u}{\partial x_1^2} (x_1, x_2) \right| &\leq C(1 + \varepsilon^{-2} e^{-\frac{x_1}{\varepsilon}}) \quad (0 \leq x_1 \leq 3/2) \\
\left| \frac{\partial^2 u}{\partial x_2^2} (x_1, x_2) \right| &\leq C(1 + \varepsilon^{-2})
\end{aligned}$$

we easily obtain

$$|I| \leq C(1 + \varepsilon^{-2}) \quad (3.2.33)$$

$$|II| \leq C(\varepsilon^{-2} + \varepsilon^{2\alpha-3}) \quad (3.2.34)$$

$$|III| \leq C(\varepsilon^{-2} + \varepsilon^{2\alpha-3}). \quad (3.2.35)$$

Now, using inequalities (3.2.33), (3.2.34) and (3.2.35) in (3.2.32) we conclude the proof.

### 3.3 An equation of convection-diffusion type

Let  $\Omega = (0, 1)^2$ . In this Section we consider the problem

$$\begin{aligned}
-\varepsilon \Delta u + b \cdot \nabla u + cu &= f \quad \text{in } \Omega \\
u &= u_D \quad \text{on } \Gamma_D \\
\varepsilon \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_N
\end{aligned} \quad (3.3.1)$$

with  $\varepsilon \in (0, 1]$  a small parameter. Only in some numerical examples we will consider  $\Gamma_N \neq \emptyset$ . Then, in what follows, we suppose  $\Gamma_D = \partial\Omega$  and  $\Gamma_N = \emptyset$ , and we make the simplifying assumption  $u_D \equiv 0$ . Also we suppose that the functions  $b = (b_1, b_2)$ ,  $c$  and  $f$  are smooth on  $\bar{\Omega}$ , and that  $b_i < -\gamma$  with  $\gamma > 0$  for  $i = 1, 2$ . Then the solution will in general vary rapidly in a layer region of width  $O(\varepsilon \log \frac{1}{\varepsilon})$  at the outflow boundary  $\{(x_1, x_2) \in \partial\Omega : x_1 = 0 \text{ or } x_2 = 0\}$  [36]. We assume the following compatibility conditions

$$\begin{aligned}
f(0, 0) &= f(1, 0) = f(1, 1) = f(0, 1) = 0 \\
\frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j} (1, 1) &= 0 \quad \text{for } 0 \leq i + j \leq 2,
\end{aligned}$$

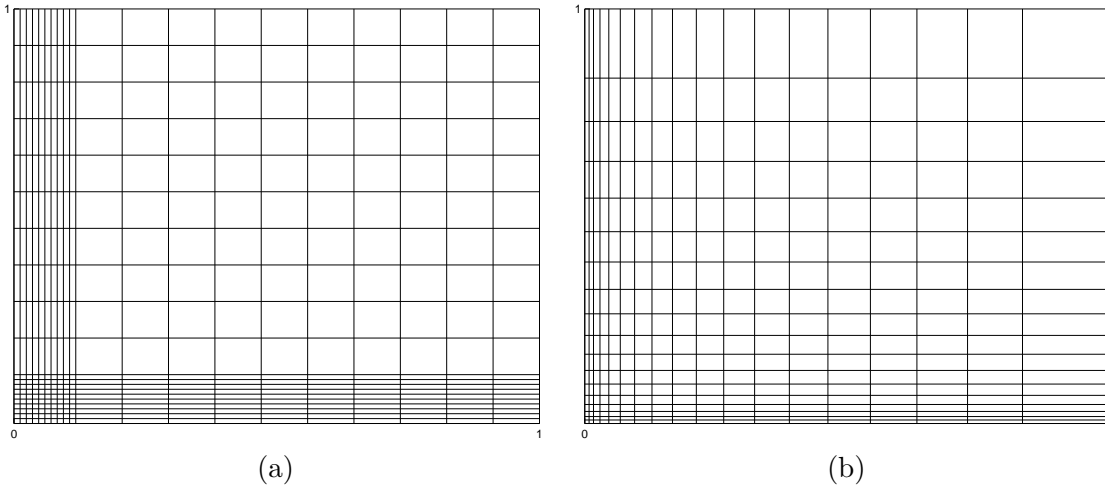


Figure 3.2: Examples of (a) Shishkin's and (b) graded meshes

in order to have precise estimates on certain derivatives of the solution  $u$ . Besides, in this case,  $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ .

This problem is considered, for example, by Stynes and O'Riordan [39]. They obtain uniform (in  $\varepsilon$ ) convergence for a finite element method that uses piecewise bilinears on a rectangular Shishkin mesh. Given an even natural  $M$ , set  $\sigma = \frac{2}{\gamma}\varepsilon \log M$  where  $\gamma$  is the constant introduced on page 71 (suppose  $\varepsilon$  small enough to have  $\sigma \leq \frac{1}{2}$ ). Then set  $t_i$ ,  $i = 0, \dots, M$  such that  $\{t_i\}$  is a partition of  $[0, 1]$  verifying that  $\{t_i\}_{i=0}^{\frac{M}{2}}$  is an equidistant partition of  $[0, \sigma]$  and  $\{t_i\}_{i=\frac{M}{2}}^M$  is an equidistant partition of  $[\sigma, 1]$ . The Shishkin mesh is then, the piecewise uniform rectangular mesh on  $\Omega$ , with nodes  $(t_i, t_j)$ ,  $i, j = 0, \dots, M$  (see Figure 3.2 (a)). If  $u^M$  is the piecewise bilinear finite element solution obtained on such a mesh, then Theorem 4.3 of [39] establishes that

$$\|u - u^M\|_{\varepsilon, \Omega} \leq C \frac{1}{M} \log M,$$

where, for a domain  $D$  we define

$$\|v\|_{\varepsilon, D}^2 = \|v\|_{0, D}^2 + \varepsilon \|\nabla v\|_{0, D}^2.$$

We will show, that if we take a graded mesh like that used in the previous Section, but with a more restrictive condition on the selection of the parameter  $\alpha$ , we can obtain a quasi-optimal estimate of the interpolation error  $\|u - \Pi u\|_{\varepsilon, \Omega}$ , with  $u$  the solution of (3.3.1)

and  $\Pi$  the average interpolation operator considered in Chapter 2. The parameter  $\alpha$ , in this case, will be taken depending on  $\varepsilon$ . As will be clear from what follows, this is because the boundary layers presented in general for the solution of the convection diffusion problem considered here, is considerably stronger than those that appear in solutions of reaction diffusion problems. Indeed, it is sufficient to compare the following *a priori* estimates with those of equations (3.2.30) and (3.2.31) (note that the coefficient of  $\Delta u$  in equation (3.2.1) is  $\varepsilon^2$ , whereas the one in equation (3.3.1) is  $\varepsilon$ ). We have (see [39])

$$\begin{aligned} \left| \frac{\partial^k u}{\partial x_1^k}(x_1, x_2) \right| &\leq C \left( 1 + \frac{1}{\varepsilon^k} e^{-\frac{\gamma x_1}{\varepsilon}} \right) \\ \left| \frac{\partial^k u}{\partial x_2^k}(x_1, x_2) \right| &\leq C \left( 1 + \frac{1}{\varepsilon^k} e^{-\frac{\gamma x_2}{\varepsilon}} \right) \end{aligned}$$

for all  $(x_1, x_2) \in \Omega$ ,  $0 \leq k \leq 2$ . As a consequence of these pointwise estimates we obtain the following Lemmas.

**Lemma 3.3.1.** *There exists a constant  $C$  such that if  $\alpha \geq \frac{1}{2}$  then*

$$\left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{0,\Omega} \leq C \quad \text{and} \quad \left\| x_2^\alpha \frac{\partial u}{\partial x_2} \right\|_{0,\Omega} \leq C. \quad (3.3.2)$$

**Lemma 3.3.2.** *There exists a constant  $C$  such that*

*i) if  $\alpha, \beta \geq 0$ , and  $\alpha + \beta \geq \frac{3}{2}$  then*

$$\varepsilon^\beta \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,\Omega} \leq C \quad \text{and} \quad \varepsilon^\beta \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,\Omega} \leq C, \quad (3.3.3)$$

*ii) if  $\alpha \geq \frac{1}{2}$  then*

$$\varepsilon^{\frac{1}{2}} \left\| x_1^\alpha \frac{\partial u}{\partial x_1 \partial x_2} \right\|_{0,\Omega} \leq C \quad \text{and} \quad \varepsilon^{\frac{1}{2}} \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,\Omega} \leq C. \quad (3.3.4)$$

**Lemma 3.3.3.** *There exists a constant  $C$  such that*

$$\left\| \frac{\partial u}{\partial x_1} \right\|_{1 \times \infty, \Omega} \leq C \quad \text{and} \quad \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times 1, \Omega} \leq C.$$

We now set

$$\alpha = 1 - \frac{1}{\log \frac{1}{\varepsilon}},$$

and, for  $0 < h < 1$ , consider the graded mesh  $\mathcal{T}_{h,\alpha}$  defined in Section 3.2, but restricted to the domain  $\Omega$  (see Figure 3.2 (b)). From Theorem 3.2.4 we have

$$\begin{aligned} \|u - \Pi u\|_{L^2(\Omega)} \leq Ch & \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|_{L^2(\tilde{\Omega})} + \left\| x_2^\alpha \frac{\partial u}{\partial x_2} \right\|_{L^2(\tilde{\Omega})} \right\} \\ & + Ch^{\frac{1}{2} \log \frac{1}{\varepsilon}} \left\{ \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty \times \underline{1}, \tilde{\Omega}} + \left\| \frac{\partial u}{\partial x_1} \right\|_{\underline{1} \times \infty, \tilde{\Omega}} \right\}. \end{aligned}$$

Assume  $\varepsilon \leq \frac{1}{e^2} \approx 0.1353$ . Then  $\alpha \geq \frac{1}{2}$  and  $\frac{1}{2} \log \frac{1}{\varepsilon} \geq 1$ , and so, we can apply Lemmas 3.3.1 and 3.3.3 to obtain

$$\|u - \Pi u\|_{L^2(\Omega)} \leq Ch.$$

Also from Theorem 3.2.4 we have

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \left\| \frac{\partial(u - \Pi u)}{\partial x_1} \right\|_{L^2(\Omega)} \leq Ch & \left( \varepsilon^{-\frac{1}{\log \frac{1}{\varepsilon}}} \varepsilon^{\frac{1}{2} + \frac{1}{\log \frac{1}{\varepsilon}}} \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\tilde{\Omega})} \right. \\ & \left. + \varepsilon^{\frac{1}{2}} \left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\tilde{\Omega})} \right). \end{aligned}$$

Since

$$\alpha \geq \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{\log \frac{1}{\varepsilon}} + \alpha = \frac{3}{2}, \quad \varepsilon^{-\frac{1}{\log \frac{1}{\varepsilon}}} = e,$$

we can use Lemma 3.3.2 to obtain

$$\varepsilon^{\frac{1}{2}} \left\| \frac{\partial(u - \Pi u)}{\partial x_1} \right\|_{L^2(\Omega)} \leq Ch.$$

Clearly, a similar estimate holds for  $\frac{\partial(u - \Pi u)}{\partial x_2}$ . We resume these results in the following theorem.

**Theorem 3.3.4.** *Let  $u$  be the solution of 3.3.1, with  $\Gamma_D = \partial\Omega$  and  $u_D \equiv 0$ . Let  $\Pi u$  be the average interpolation of  $u$  on the restriction to  $\Omega$  of the mesh  $\mathcal{T}_{h,\alpha}$  with  $0 < h < 1$  and  $\alpha = 1 - \frac{1}{\log \frac{1}{\varepsilon}}$ . If  $\varepsilon \leq \frac{1}{e^2}$  then*

$$\|u - \Pi u\|_{\varepsilon, \Omega} \leq Ch \tag{3.3.5}$$

where  $C$  depends on  $b, c$  and  $f$ , but it is independent of  $h$  and  $\varepsilon$ .

Note that since

$$\frac{1}{1 - \alpha} = \log \frac{1}{\varepsilon},$$

as in equation (3.2.29) in the proof of Corollary 3.2.5, we have

$$h \leq C \frac{1}{\sqrt{N}} \log \frac{1}{\varepsilon} \log N,$$

where  $N$  is the number of nodes in the mesh  $\mathcal{T}_{h,\alpha}$  restricted to the domain  $\Omega$  (in this case,  $N = M^2$  if  $M$  is as in Corollary 3.2.5). Then we can restate the inequality (3.3.5) as

$$\|u - \Pi u\|_{\varepsilon,\Omega} \leq C \frac{1}{\sqrt{N}} \log \frac{1}{\varepsilon} \log N.$$

This estimate is quasi-optimal, having in mind that  $\log \frac{1}{\varepsilon}$  and  $\log N$  are depreciable in practice.

It would be desirable to have an estimate like (3.3.5) with  $\Pi u$  replaced by  $u_h$ , the piecewise bilinear finite element solution on the particular mesh considered here. For the moment, we do not know how to deduce it. It must be noted that the Cea's Lemma can not be applied (in order to obtain an estimate quasi-uniformly in  $\varepsilon$ ) because the bilinear form associated with (3.3.1)

$$\varepsilon \int_{\Omega} (\nabla v \nabla w \, dx + b \cdot \nabla v w + c v w) \, dx$$

is (under suitable conditions on  $b$  and  $c$ ) coercive but not continuous uniformly in  $\varepsilon$ , for the norm  $\|\cdot\|_{\varepsilon,\Omega}$ . However, in some numerical examples we observe a good adequation of the discrete solution  $u_h$  that would indicate that such an estimate could hold true. In Figure 3.3 we present some pictures of numerical solutions of singularly perturbed convection diffusion problems obtained using continuous piecewise bilinears on graded meshes like those previously described. For the examples presenting boundary layers only on  $0 \times [0, 1]$ , we have used meshes graded only along the  $x_1$  axis. For all the examples we have considered  $\varepsilon = 10^{-6}$  and the following data: (a)  $b = (-1, 0)$ ,  $\Gamma_D = \{0, 1\} \times [0, 1]$ ,  $\Gamma_N = [0, 1] \times \{0, 1\}$ ,  $u_D = 0$ ,  $g = 0$  and  $f = 1$ ; (b)  $b, \Gamma_D$  and  $\Gamma_N$  as in (a),  $u_D = 0$  on  $\{0\} \times [0, 1]$  and  $u_D = 1$  on  $\{1\} \times [0, 1]$ ,  $g = 0$  and  $f = 0$ ; (c)  $b = (-\frac{1}{2}, -1)$ ,  $\Gamma_D = \partial\Omega$ ,  $g = 0$ , and  $f = 1$ ; (d)  $b = (-1, -1)$ ,  $\Gamma_D = \partial\Omega$ ,  $g = 0$  and  $f(x, y) = -x + (1 - \exp(-x))(1 - \exp(-\frac{1}{\varepsilon}))^{-1} - y + (1 - \exp(-y))(1 - \exp(-\frac{1}{\varepsilon}))^{-1}$ . The exact solution of example (d) is give by

$$u(x, y) = \left( x - \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \left( y - \frac{1 - e^{-\frac{y}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right).$$

Using this exact solution, for  $\varepsilon = 10^{-4}$ , in Tables 3.1 and 3.2 we study the numerical convergence order of the finite element method, using graded and Shishkin's meshes respectively.

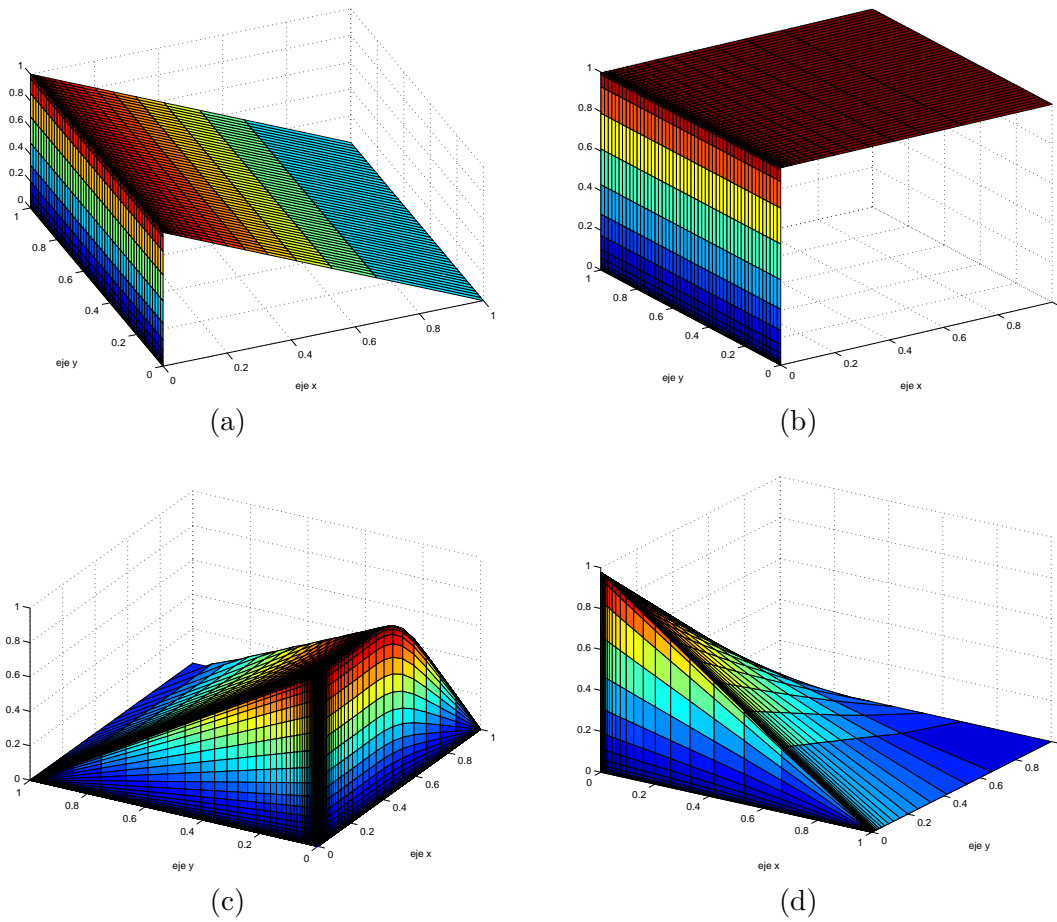


Figure 3.3: Examples of singularly perturbed equations solved using graded meshes

*Remark 3.3.1.* As in Remark 3.2.1 we can note that on a uniform mesh with elements of side  $h$ , inequalities (2.6.1), (2.6.2) and (2.6.3) can be used to estimate the interpolation error. For the  $L^2$  norm we have for  $\frac{1}{2} \leq \alpha \leq 1$

$$\|u - \Pi u\|_{0,\Omega} \leq Ch^{1-\alpha}$$

where  $D = [0, 2]^2$ , and  $C$  is a constant independent of  $\alpha$  and  $\varepsilon$ . Notice that in this case we have again  $|\tilde{D}_\partial| \leq 4h$  and  $\|u\|_\infty \leq C$ , and so  $\|u\|_{0,\tilde{D}_\partial}^2 \leq Ch$ . By the other hand, for the  $H^1$

N	Error	Order of Conv.
1156	0.0667	-
5476	0.0347	0.42
22801	0.0183	0.45
90601	0.0095	0.48

Table 3.1: Convergence on graded meshes in the energy norm

N	Error	Order of Conv.
1369	0.0660	-
5929	0.0379	0.38
23409	0.0220	0.40
91809	0.0126	0.41

Table 3.2: Convergence on Shishkin's meshes in the energy norm

seminorm we have also for  $\frac{1}{2} \leq \alpha \leq 1$

$$\varepsilon^{\frac{1}{2}} |u - \Pi u|_{1,D} \leq Ch^{1-\alpha} \left[ \varepsilon^{-(1-\alpha)} \left( \varepsilon^{\frac{3}{2}-\alpha} \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{0,\Omega} + \varepsilon^{\frac{3}{2}-\alpha} \left\| x_2^\alpha \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,\Omega} \right) + \varepsilon^{\frac{1}{2}} \left\| (x_1^\alpha + x_2^\alpha) \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{0,\Omega} \right].$$

Now, in order to eliminate the factor  $\varepsilon^{-(1-\alpha)}$  of the right hand side, we can take  $\alpha = 1 - \frac{1}{\log \frac{1}{\varepsilon}}$ , making the error of the order  $h^{\frac{1}{\log \frac{1}{\varepsilon}}}$ . So, if we want to achieve a tolerance of  $s$ , for example, we need  $h^{\frac{1}{\log \frac{1}{\varepsilon}}} = s$ , that is,  $h = \varepsilon^{\log \frac{1}{s}}$ . If  $\varepsilon$  is small, the mesh may result too fine to work in practice.

### 3.4 A fourth order equation

Let  $\Omega = (0, 1)^2$  and  $f \in L^2(\Omega)$ . In this Section we consider the problem

$$\begin{aligned} \varepsilon^2 \Delta^2 u - \Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4.1}$$

We introduce the following bilinear forms defined for functions  $v, w \in H^2(\Omega)$

$$a(v, w) = \int_{\Omega} D^2v : D^2w \, dx, \quad b(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx.$$

where  $D^2v$  is the  $2 \times 2$  matrix  $\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{ij}$ , and if  $A$  and  $B$  are  $2 \times 2$  matrices we set  $A : B = \sum_{i,j=1}^2 A_{ij} B_{ij}$ . Note that if  $v, w \in H_0^2(\Omega)$  then

$$a(v, w) = \int_{\Omega} \Delta v \Delta w \, dx.$$

Therefore, a function  $u \in H_0^2(\Omega)$  is a weak solution of (3.4.1) if

$$\varepsilon^2 a(u, v) + b(u, v) = \int_{\Omega} f v \, dx \quad \forall v \in H_0^2(\Omega) \quad (3.4.2)$$

It follows from the regularity theory for elliptic problems in nonsmooth domains (see [23] Corollary 7.3.2.5) that if  $f \in H^{-1}(\Omega)$  then  $u \in H_0^2(\Omega) \cap H^3(\Omega)$ , and

$$\|u\|_{3,\Omega} \leq c \|f\|_{-1,\Omega}.$$

However, the constant  $c$  will in general depend on  $\varepsilon$  and will blow up as  $\varepsilon$  goes to zero. We will need to consider the solution of the reduced problem, that is,  $u^0 \in H_0^1(\Omega) \cap H^2(\Omega)$  such that

$$\begin{aligned} -\Delta u^0 &= f & \text{in } \Omega \\ u^0 &= 0 & \text{on } \partial\Omega \end{aligned} \quad (3.4.3)$$

Since  $\Omega$  is convex we know that

$$\|u^0\|_2 \leq C \|f\|_0. \quad (3.4.4)$$

The following Lemma is proved in [33].

**Lemma 3.4.1** ([33], Lemma 5.1). *There is a constant  $C$ , independent of  $\varepsilon$  and  $f$ , such that*

$$|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} \leq C \varepsilon^{-\frac{1}{2}} \|f\|_{0,\Omega}, \quad |u - u^0|_{1,\Omega} \leq C \varepsilon^{\frac{1}{2}} \|f\|_{0,\Omega}.$$

To approximate the solution  $u$ , we will consider a nonconforming finite element, the Adini rectangle [17], that corresponds to the following data  $K, P_K$  and  $\Sigma_K$ : the set  $K$  is a rectangle whose vertices  $a_i$ ,  $1 \leq i \leq 4$ , are counted as in Figure 3.4,  $P_K$  is defined by

$$P_K = P_3(K) \oplus \langle \{x_1 x_2^3, x_1^3 x_2\} \rangle$$



and the degrees of freedom  $\Sigma_K$  are given by

$$\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), 1 \leq i \leq 4\}.$$

Notice that the inclusion  $P_3(K) \subset P_K$  holds,  $\dim(P_K) = 12$ , and the set  $\Sigma_K$  is  $P_K$ -unisolvent.

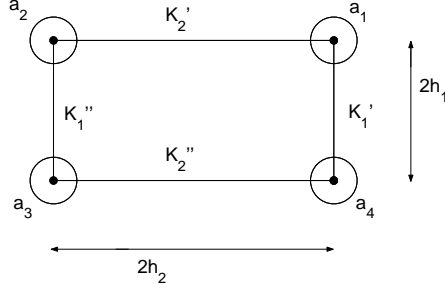


Figure 3.4: Adini's rectangle  $K$ .

Let  $\mathcal{T}_h$  be a triangulation of  $\Omega$  made of rectangles. We assume that  $\mathcal{T}_h$  is quasi-uniform, that is, there exist positive constants  $c_1 \leq c_2$  such that

$$c_1 h \leq h_{T,j} \leq c_2 h \quad \forall T \in \mathcal{T}_h,$$

where  $h_{T,j}$  is the length of the side of  $T$  in the  $x_j$ -direction. So, the length of the sides of all the rectangles  $T \in \mathcal{T}_h$  are of order  $h$ . We will use this fact repeatedly without make explicit mention.

With such a triangulation  $\mathcal{T}_h$ , we associate a finite element space  $X_h$ , such that,  $v_h \in X_h$  if  $v_h|_K \in P_K$  for all  $K \in \mathcal{T}_h$ , and  $v_h$  is defined by its values and the values of its first derivatives at all the vertices of the triangulation. Finally, let  $V_h$  be the space of all functions in  $X_h$  such that  $v_h(b) = \partial_1 v_h(b) = \partial_2 v_h(b) = 0$  at all the boundary nodes  $b$ . Functions in  $V_h$  are continuous and they vanish on  $\partial\Omega$ , but in general, they are not in  $H^2(\Omega)$  and their normal derivatives do not vanish along  $\partial\Omega$ , although  $\partial_n v_h(b) = 0$  for all boundary node  $b$  and  $v_h \in V_h$ .

Now, we consider the following discrete version of problem (3.4.2): Find  $u_h \in V_h$  such that

$$\varepsilon^2 a_h(u_h, v) + b(u_h, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_h \quad (3.4.5)$$

where the bilinear form  $a_h$  is defined by

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 w \, dx.$$

If we consider the norm  $\|\cdot\|_{\varepsilon, h}$  given by

$$\|v\|_{\varepsilon, h}^2 = \varepsilon^2 a_h(v, v) + b(v, v) \quad v \in H^2(\Omega) \cap H_0^1(\Omega)$$

it follows that the bilinear form  $\varepsilon^2 a_h(\cdot, \cdot) + b(\cdot, \cdot)$  is  $V_h$ -coercive, uniformly in  $h$  and  $\varepsilon$ . As we are considering a nonconforming method, the error estimate follows from the Second Strang Lemma [17], which establishes that

$$\|u - u_h\|_{\varepsilon, h} \leq \inf_{v \in V_h} \|u - v\|_{\varepsilon, h} + \sup_{w \in V_h} \frac{|D_{\varepsilon, h}(u, w)|}{\|w\|_{\varepsilon, h}}, \quad (3.4.6)$$

where the consistency error  $D_{\varepsilon, h}(v, w)$  is given by

$$D_{\varepsilon, h}(v, w) = \varepsilon^2 a_h(v, w) + b(v, w) - \int_{\Omega} f w \, dx. \quad (3.4.7)$$

In order to bound the first term in the r.h.s. of (3.4.6), the approximation error, we could estimate  $\|u - \Pi_h u\|_{\varepsilon, h}$ , where  $\Pi_h$  is the associated  $X_h$ -interpolation operator. It verifies

$$v \in H_0^2(\Omega) \cap H^3(\Omega) \Rightarrow \Pi_h v \in V_h,$$

and, as a consequence of the Bramble-Hilbert Lemma, we have

$$\|v - \Pi_h v\|_{\varepsilon, h} \leq C(h + \varepsilon h^2)|v|_3 \quad \forall v \in H^3(\Omega). \quad (3.4.8)$$

However, in view of the *a priori* estimates given by Lemma (3.4.1), we would not obtain results independent of  $\varepsilon$ . Then we need to obtain interpolation error estimates like (3.4.8), but with a weaker dependence on the function  $v$ . In order to do that, we consider the interpolation operator introduced in [22].

Each function in  $X_h$  is determined univocally by evaluations of the function and its first derivatives at all the vertices of the triangulation  $\mathcal{T}_h$ . We denote these degrees of freedom (nodes) as  $z_i = (z_i, D_i)$ ,  $i = 1 \dots N$ , where for each  $i$ ,  $z_i$  is a vertex, and  $D_i$  is the differential operator associated with the corresponding nodal variable, i. e., for a function  $v$ ,  $D_i(v)$  is either the same function  $v$  or  $\frac{\partial v}{\partial x_1}$  or  $\frac{\partial v}{\partial x_2}$ , and so the nodal variable becomes  $D_i v(z_i)$ . Let  $\{\phi_i, i = 1 \dots N\}$  be the corresponding nodal basis. Therefore, we have

$$v = \sum_{i=1}^N D_i v(z_i) \phi_i \quad \forall v \in X_h.$$

For each node  $z_i$  we choose an element  $T_i$  and an edge  $e_i$ , such that,  $z_i \in e_i \subset T_i$ , with the additional condition that  $e_i \subset \partial\Omega$  if  $z_i \in \partial\Omega$ . By the Riesz Representation Theorem, for each  $i$ , there exists a polynomial  $\psi_{z_i}^{e_i} \in \mathcal{P}_3(e_i)$  such that

$$\int_{e_i} f \psi_{z_i}^{e_i} ds = f(z_i) \quad \forall f \in \mathcal{P}_3(e_i).$$

Taking into account that  $\{f|_{e_i} : f \in P_{T_i}\} = \mathcal{P}_3(e_i)$ , we can write

$$v = \sum_{i=1}^N \left( \int_{e_i} D_i(v|_{T_i}) \psi_{z_i}^{e_i} ds \right) \phi_i \quad \forall v \in X_h.$$

Now, for a function  $v \in H^2(\Omega) + X_h$  we define the interpolation

$$I_h v = \sum_{i=1}^N \left( \int_{e_i} D_i(v|_{T_i}) \psi_{z_i}^{e_i} ds \right) \phi_i.$$

It follows that

$$I_h v \in X_h \quad \forall v \in H^2(\Omega), \quad I_h v = v \quad \forall v \in X_h$$

and that

$$I_h v \in V_h \quad \forall v \in H_0^2(\Omega).$$

The following lemmas can be obtained by standard scaling arguments, and therefore we omit their proofs. For the operator  $D_i$ , we put  $|D_i| = 0$  if  $D_i v(z_i) = v(z_i)$  or  $|D_i| = 1$  if  $D_i v(z_i) = \frac{\partial v}{\partial x_j}(z_i)$ , with  $j = 1, 2$ .

**Lemma 3.4.2.** *There exists a constant  $C$ , such that if  $\phi_i$  is a nodal basis function corresponding to the node  $(z_i, D_i)$ , then*

$$\|\phi_i\|_{0,K} \leq Ch^{|D_i|+1}, \quad \left\| \frac{\partial \phi_i}{\partial x_j} \right\|_{0,K} \leq Ch^{|D_i|}, \quad \left\| \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} \right\|_{0,K} \leq Ch^{|D_i|-1},$$

for all  $K \in \mathcal{T}_h$  and  $j, k = 1, 2$ .

**Lemma 3.4.3.** *If  $h_{e_i}$  is the length of the edge  $e_i$ , then there exists a constant  $C$  such that*

$$\|\psi_{z_i}^{e_i}\|_{0,e_i} \leq Ch_{e_i}^{-\frac{1}{2}}.$$

If  $\hat{e}$  is an edge of the reference element  $\hat{K}$  we have the following two trace inequalities

$$\|v\|_{0,\hat{e}} \leq C\|v\|_{1,\hat{K}}, \quad \|v\|_{0,\hat{e}} \leq C\|v\|_{0,\hat{K}}^{\frac{1}{2}}\|v\|_{1,\hat{K}}^{\frac{1}{2}},$$

for all  $v \in H^1(\hat{K})$ , see for example [23]. Then we obtain the following Lemma.

**Lemma 3.4.4.** *If  $e$  is an edge of an element  $K \in \mathcal{T}_h$  then we have the following trace inequalities*

$$\|v\|_{0,e} \leq C \left( h^{-\frac{1}{2}} \|v\|_{0,K} + h^{\frac{1}{2}} |v|_{1,K} \right), \quad (3.4.9)$$

$$\|v\|_{0,e} \leq C \|v\|_{0,K}^{\frac{1}{2}} \left( h^{-1} \|v\|_{0,K} + |v|_{1,K} \right)^{\frac{1}{2}}, \quad (3.4.10)$$

for all  $v \in H^1(K)$ .

Now we deal with the stability of the interpolation operator  $I_h$ . Given  $T \in \mathcal{T}_h$ , let  $\omega_T = \cup\{K \in \mathcal{T}_h : K \cap T \neq \emptyset\}$ . Suppose that the firsts  $n_1$  nodes belong to the element  $T$ . We have, using the Lemmas (3.4.2) and (3.4.3),

$$\begin{aligned} \|I_h v\|_{0,T} &\leq \sum_{i=1}^{n_1} \left| \int_{e_i} D_i v \psi_{z_i}^{e_i} ds \right| \|\phi_i\|_{0,T} \\ &\leq \sum_{i=1}^{n_1} \|D_i v\|_{0,e_i} \|\psi_{z_i}^{e_i}\|_{0,e_i} \|\phi_i\|_{0,T} \\ &\leq C \sum_{i=1}^{n_1} h^{|D_i| + \frac{1}{2}} \|D_i v\|_{0,e_i} \end{aligned}$$

Applying the trace inequality (3.4.9) we obtain

$$\|I_h v\|_{0,T} \leq C \left( \|v\|_{0,\omega_T} + h|v|_{1,\omega_T} + h^2|v|_{2,\omega_T} \right) \quad (3.4.11)$$

while, using the inequality (3.4.10) we obtain

$$\|I_h v\|_{0,T} \leq C \left\{ h^{\frac{1}{2}} \|v\|_{0,\omega_T}^{\frac{1}{2}} \left( h^{-1} \|v\|_{0,\omega_T} + |v|_{1,\omega_T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} |v|_{1,\omega_T} \left( h^{-1} |v|_{1,\omega_T} + |v|_{2,\omega_T} \right)^{\frac{1}{2}} \right\}. \quad (3.4.12)$$

Similarly, we have

$$\begin{aligned} |I_h v|_{1,T} &\leq \sum_{i=1}^{n_1} \left| \int_{e_i} D_i v \psi_{z_i}^{e_i} ds \right| |\phi_i|_{1,T} \\ &\leq C \sum_{i=1}^{n_1} h^{|D_i| - \frac{1}{2}} \|D_i v\|_{0,e_i}, \end{aligned}$$

and using the trace inequalities (3.4.9) and (3.4.10) we obtain, respectively,

$$|I_h v|_{1,T} \leq C \left( h^{-1} \|v\|_{0,\omega_T} + |v|_{1,\omega_T} + h|v|_{2,\omega_T} \right) \quad (3.4.13)$$

$$\begin{aligned} |I_h v|_{1,T} &\leq C \left\{ h^{-\frac{1}{2}} \|v\|_{0,\omega_T}^{\frac{1}{2}} \left( h^{-1} \|v\|_{0,\omega_T} + |v|_{1,\omega_T} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + h^{\frac{1}{2}} |v|_{1,\omega_T} \left( h^{-1} |v|_{1,\omega_T} + |v|_{2,\omega_T} \right)^{\frac{1}{2}} \right\}. \end{aligned} \quad (3.4.14)$$

Finally, from the Lemmas (3.4.2) and (3.4.3) we have

$$\begin{aligned} |I_h v|_{2,T} &\leq \sum_{i=1}^{n_1} \left| \int_{e_i} D_i v \psi_{z_i}^{e_i} ds \right| |\phi_i|_{2,T} \\ &\leq C \sum_{i=1}^{n_1} h^{|D_i| - \frac{3}{2}} \|D_i v\|_{0,e_i}, \end{aligned}$$

and after applying the inequality (3.4.9) it results

$$|I_h v|_{2,T} \leq C (h^{-2} \|v\|_{0,\omega_T} + h^{-1} |v|_{1,\omega_T} + |v|_{2,\omega_T}) \quad (3.4.15)$$

(in this case we do not need to use the other trace inequality).

Now we are ready to deal with the interpolation error for the operator  $I_h$ . In the proofs of the following two Lemmas we will use properties of the averaged Taylor polynomials, that can be found in [12]. If  $B = \{x \in \mathbb{R}^2 : |x - x_0| \leq r\}$ , we consider the function

$$\rho_B(x) = \begin{cases} \frac{1}{c} e^{-\left(1 - \frac{|x-x_0|}{r}\right)^{-1}} & \text{if } |x - x_0| < r \\ 0 & \text{if } |x - x_0| \geq r, \end{cases}$$

with  $c = \int_{|x| < r} e^{-\left(1 - \frac{x}{r}\right)^{-1}} dx$ . The function  $\rho_B$  satisfies

$$\text{supp } \rho_B \subset B, \quad \int_{\mathbb{R}^2} \rho_B dx = 1, \quad \max |\rho_B| \leq \text{const.} r^{-2}.$$

Following [12], the Taylor polynomial of order  $m$  evaluated at  $y$  of a function  $v$  is given by

$$T_y^m v(x) = \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha v(y) (x - y)^\alpha$$

where  $\alpha = (\alpha_1, \alpha_2)$  is a multi-index. Suppose  $v \in H^{m-1}(D)$  for a region  $D$ , and that  $B \subset\subset D$ , then the Taylor polynomial of order  $m$  of  $v$  averaged over  $B$  is defined as

$$Q^m v(x) = \int_B T_y^m v(x) \rho_B(y) dy.$$

It results that  $Q^m v$  is a polynomial of degree less than  $m$  in  $x$ . Let  $d$  be the diameter of  $D$ , and suppose that  $D$  is star-shaped with respect to the ball  $B$ . Let  $r_{\max} = \sup\{r : D \text{ is star-shaped with respect to a ball of radius } r\}$ . Then the chunkiness parameter of  $D$  is defined by

$$\gamma = \frac{d}{r_{\max}}.$$

If further the radius  $r$  of  $B$  verifies  $r > \frac{1}{2}r_{\max}$ , then the version of the Bramble-Hilbert Lemma given in [12] states that for all  $v \in H^m(D)$

$$|v - Q^m v|_{k,D} \leq C_{m,n,\gamma} d^{m-k} |v|_{m,D}, \quad k = 0, 1, \dots, m.$$

It follows that for each element  $T$  of the quasi-uniform mesh  $\mathcal{T}_h$ , it is possible to choose a ball  $B_T \subset \omega_T$  such that

$$|v - Q^m v|_{k,\omega_T} \leq Ch^{m-k} |v|_{m,\omega_T}, \quad k = 0, 1, \dots, m,$$

if  $Q^m v$  is the averaged polynomial of  $v$  of degree  $m$  with respect to the ball  $B_T$ , where the constant  $C$  is independent of  $T$  (see [12], Sections 3.3 and 3.4 for details).

**Lemma 3.4.5.** *There exists a constant  $C$  independent of  $h$  such that*

$$\left( \sum_{T \in \mathcal{T}_h} \|v - I_h v\|_{j,T}^2 \right)^{\frac{1}{2}} \leq Ch^{k-j} |v|_k \quad \forall v \in H^k(\Omega) \quad (3.4.16)$$

where  $j = 1, 2$  and  $k = 2, 3$ .

*Proof.* Let  $T \in \mathcal{T}_h$  fixed, and let  $Q^2 v$  be the averaged Taylor polynomial of order 2 with respect to a ball, like was introduced above. Since  $Q^2 v$  a polynomial of degree less than 2, then

$$(v - I_h v)|_T = (v - Q^2 v)|_T + I_h(v - Q^2 v)|_T. \quad (3.4.17)$$

It follows from inequalities (3.4.11) and (3.4.13), that

$$\|I_h(v - Q^2 v)\|_{1,T} \leq C (h^{-1} \|v - Q^2 v\|_{0,\omega_T} + |v - Q^2 v|_{1,\omega_T} + h|v - Q^2 v|_{2,\omega_T})$$

Then it results

$$\|I_h(v - Q^2 v)\|_{1,T} \leq Ch|v|_{2,\omega_T}.$$

In a similar way, from inequalities (3.4.11), (3.4.13) and (3.4.15) it follows that

$$\|I_h(v - Q^2 v)\|_{2,T} \leq C (h^{-2} \|v - Q^2 v\|_{0,\omega_T} + h^{-1} |v - Q^2 v|_{1,\omega_T} + |v - Q^2 v|_{2,\omega_T})$$

and then

$$\|I_h(v - Q^2 v)\|_{2,T} \leq C|v|_{2,\omega_T}.$$

Then using the triangle inequality and the splitting (3.4.17), we easily obtain the inequality (3.4.16) for  $j = 1, 2$  and  $k = 2$ . Case  $k = 3$  can be proved analogously, using now the averaged polynomial  $Q^3 v$ .  $\square$

**Lemma 3.4.6.** *There exists a constant  $C$  such that*

$$\|v - I_h v\|_1 \leq Ch^{\frac{1}{2}} |v|_1^{\frac{1}{2}} |v|_2^{\frac{1}{2}} \quad \forall v \in H^1(\Omega). \quad (3.4.18)$$

*Proof.* For each element  $T \in \mathcal{T}_h$ , let  $Q^1 v$  and  $Q^2 v$  be the averaged Taylor polynomials with respect to a ball  $B_T$  introduced before the Lemma 3.4.5. We consider the splitting (3.4.17). From inequalities (3.4.12) and (3.4.14) we have

$$\begin{aligned} \|I_h(v - Q^2 v)\|_{1,T} \leq & C \left\{ h^{-\frac{1}{2}} \|v - Q^2 v\|_{0,\omega_T}^{\frac{1}{2}} (h^{-1} \|v - Q^2 v\|_{0,\omega_T} + |v - Q^2 v|_{1,\omega_T})^{\frac{1}{2}} \right. \\ & \left. + h^{\frac{1}{2}} |v - Q^2 v|_{1,\omega_T} (h^{-1} |v - Q^2 v|_{1,\omega_T} + |v - Q^2 v|_{2,\omega_T})^{\frac{1}{2}} \right\}. \end{aligned} \quad (3.4.19)$$

It is easy to prove that

$$\|Q^1 v - Q^2 v\|_{0,\omega_T} \leq Ch |v|_{1,\omega_T}$$

and then it follows

$$\|v - Q^2 v\|_{0,\omega_T} \leq \|v - Q^1 v\|_{0,\omega_T} + \|Q^1 v - Q^2 v\|_{0,\omega_T} \leq Ch |v|_{1,\omega_T}.$$

Also one can check that

$$|Q^2 v|_{1,\omega_T} \leq C |v|_{1,\omega_T}.$$

Inserting these inequalities and

$$\|v - Q^2 v\|_{0,\omega_T} \leq Ch^2 |v|_{2,\omega_T}, \quad |v - Q^2 v|_{1,\omega_T} \leq Ch |v|_{2,\omega_T}$$

in (3.4.19) we obtain

$$\|I_h(v - Q^2 v)\|_{1,T} \leq Ch^{\frac{1}{2}} |v|_{1,\omega_T}^{\frac{1}{2}} |v|_{2,\omega_T}^{\frac{1}{2}}.$$

By the other hand we have

$$\begin{aligned} \|v - Q^2 v\|_{1,T} &= \|v - Q^2 v\|_{1,T}^{\frac{1}{2}} \|v - Q^2 v\|_{1,T}^{\frac{1}{2}} \\ &\leq Ch^{\frac{1}{2}} |v|_{2,T}^{\frac{1}{2}} \|v - Q^2 v\|_{1,T}^{\frac{1}{2}}. \end{aligned} \quad (3.4.20)$$

It is not difficult to prove that

$$\|Q^2 v - Q^1 v\|_{1,T} \leq C |v|_{1,T}$$

and then it follows

$$\|v - Q^2 v\|_{1,T} \leq \|v - Q^1 v\|_{1,T} + \|Q^1 v - Q^2 v\|_{1,T} \leq C |v|_{1,T}.$$

So, inserting this inequality in (3.4.20) we obtain

$$\|v - Q^2 v\|_{1,T} \leq Ch^{\frac{1}{2}} |v|_{1,\omega_T}^{\frac{1}{2}} |v|_{2,\omega_T}^{\frac{1}{2}}.$$

It follows from the triangle inequality that

$$\|v - I_h v\|_{1,T} \leq Ch^{\frac{1}{2}} |v|_{1,\omega_T}^{\frac{1}{2}} |v|_{2,\omega_T}^{\frac{1}{2}}.$$

Now by adding this inequality on all the elements  $T$ , we easily arrive to (3.4.18).  $\square$

Now we prove the following Lemma concerning with the consistency error defined in (3.4.7).

**Lemma 3.4.7.** *There exists a constant  $C$  such that*

$$D_{\varepsilon,h}(u, w_h) \leq C\varepsilon h^{\frac{1}{2}} |u|_{\frac{1}{2}}^{\frac{1}{2}} |u|_{\frac{2}{3}}^{\frac{1}{2}} \|w_h\|_{\varepsilon,h} \quad \forall w_h \in V_h \quad (3.4.21)$$

where  $u \in H_0^2(\Omega) \cap H^3(\Omega)$  is the solution of Problem (3.4.2).

*Proof.* First we prove that

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \Delta u - \frac{\partial^2 u}{\partial \tau^2} \right) \frac{\partial w_h}{\partial n} ds.$$

Since  $V_h \subset H_0^1(\Omega)$ , given  $w_h \in V_h$  there exists a sequence  $\{w_h^k\}_k \subset C_C^\infty(\Omega)$  such that  $w_h^k \rightarrow w_h$  in  $H^1(\Omega)$  when  $k \rightarrow \infty$ . We have

$$\sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 w_h^k dx = \int_\Omega D^2 u : D^2 w_h^k dx = \int_\Omega \Delta u \Delta w_h^k dx.$$

Then, after integrate by parts, we have

$$\sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 w_h^k dx = - \int_\Omega \nabla(\Delta u) \cdot \nabla w_h^k dx.$$

So, from the weak formulation (3.4.2) with  $v = w_h^k$  we obtain

$$-\varepsilon^2 \int_\Omega \nabla(\Delta u) \cdot \nabla w_h^k dx + \int_\Omega \nabla u \cdot \nabla w_h^k dx = \int_\Omega f w_h^k dx.$$

Letting  $k \rightarrow \infty$ , and taking into account that  $\|w_h - w_h^k\|_{1,\Omega} \rightarrow 0$  we arrive at

$$-\varepsilon^2 \int_\Omega \nabla(\Delta u) \cdot \nabla w_h dx + \int_\Omega \nabla u \cdot \nabla w_h dx = \int_\Omega f w_h dx.$$



Then, we have

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 \left( \sum_{T \in \mathcal{T}_h} \int_T D_2 u : D_2 w_h \, dx + \int_{\Omega} \nabla(\Delta u) \cdot \nabla w_h \, dx \right).$$

Let  $C^2 u = -\text{curl curl } u$ , that is

$$C^2 u = -\text{curl} \begin{pmatrix} -\frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 u}{\partial x_2^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} & -\frac{\partial^2 u}{\partial x_1^2} \end{pmatrix},$$

therefore  $D^2 u = \Delta u \mathbb{I} + C^2 u$ . For each  $T \in \mathcal{T}_h$  we have

$$\int_T \Delta u \mathbb{I} : D^2 w_h \, dx = \int_T \nabla(\Delta u) \cdot \nabla w_h \, dx + \int_{\partial T} \Delta u \frac{\partial w_h}{\partial n} \, ds$$

and

$$\begin{aligned} \int_T C^2 u : D^2 w_h \, dx &= - \int_T \text{curl} \begin{pmatrix} -\frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_1} \end{pmatrix} \cdot \nabla \frac{\partial w_h}{\partial x_1} \, dx - \int_T \text{curl} \frac{\partial u}{\partial x_1} \cdot \nabla \frac{\partial w_h}{\partial x_2} \, dx \\ &= - \int_{\partial T} \text{curl} \begin{pmatrix} -\frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_1} \end{pmatrix} \cdot n \frac{\partial w_h}{\partial x_1} \, ds - \int_{\partial T} \text{curl} \frac{\partial u}{\partial x_2} \cdot n \frac{\partial w_h}{\partial x_2} \, ds \\ &= \int_{\partial T} n^t C^2 u \nabla w_h \, ds. \end{aligned}$$

where  $n = n_T$  is the unit normal to  $\partial T$  pointing outward. Then

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \Delta u \frac{\partial w_h}{\partial n} + n^t C^2 u \nabla w_h \right) \, ds.$$

Let  $\tau$  a unit tangent vector to  $\partial T$  (which is piecewise continuous on  $\partial T$ ). Since  $w_h$  is a continuous function on  $\overline{\Omega}$  vanishing on  $\partial\Omega$ , we have that  $\frac{\partial w_h}{\partial \tau}$  is continuous along the interelements and besides  $\frac{\partial w_h}{\partial \tau} = 0$  on  $\partial\Omega$ . Then

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} n^t C^2 u \cdot \frac{\partial w_h}{\partial \tau} \tau \, ds = 0.$$

Taking into account that  $\nabla w_h = (\nabla w_h \cdot n)n + (\nabla w_h \cdot \tau)\tau$  we obtain

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \Delta u \frac{\partial w_h}{\partial n} + n^t C^2 u \cdot n \frac{\partial w_h}{\partial n} \right) \, ds.$$

Noting that

$$n^t C^2 u \cdot n = \frac{\partial^2 u}{\partial \tau^2}$$

we can write

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \Delta u - \frac{\partial^2 u}{\partial \tau^2} \right) \frac{\partial w_h}{\partial n} ds$$

as we wanted.

We decompose  $D_{\varepsilon,h}(u, w_h)$  as follows:

$$D_{\varepsilon,h}(u, w_h) = \varepsilon^2 D_h^1 \left( u, \frac{\partial w_h}{\partial x_1} \right) + \varepsilon^2 D_h^2 \left( u, \frac{\partial w_h}{\partial x_2} \right)$$

where (see the notation of Figure 3.4)

$$D_h^j \left( v, \frac{\partial w_h}{\partial x_j} \right) = \sum_{T \in \mathcal{T}_h} \left\{ \int_{T'_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) \frac{\partial w_h}{\partial x_j} ds - \int_{T''_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) \frac{\partial w_h}{\partial x_j} ds \right\}.$$

Let  $Z_h$  be the finite element space associated with the triangulation  $\mathcal{T}_h$  of continuous piecewise bilinear functions vanishing on  $\partial\Omega$ . Then  $Z_h \subset C^0(\Omega) \cap H_0^1(\Omega)$  and consequently

$$D_h^j(v, z_h) = \sum_{T \in \mathcal{T}_h} \left\{ \int_{T'_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) z_h ds - \int_{T''_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) z_h ds \right\} = 0$$

for all  $z_h \in Z_h, v \in H^3(\Omega)$ . Now we define

$$D_T(v, p) = \Delta_{1,T} \left( v, \frac{\partial p}{\partial x_1} \right) + \Delta_{2,T} \left( v, \frac{\partial p}{\partial x_2} \right), \quad v \in H^3(\Omega), p \in V_h,$$

with

$$\begin{aligned} \Delta_{j,T} \left( v, \frac{\partial p}{\partial x_j} \right) &= \int_{T'_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) \left( \frac{\partial p}{\partial x_j} - \Lambda_T \frac{\partial p}{\partial x_j} \right) ds - \\ &\quad \int_{T''_j} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2} \right) \left( \frac{\partial p}{\partial x_j} - \Lambda_T \frac{\partial p}{\partial x_j} \right) ds \end{aligned}$$

where  $\Lambda_T$  is the  $\mathcal{Q}_1$ -interpolant operator over  $T$ . So

$$D_{\varepsilon,h}(v, w_h) = \varepsilon^2 \sum_{T \in \mathcal{T}_h} D_T(v, w_h) \quad v \in H^3(\Omega), w_h \in V_h.$$

Clearly we have a *first polynomial invariance*

$$\Delta_{j,T}(v, q) = 0, \quad \forall q \in \mathcal{Q}(T).$$

But, we also have a *second polynomial invariance*

$$\Delta_{j,T}(v, q) = 0, \quad \forall v \in \mathcal{P}_2(T), q \in \partial_j P_T$$

$(\partial_j P_T = \{\partial_j v : v \in P_T\})$ . Indeed, it follows by noting that if  $v \in \mathcal{P}_2(T)$  then  $\Delta v - \frac{\partial^2 v}{\partial \tau^2} = cte$  on  $T$ , and that if  $q \in \partial P_T$  then

$$\int_{T'_j} (q - \Lambda_T q) ds = \int_{T''_j} (q - \Lambda_T q) ds, \quad j = 1, 2,$$

as can easily be checked (see also [17]).

Let  $\delta_{j,T}$  be defined analogously to  $\Delta_{j,T}$  by

$$\delta_{j,T}(\phi, q) = \int_{T'_j} \phi(q - \Lambda_T q) ds - \int_{T''_j} \phi(q - \Lambda_T q) ds, \quad \phi \in H^1(\Omega), q \in \partial_j P_T$$

By standards scaling arguments we can see

$$\hat{q} \in \partial_j P_{\hat{T}} \Leftrightarrow q \in \partial_j P_T, \quad (3.4.22)$$

$$\delta_{j,T}(\phi, q) = h_{j,T} \delta_{j,\hat{T}}(\hat{\phi}, \hat{q}), \quad j = 1, 2, \quad (3.4.23)$$

and also one can check that

$$\delta_{j,\hat{T}}(\hat{\phi}, \hat{q}) = 0 \quad \forall \hat{\phi} \in H^1(\hat{T}), \hat{q} \in \mathcal{P}_0(\hat{T}),$$

$$\delta_{j,\hat{T}}(\hat{\phi}, \hat{q}) = 0 \quad \forall \hat{\phi} \in \mathcal{P}_0(\hat{T}), \hat{q} \in \partial_j P_{\hat{T}}.$$

Now we estimate  $|\delta_{j,\hat{T}}(\hat{\phi}, \hat{q})|$ . If  $m_{\hat{T}}\hat{\phi}$  is the mean value of  $\hat{\phi}$  on the element  $\hat{T}$ , we have

$$\begin{aligned} |\delta_{j,\hat{T}}(\hat{\phi}, \hat{q})| &= |\delta_{j,\hat{T}}(\hat{\phi} - m_{\hat{T}}\hat{\phi}, \hat{q} - c_0)| \\ &\leq C \|\hat{\phi} - m_{\hat{T}}\hat{\phi}\|_{0,\partial\hat{T}} \|(\hat{q} - c) - \Lambda_{\hat{T}}(\hat{q} - c)\|_{0,\partial\hat{T}} \\ &= C \|\hat{\phi} - m_{\hat{T}}\hat{\phi}\|_{0,\hat{T}}^{\frac{1}{2}} \|\hat{\phi} - m_{\hat{T}}\hat{\phi}\|_{1,\hat{T}}^{\frac{1}{2}} \|(\hat{q} - c) - \Lambda_{\hat{T}}(\hat{q} - c)\|_{1,\hat{T}} \end{aligned} \quad (3.4.24)$$

for all  $\hat{\phi} \in H^1(\hat{T})$  and  $\hat{q} \in \partial_j P_{\hat{T}}$ . By the Poincare inequality it holds

$$\|\hat{\phi} - m_{\hat{T}}\hat{\phi}\|_{1,\hat{T}} \leq |\hat{\phi}|_{1,\hat{T}},$$

besides, using the equivalence of norms in  $\partial_j P_{\hat{T}}$  it results

$$\|\hat{p} - \Lambda_{\hat{T}}\hat{p}\|_{1,\hat{T}} \leq C \|\hat{p}\|_{2,\hat{T}} \leq C \|\hat{p}\|_{1,\hat{T}}, \quad \forall \hat{p} \in \partial_j P_{\hat{T}}$$

and taking into account that  $|m_{\hat{T}}\hat{\phi}| \leq |\hat{T}|^{\frac{1}{2}} \|\hat{\phi}\|_{0,\hat{T}}$  we also have

$$\|\hat{\phi} - m_{\hat{T}}\hat{\phi}\|_{0,\hat{T}} \leq \|\hat{\phi}\|_{0,\hat{T}} + |\hat{T}|^{\frac{1}{2}} |m_{\hat{T}}\hat{\phi}| \leq C \|\hat{\phi}\|_{0,\hat{T}}.$$

Now, inserting these inequalities in (3.4.24) we obtain

$$\left| \delta_{j,\hat{T}}(\hat{\phi}, \hat{q}) \right| \leq C \|\hat{\phi}\|_{0,\hat{T}}^{\frac{1}{2}} |\hat{\phi}|_{1,\hat{T}}^{\frac{1}{2}} |\hat{q}|_{1,\hat{T}}, \quad \forall \hat{\phi} \in H^1(\hat{T}), \hat{q} \in \partial_j P_{\hat{T}},$$

where we have use that there exists a constant  $C$  such that  $\|\hat{q} - c_0\|_{1,\hat{T}} \leq C|\hat{q}|_{1,\hat{T}}$  for all constant  $c_0$ . From the property (3.4.23) and scaling arguments we then obtain

$$|\delta_{j,T}(\phi, q)| \leq Ch^{\frac{1}{2}} \|\phi\|_{0,T}^{\frac{1}{2}} |\phi|_{1,T}^{\frac{1}{2}} |q|_{1,T}, \quad \forall \phi \in H^1(T), q \in \partial_j P_T.$$

Since

$$\Delta_{j,T}(v, \frac{\partial p}{\partial x_j}) = \delta_{j,T} \left( \Delta v - \frac{\partial^2 v}{\partial \tau^2}, \frac{\partial p}{\partial x_j} \right)$$

it results that

$$\left| \Delta_{j,T}(v, \frac{\partial p}{\partial x_j}) \right| \leq Ch^{\frac{1}{2}} |v|_{2,T}^{\frac{1}{2}} |v|_{3,T}^{\frac{1}{2}} |p|_{2,T} \quad \forall v \in H^3(\Omega), p \in V_h,$$

and then for all element  $T \in \mathcal{T}_h$

$$|D_T(v, p)| \leq Ch^{\frac{1}{2}} |v|_{2,T}^{\frac{1}{2}} |v|_{3,T}^{\frac{1}{2}} |p|_{2,T} \quad \forall v \in H^3(\Omega), p \in V_h.$$

Then it follows that

$$\begin{aligned} D_{\varepsilon,h}(u, w_h) &\leq C\varepsilon^2 h^{\frac{1}{2}} \sum_{T \in \mathcal{T}_h} Ch^{\frac{1}{2}} |v|_{2,T}^{\frac{1}{2}} |v|_{3,T}^{\frac{1}{2}} |p|_{2,T} \\ &\leq C\varepsilon^2 h^{\frac{1}{2}} \left( \sum_T |u|_{2,T} |u|_{3,T} \right)^{\frac{1}{2}} \left( \sum_T |w_h|_{2,T}^2 \right)^{\frac{1}{2}} \\ &\leq C\varepsilon^2 h^{\frac{1}{2}} \left[ \left( \sum_T |u|_{2,T}^2 \right)^{\frac{1}{2}} \left( \sum_T |u|_{3,T}^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left( \sum_T |w_h|_{2,T}^2 \right)^{\frac{1}{2}} \\ &\leq C\varepsilon^2 h^{\frac{1}{2}} |u|_{\frac{1}{2}} |u|_{\frac{1}{3}} |w_h|_2. \end{aligned}$$

Finally, using that  $|w_h|_2 \leq \frac{1}{\varepsilon} \|w_h\|_{\varepsilon,h}$  we have

$$D_{\varepsilon,h}(u, w_h) \leq C\varepsilon h^{\frac{1}{2}} |u|_{\frac{1}{2}} |u|_{\frac{1}{3}} \|w_h\|_{\varepsilon,h}$$

as we wanted to prove.  $\square$

The following theorem is proved in [33] for a different non-conforming finite element space. The basic ingredients in that proof are the existence of an interpolation operator  $I_h$  verifying the inequalities (3.4.16) and (3.4.18), the inequality (3.4.21) for the consistency error and Lemma 3.4.1, and then it can be reproduced in our case. For sake of completeness we include that proof here.

**Theorem 3.4.8.** *Assume that  $f \in L^2(\Omega)$  and  $u \in H_0^2(\Omega) \cap H^3(\Omega)$  is the corresponding weak solution of (3.4.2). Let  $u_h \in V_h$  the finite element solution of (3.4.5). Then there is a constant  $C$  independent of  $\varepsilon$  and  $h$  such that*

$$\|u - u_h\|_{\varepsilon,h} \leq Ch^{\frac{1}{2}} \|f\|_{0,\Omega}.$$

*Proof.* We first show that

$$\inf_{v \in V_h} \|u - v\|_{\varepsilon,h} \leq \|u - I_h u\|_{\varepsilon,h} \leq Ch^{\frac{1}{2}} \|f\|_0.$$

From (3.4.16) with  $j = k = 2$  in one case, and  $j = 2, k = 3$  in the second case, and Lemma 3.4.1 we obtain

$$\begin{aligned} \varepsilon \|u - I_h u\|_2 &\leq C\varepsilon \|u\|_2^{\frac{1}{2}} \|u - I_h u\|_2^{\frac{1}{2}} \\ &\leq C\varepsilon h^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} \|u\|_3^{\frac{1}{2}} \\ &\leq Ch^{\frac{1}{2}} \|f\|_0. \end{aligned}$$

By the other hand, we have (recall that  $u^0$  was defined before Lemma 3.4.1)

$$\|u - I_h u\|_1 \leq \|u - u^0 - I_h(u - u^0)\|_1 + \|u^0 - I_h u^0\|_1.$$

From (3.4.18), (3.4.4) and Lemma 3.4.1 it follows that

$$\begin{aligned} \|u - u^0 - I_h(u - u^0)\|_1 &\leq Ch^{\frac{1}{2}} \|u - u^0\|_1^{\frac{1}{2}} \|u - u^0\|_2^{\frac{1}{2}} \\ &\leq Ch^{\frac{1}{2}} \|f\|_0 \end{aligned}$$

while from (3.4.16) with  $j = 1, k = 2$  and (3.4.4) gives

$$\|u - u^0\|_1 \leq Ch \|u^0\|_2 \leq Ch \|f\|_0.$$

Hence we have the estimate

$$\|u - I_h u\|_{\varepsilon,h} \leq Ch^{\frac{1}{2}} \|f\|_0. \quad (3.4.25)$$

Furthermore, applying Lemma 3.4.1 in the inequality (3.4.21) for the consistency error, we obtain

$$D_{\varepsilon,h}(u, w_h) \leq Ch^{\frac{1}{2}} \|f\|_0 \|w_h\|_{\varepsilon,h}. \quad (3.4.26)$$

Taking into account (3.4.25) and (3.4.26), the proof concludes by using the inequality 3.4.6 given by the Second Strang Lemma.  $\square$

In what follows we show a numerical example. We consider the problem (3.4.1) with  $f \equiv 1$ , and  $\varepsilon = 10^{-3}$ . In Figure 3.5 we have plotted a numerical approximation of the solution, and in Figure 3.6 we present the first derivatives of that approximation. As we can see, the derivatives of the solution present boundary layers.

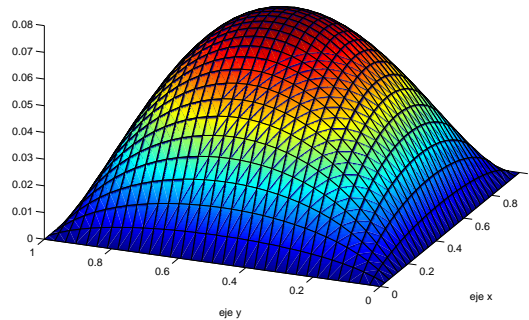


Figure 3.5: Numerical Solution using Adini's Rectangular Finite Elements.  $\varepsilon = 10^{-3}$

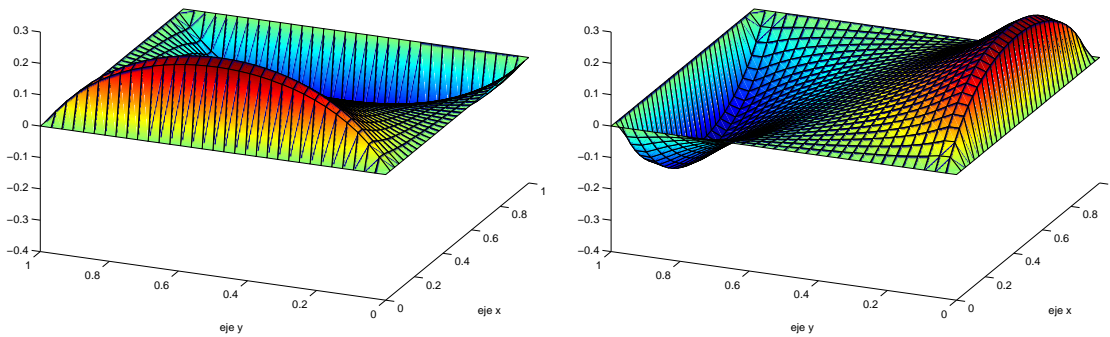


Figure 3.6: Derivatives of the solution

In Table 3.3 we study the convergence order when the Adini's rectangular finite element method is used on uniform meshes. We observe that the convergence order (with respect to the number of elements) obtained is  $\frac{1}{4}$ , as is predicted by the previous Theorem.

The one dimensional analogous equation

$$\begin{aligned}\varepsilon^2 u^{(iv)} - u'' &= f \quad \text{in } (0, 1) \\ u(0) = u(1) = u'(0) = u'(1) &= 0\end{aligned}$$

admits solutions of the form

$$u(x) = \varepsilon e^{-\frac{x}{\varepsilon}} - p(x)$$

where  $p$  is a cubic polynomial which is bounded independently of  $\varepsilon$ , chosen such that the boundary conditions hold, and  $f = p''$ . Then we see that the derivative  $u'$  of such a solution is of the form  $e^{-\frac{x}{\varepsilon}} - p'(x)$  and then, it presents boundary layers like those that appear in the solutions of the one dimensional reaction diffusion equation analogous to the one considered in Section 3.1. This fact, motive us to investigate what happens if the Adini finite element method is used on graded meshes like those described for the reaction diffusion problem (we use  $\alpha = \frac{2}{3}$  and different values of  $h$ ). In fact, for the example considered above, we see in Table 3.4 that the optimal order  $\frac{1}{2}$  seems to be recovered.

Number of elem.	Est. Error	Est. Order of Conv.
16	0.0646	-
64	0.0455	0.25
256	0.0320	0.25
1024	0.0221	0.27

Table 3.3: Estimated Convergence order in the energy norm using uniform meshes

Number of elem.	Est. Error	Est. Order of Conv.
64	0.0442	-
256	0.0237	0.45
576	0.0144	0.62
961	0.0091	0.90

Table 3.4: Estimated Convergence order in the energy norm using graded meshes

## Chapter 4

# Discontinuous Galerkin Method with Exponential Fitting Stabilization

### 4.1 Introduction

In this Chapter we consider again the approximation of singularly perturbed problems, but now we follow an approach that is different to that one of the previous Chapter. We seek numerical methods that are stable even for meshes that are not locally refined. More precisely, we consider a Discontinuous Galerkin method with stabilization of exponential fitting type.

We consider the approximation of a singularly perturbed convection diffusion problem that can be written in a symmetric variational form (see eq. (4.2.1)). This kind of partial differential equation can appear, for example, when the Gummel iterative algorithm [13] is applied to solve the Drift-Diffusion model for Semiconductors. In fact, in the stationary case, the scaled Drift-Diffusion model reads [13]

$$\begin{aligned} \operatorname{div}(\lambda^2 E) &= p - n + C(x) \\ -\operatorname{div} J_n &= -R \\ -\operatorname{div} J_p &= -R \\ E &= -\nabla\psi \\ J_n &= \mu_n(\nabla n - n\nabla\psi) \\ J_p &= \mu_p(\nabla p - p\nabla\psi) \end{aligned} \tag{4.1.1}$$



where  $E$  is the scaled electric field,  $\lambda^2$  is the dielectric constant of the materials,  $J_n$  and  $J_p$  are the electron and hole current densities respectively,  $\psi$  is the electrostatic potential,  $n$  and  $p$  are the electron and hole concentrations inside the semiconductor,  $C(x)$  is the doping profile. Suitable boundary conditions must be imposed.

The Gummel map is a nonlinear block iterative algorithm that splits the semiconductor device equations (4.1.1) into the successive solution of a nonlinear Poisson equation for the electric potential  $\psi$  and two linearized continuity equations for the electron and hole densities  $n$  and  $p$ . There is a considerable amount of mathematical work carried out to analyze the convergence of Gummel's iteration (see, e.g., the papers [26, 27], the book [32], and for an overview [13]). In this Chapter we consider only a possible discretization of the continuity equations mentioned above, that we write in the general form (4.2.1). Our approach is very close to those analyzed in [13, 14, 15, 16], we also consider a finite element method with stabilization of exponential fitting type, the difference is that here we consider a Discontinuous Galerkin finite element method. This may be useful in considering adaptive schemes. Indeed, a big advantage in using discontinuous elements is the possibility of dealing with decompositions highly unstructured, with possible hanging nodes, with elements of different shapes or with local approximation spaces of different types. We do not treat this subject here.

In Sections 2 and 3 we introduce the method in mixed and primal form, making an asymptotic analysis of the stiffness matrix coefficients in terms of the (large) potential gradient. In Section 4 we present some numerical examples showing adequate solutions even on meshes that are not locally refined when internal and boundary layers are present. Finally, in Section 5, we make a theoretical analysis of the modified Interior Penalty method introduced in Sections 2 and 3, obtaining optimal error estimates for symmetric regular problems.

We will use the following notation (taken in general from [8]). Let  $\mathcal{T}_h$  be a partition of a polygonal domain  $\Omega$  made of triangles. We denote by  $\Gamma$  the union of all the edges of  $\mathcal{T}_h$ , and by  $\Gamma^0$  the union of all the internal edges of  $\mathcal{T}_h$ , and also we denote by  $\mathcal{E}$  the set of all the edges of  $\mathcal{T}_h$  and by  $\mathcal{E}^0$  the set of the internal edges of  $\mathcal{T}_h$ . Let  $H^l(\mathcal{T}_h)$  be the space of functions whose restriction to each element  $K$  belong to the Sobolev space  $H^l(K)$ . The space of traces of functions in  $H^1(\mathcal{T}_h)$  is contained in  $T(\Gamma)$ , which is defined as  $T(\Gamma) = \Pi_K L^2(\partial K)$ . Thus, functions in  $T(\Gamma)$  and double valued on  $\Gamma^0$  and single valued

on  $\partial\Omega$ . For a function  $q \in T(\Gamma)$  we define the average  $\{q\}$  and the jump  $\llbracket q \rrbracket$  of  $q$  on  $\Gamma^0$  as follows. Let  $e$  be an interior edge shared by elements  $K_1$  and  $K_2$ , and let  $n_1$  and  $n_2$  be the outward normals to  $K_1$  and  $K_2$  respectively. If  $q_i = q|_{\partial K_i}$  then we set

$$\{q\} = \frac{1}{2}(q_1 + q_2), \quad \llbracket q \rrbracket = q_1 n_1 + q_2 n_2, \quad \text{on } e \subset \Gamma^0.$$

For  $\phi \in T(\Gamma)^2$  we define  $\phi_1$  and  $\phi_2$  analogously and we set

$$\{\phi\} = \frac{1}{2}(\phi_1 + \phi_2), \quad \llbracket \phi \rrbracket = \phi_1 \cdot n_1 + \phi_2 \cdot n_2, \quad \text{on } e \subset \Gamma^0$$

Notice that these definitions do not depend on assigning an ordering to the elements  $K_1$  and  $K_2$ . Also note that the jump of a scalar function is a vector parallel to the normal, and the jump of a vector function is a scalar quantity. On boundary edges we set

$$\llbracket q \rrbracket = qn, \quad \{\phi\} = \phi, \quad \text{on } e \subset \partial\Omega$$

where  $n$  is the exterior normal of  $\Omega$ . We will not require either of the quantities  $\{q\}$  or  $\llbracket \phi \rrbracket$  on boundary edges. Also we define the projection  $\Pi_h : H^1(\mathcal{T}_h) \rightarrow T(\Gamma)$  as

$$\Pi_h v|_{\partial K} = \prod_{e \subset K} \Pi_0^{e,K} v$$

where the product in the right hand side is taken on the edges  $e$  of  $K$ , and  $\Pi_0^{e,K} v$  is the  $L^2$ -projection of the trace on  $e$  of  $v|_K$  on the space  $\mathcal{P}_0(e)$  of constant functions on  $e$  (we will not make explicit the dependence on  $K$ , if it is clear from the context, or in those cases where there is no dependence on  $K$ , for example, when  $v \in H^1(\Omega)$ ).

Let  $C(\mathcal{T}_h) = \prod_{K \in \mathcal{T}_h} C(K)$ , where  $C(K)$  denotes the set of continuous functions on  $K$ . If  $v \in C(\mathcal{T}_h)$  we have  $v = (v_K)_{K \in \mathcal{T}_h}$ , and we will write  $v|_K = v_K$ . We set  $v_{m,K} = \min\{v_K(x) : x \in K\}$  and if  $e$  is an edge of  $K$  we put  $v_{m,e,K} = \min\{v_K(s) : s \in e\}$ .

For a function  $v \in H^1(\mathcal{T}_h)$ ,  $\nabla_h v$  mean the function in  $\prod_{K \in \mathcal{T}_h} (L^2(K))^2$  given by  $(\nabla_h v)_K = \nabla v_K$ , and for a function  $\tau \in [H^1(\mathcal{T}_h)]^2$ ,  $\text{div}_h \tau$  denotes the function in  $\prod_{K \in \mathcal{T}_h} L^2(K)$  defined by  $(\text{div}_h \tau)_K = \text{div } \tau_K$ . Also we define  $|v|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v_K\|_{0,K}^2$  and  $\|v\|_{1,h}^2 = \|v\|_0^2 + |v|_{1,h}^2$ .

We also use the standard notation

$$\int_e f ds = \frac{1}{|e|} \int_e f ds, \quad \int_K f dx = \frac{1}{|K|} \int_K f dx$$

for edges  $e$  and elements  $K$ .

## 4.2 A stabilized discontinuous Galerkin method. The mixed formulation

Given a continuous piecewise smooth function  $\psi$  defined on a polygonal domain  $\Omega \subset \mathbb{R}^2$  we consider the problem

$$\begin{aligned} -\operatorname{div}(\varepsilon \nabla u - \nabla \psi u) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma_D \\ (\varepsilon \nabla u - \nabla \psi u) \cdot n &= 0 \quad \text{on } \Gamma_N \end{aligned} \tag{4.2.1}$$

with  $\Gamma_D \cup \Gamma_N = \partial\Omega$ ,  $\Gamma_D \cap \Gamma_N = \emptyset$  and  $\varepsilon$  is a positive parameter. This is a singularly perturbed problem, in fact, if  $\varepsilon \ll 1$  the solution  $u$  can exhibit internal and boundary layers. We are interested in obtaining a stable finite element method even on meshes which are not locally refined. Here, we consider a kind of stabilization known as ‘‘Exponential Fitting’’. For an extensive study of this kind of stabilization we refer to [13].

Now, we briefly describe this approach. Introducing the variable (known as Slotboom variable in the context of semiconductor problems)

$$\rho = ue^{-\frac{\psi}{\varepsilon}} \tag{4.2.2}$$

we can rewrite (4.2.1) as

$$\begin{aligned} -\operatorname{div}\left(\varepsilon e^{\frac{\psi}{\varepsilon}} \nabla \rho\right) &= f \quad \text{in } \Omega \\ \rho &= \chi \quad \text{on } \Gamma_D \\ \varepsilon e^{\frac{\psi}{\varepsilon}} \frac{\partial \rho}{\partial n} &= 0 \quad \text{on } \Gamma_N \end{aligned} \tag{4.2.3}$$

with  $\chi = ge^{-\frac{\psi}{\varepsilon}}$ . In order to numerically approximate the solution  $u$  of Problem (4.2.1) we will discretize Problem (4.2.3) for the variable  $\rho$ , obtaining a linear system

$$\widetilde{M}_p \rho_h = \widetilde{b}. \tag{4.2.4}$$

Notice that the coefficient  $\varepsilon e^{\frac{\psi}{\varepsilon}}$  in the left hand side of the first line of (4.2.3), may be a source of numerical problems when the matrix  $\widetilde{M}_p$  is constructed. For this reason, and since we are interested in obtaining an approximation of  $u$ , we will perform, at the matrix level (4.2.4), a discrete inverse transformation  $u_h = \mathcal{R}(\rho_h)$ , obtaining the desirable form

$$M_p u_h = b. \tag{4.2.5}$$

Here we will study a Discontinuous Galerkin Finite Element Method in order to discretize Problem (4.2.3).

In what follows, we state in detail our method. Firstly, we consider a mixed formulation. The primal formulation is considered in the next Section.

Let  $a = \varepsilon e^{\frac{\psi}{\varepsilon}}$ . If we put  $\sigma = a\nabla\rho$  we can write Problem (4.2.3) as

$$\begin{aligned} a^{-1}\sigma &= \nabla\rho \quad \text{in } \Omega \\ -\operatorname{div}\sigma &= f \quad \text{in } \Omega \\ \rho &= \chi \quad \text{on } \Gamma_D \\ \sigma \cdot n &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{4.2.6}$$

Let  $\mathcal{T}_h$  be a triangulation of  $\Omega$ . For any element  $K \in \mathcal{T}_h$  we multiply the first equation of 4.2.6 by a function  $\tau \in C^\infty(K)^2$  and the second equation by a function  $v \in C^\infty(K)$ , and then, integrating by parts on  $K$  we obtain

$$\begin{aligned} \int_K a^{-1}\sigma \cdot \tau \, dx + \int_K \rho \operatorname{div} \tau \, dx - \int_{\partial K} \rho \tau \cdot n_K \, ds &= 0 \\ \int_K \sigma \cdot \nabla v \, dx - \int_{\partial K} \sigma \cdot n_K v \, ds &= \int_K f v \, dx \end{aligned} \tag{4.2.7}$$

where  $n = n_K$  is the exterior normal to  $K$ . For each  $K \in \mathcal{T}_h$  let  $V(K) = \mathcal{P}_1(K)$  and  $\Sigma(K) = V(K)^2$  and then consider

$$V_h = \{v \in H^2(\mathcal{T}_h) : v|_K \in V(K)\}, \quad \Sigma_h = V_h^2.$$

Following [8], a general formulation of a Discontinuous Galerkin finite element method consists of: Find  $\sigma_h \in \Sigma_h$  and  $\rho_h \in V_h$  such that for all  $K \in \mathcal{T}_h$

$$\begin{aligned} \int_K a_h^{-1}\sigma_h \cdot \tau \, dx + \int_K \rho_h \operatorname{div} \tau \, dx - \int_{\partial K} \hat{\rho} \tau \cdot n_K \, ds &= 0 \quad \forall \tau \in \Sigma(K) \\ \int_K \sigma_h \cdot \nabla v \, dx - \int_{\partial K} \hat{\sigma} \cdot n_K v \, ds &= \int_K f v \, dx \quad \forall v \in V(K) \end{aligned} \tag{4.2.8}$$

where  $a_h \in H^1(\mathcal{T}_h)$  is an approximation of  $a$ , and  $\hat{\rho}$  and  $\hat{\sigma}$  are the numerical fluxes, which must be defined. The definition of such numerical fluxes identify the Discontinuous Galerkin method under consideration. Here, we take  $\hat{\rho} : H^1(\mathcal{T}_h) \rightarrow T(\Gamma)$  and  $\hat{\sigma} : [H^2(\mathcal{T}_h)]^2 \rightarrow T(\Gamma)$

as

$$\hat{\rho}|_e = \begin{cases} \{\Pi_h \rho_h\} & \text{if } e \in \mathcal{E}^0 \text{ or } e \subset \Gamma_N \\ \chi_h & \text{if } e \subset \Gamma_D \end{cases} \quad (4.2.9)$$

$$\hat{\sigma}|_e = \begin{cases} \{a_h \nabla_h \rho_h\} - \mu [\Pi_h \rho_h] & \text{if } e \in \mathcal{E}^0 \\ a_h \nabla_h \rho_h - \mu (\Pi_h \rho_h - \chi_h) n & \text{if } e \subset \Gamma_D \\ 0 & \text{if } e \subset \Gamma_N \end{cases} \quad (4.2.10)$$

where  $\chi_h$  is a convenient approximation of  $\chi$  on  $\Gamma_D$ . This choice of the fluxes corresponds to a modification of the Interior Penalty Method, where we have changed  $\rho_h$  by its projection  $\Pi_h \rho_h$  (see [7, 8]). The reason for that modification will be clear in the next paragraphs.

The penalty function  $\mu$  which appears in the definition of  $\hat{\sigma}$  is taken on each edge  $e$  as

$$\mu|_e = \frac{\eta_e}{h_e}$$

with  $\eta_e$  a constant and where  $h_e$  is the length of  $e$ .

We have introduced a scheme to approximate the unknowns  $\sigma$  and  $\rho$ . Now we must recover the variable  $u$ , for which we should define the discrete inverse transformation  $\mathcal{R}$ . In order to do that we need to introduce the basis of the discontinuous finite element spaces  $V_h$  and  $\Sigma_h$ .

For any  $K \in \mathcal{T}_h$  we consider three piecewise linear basis functions of  $V_h$ ,  $v_i^K$ , such that  $(v_i^K)|_{K'} = 0$  if  $K \neq K' \in \mathcal{T}_h$  and  $(v_i^K)|_K(m_j^K) = \delta_{ij}$  where  $m_i^K$  is the midpoint of the edge  $e_i^K$  of  $K$ ,  $i = 1, 2, 3$  (see Figure 4.1). Clearly, if  $Nel$  is the number of elements in  $\mathcal{T}_h$ , we have  $\dim V_h = 3Nel$ . A set of basis functions for the space  $\Sigma_h$  can be defined analogously, and we will have  $\dim \Sigma_h = 6Nel$ . Then,  $v \in V_h$  and  $\tau \in \Sigma_h$  are defined by the following degrees of freedom:

$$v_K(m_i^K), \quad \tau_K(m_i^K), \quad K \in \mathcal{T}_h, i = 1, 2, 3.$$

Finally, we denote by  $\{v_i\}_1^{3Nel}$  and  $\{\tau_i\}_1^{6Nel}$  the basis of  $V_h$  and  $\Sigma_h$  described above.

We want to remark that if  $v_j$  is one of the basis functions for  $V_h$ , with  $K = \text{supp } v_j$ , and if it assumes the value 1 at the midpoint of the edge  $e$  of  $K$  and 0 at the midpoints of the other two edges of  $K$ , then since  $v_j$  is linear on  $K$  it follows that it is identically 1 on all the edge  $e$ .

Now, it is natural to consider the discrete inverse transformation of (4.2.2) given by

$$u_{h|K}(m_i^K) = \rho_{h|K}(m_i^K) \left( \int_{e_i^K} e^{-\frac{\psi}{\varepsilon}} ds \right)^{-1} \quad (4.2.11)$$

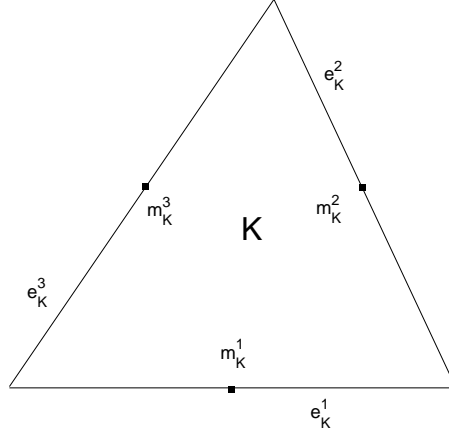


Figure 4.1: Notation on a generic element

We need to define  $a_h$  on each element in order to perform the computations. We consider an approximation  $a_h \in \mathcal{P}_0(\mathcal{T}_h)$  of  $a$ , such that in each element  $K$  is defined by

$$a_{h|K}^{-1} = \fint_K a^{-1} dx.$$

That is, we are taking

$$a_{h|K} = \frac{1}{a_{h|K}^{-1}} = \mathcal{H}_K(a) \quad (4.2.12)$$

( $\mathcal{H}_K(a)$  is the harmonic average of  $a$  on  $K$ ). Also, if  $e \in \Gamma_D$  we define

$$\chi_{h|e} = g_{h|e} \fint_e e^{-\frac{\psi}{\varepsilon}} ds \quad (4.2.13)$$

with  $g_h$  a piecewise constant approximation of  $g$ ,  $g_{h|e} = \Pi_0^e g$ .

The mixed formulation (4.2.8), together with the definitions of the fluxes, conduces to a linear system of the form

$$\widetilde{M}_m \begin{pmatrix} \sigma_h \\ \rho_h \end{pmatrix} = \begin{pmatrix} \widetilde{b}_m^1 \\ \widetilde{b}_m^2 \end{pmatrix}.$$

Using the inverse transformation (4.2.11), and after some matrix manipulations, we will arrive to an equivalent system

$$M_m \begin{pmatrix} \sigma_h \\ u_h \end{pmatrix} = \begin{pmatrix} b_m^1 \\ b_m^2 \end{pmatrix}.$$

In the diffusion coefficient  $a_h$ , as well as in transformation (4.2.11), appear exponential functions, and it is important to check that overflow does not occur. In what follows, we

will analyze the entries of the matrix  $M_m$  in order to make sure that they are computable quantities. As we will see, the choices of the numerical fluxes, the approximation  $a_h$  and the inverse transformation  $\mathcal{R}$ , joint with some rows operations are important ingredients in order to avoid numerical overflow.

First, write  $\widetilde{M}_m$  and  $M_m$  in the form

$$\widetilde{M}_m = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{D} & \widetilde{C} \end{pmatrix} \quad \text{and} \quad M_m = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

where

$$\begin{aligned} \widetilde{A}, A &\in \mathbb{R}^{6Nel \times 6Nel} & \widetilde{B}, B &\in \mathbb{R}^{6Nel \times 3Nel} \\ \widetilde{D}, D &\in \mathbb{R}^{3Nel \times 6Nel} & \widetilde{C}, C &\in \mathbb{R}^{3Nel \times 3Nel}. \end{aligned}$$

The coefficients in the matrix  $\widetilde{A}$  are of the form

$$\widetilde{A}_{ij} = \int_{\Omega} a_h^{-1} \tau_i \cdot \tau_j \, dx.$$

It is clear that  $A_{ij} \neq 0$  only if there exists  $K \in \mathcal{T}_h$  such that  $\text{supp } \tau_i = \text{supp } \tau_j = K$ . Then we have

$$\begin{aligned} \widetilde{A}_{ij} &= \frac{1}{\varepsilon} \int_K e^{-\frac{\psi}{\varepsilon}} \, dx \int_K \tau_i \cdot \tau_j \, dx \\ &= \frac{1}{\varepsilon} e^{-\frac{\psi_{m,K}}{\varepsilon}} \int_K e^{-\frac{\psi - \psi_{m,K}}{\varepsilon}} \, dx \int_K \tau_i \cdot \tau_j \, dx. \end{aligned}$$

We recall that  $\psi_{m,K}$  denotes the minimum of  $\psi$  on  $K$  and we refer to the end of the Introduction for its proper definition. Since the quantity  $e^{-\frac{\psi_{m,K}}{\varepsilon}}$  may be not computable in practice, we multiply the row  $i$  of  $\widetilde{M}$  by  $e^{\frac{\psi_{m,K}}{\varepsilon}}$  (recall that  $K = \text{supp } \tau_i$ ). We take

$$A_{ij} = \frac{1}{\varepsilon} \int_K e^{-\frac{\psi - \psi_{m,K}}{\varepsilon}} \, dx \int_K \tau_i \cdot \tau_j \, dx.$$

The entries of  $\widetilde{B}$  are given by

$$\widetilde{B}_{ij} = \int_{\Omega} v_j \text{div } \tau_i \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \{\Pi_h v_j\} \tau_i \cdot n \, ds.$$

In view of our construction of the matrix  $A$  and the inverse transformation (4.2.11), we must take

$$\begin{aligned} B_{ij} &= \widetilde{B}_{ij} e^{\frac{\psi_{m,K}}{\varepsilon}} \int_e e^{-\frac{\psi|_{K'}}{\varepsilon}} \, ds \\ &= \widetilde{B}_{ij} \xi_{ij} \end{aligned}$$

where  $K = \text{supp } \tau_i$  and  $K' = \text{supp } v_j$  and  $e$  is the edge of  $K'$  where  $v_j$  is identically 1. In order to compute  $\xi_{ij}$  in practice, we need to express it as

$$\xi_{ij} = e^{\frac{\psi_{m,K} - \psi_{m,K',e}}{\varepsilon}} \int_e e^{-\frac{\psi_{|K'|} - \psi_{m,K',e}}{\varepsilon}} ds.$$

If  $K = K'$  only exponentials of nonpositive functions appear. If  $K \neq K'$  it is easy to see that  $\tilde{B}_{ij} \neq 0$  only if  $e = K \cap K'$ , and then the exponentials are also computable (recall that we have assumed that  $\psi$  is continuous). Note that this is a consequence of our choice of the fluxes, and this is not the case if the standard interior penalty method is used.

The respective component  $\tilde{b}_{m,i}^1$  of the right hand side does not vanish only when  $K$  intersects  $\Gamma_D$ . If  $K \cap \Gamma_D = e_D$  then we have

$$\tilde{b}_{m,i}^1 = \int_{e_D} \chi_h \tau_i \cdot n ds.$$

Recalling the definition of  $\chi_h$  and having in mind our definition of  $A$  and  $B$ , we must put

$$b_{m,i}^1 = e^{\frac{\psi_{m,K} - \psi_{m,K,e_D}}{\varepsilon}} \int_{e_D} e^{-\frac{\psi - \psi_{m,K,e_D}}{\varepsilon}} ds \int_{e_D} g_h \tau_i \cdot n ds.$$

Also we see that those quantities are computable.

For  $D$  we simply take

$$D_{ij} = \int_{\Omega} \tau_j \cdot \nabla v_i dx.$$

Finally, we deal with the matrix  $C$ . We write  $\tilde{C} = \tilde{C}^1 + \tilde{C}^2$  with

$$\tilde{C}_{ij}^1 = - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \{a_h \nabla_h v_j\} \cdot n_K v_i ds, \quad \tilde{C}_{ij}^2 = - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu [\Pi_h v_j] \cdot n_K v_i ds.$$

and we consider the corresponding decomposition  $C = C^1 + C^2$ . Taking into account the inverse transformation (4.2.11) we have

$$C_{ij} = \tilde{C}_{ij} \int_e e^{-\frac{\psi_{|K|}}{\varepsilon}} ds$$

where  $e$  is the edge where  $v_j$  is identically 1. Let now  $K = \text{supp } v_j$ , and then  $e$  is an edge of  $K$ . Then it is easy to see that

$$C_{ij}^1 = e^{\frac{\psi_{m,K} - \psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi_{|K|} - \psi_{m,K,e}}{\varepsilon}} ds}{\frac{1}{\varepsilon} \int_K e^{-\frac{\psi_{|K|} - \psi_{m,K}}{\varepsilon}} dx} \int_{\partial K} \{\nabla_h v_j\} \cdot n_K v_i ds$$

with all the exponents of the exponentials being nonpositive.



Taking into account that  $v_i, v_j \in \mathcal{P}_1(\mathcal{T})$  we have

$$\tilde{C}_{ij}^2 = \int_e \mu \llbracket \Pi_h v_j \rrbracket \cdot n_K v_i \, ds.$$

Then, if  $e$  is an interior edge,  $e = K \cap K'$ , we can take  $\eta_e = \frac{1}{2}(a_{h|K} + a_{h|K'})$ , that is

$$\eta_e = \frac{1}{2} \left( \frac{e^{\frac{\psi_{m,K}}{\varepsilon}}}{\frac{1}{\varepsilon} \int_K e^{-\frac{\psi-\psi_{m,K}}{\varepsilon}} \, dx} + \frac{e^{\frac{\psi_{m,K'}}{\varepsilon}}}{\frac{1}{\varepsilon} \int_{K'} e^{-\frac{\psi-\psi_{m,K'}}{\varepsilon}} \, dx} \right). \quad (4.2.14)$$

Then  $C_{ij}^2$  becomes

$$\begin{aligned} C_{ij}^2 &= \frac{\eta_e}{h_e} \int_e e^{-\frac{\psi|_K}{\varepsilon}} \, ds \int_e \llbracket \Pi_h v_j \rrbracket \cdot n_K v_i \, ds \\ &= \frac{1}{2h_e} \left( e^{\frac{\psi_{m,K}-\psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi|_K-\psi_{m,K,e}}{\varepsilon}} \, ds}{\frac{1}{\varepsilon} \int_K e^{-\frac{\psi-\psi_{m,K}}{\varepsilon}} \, dx} \right. \\ &\quad \left. + e^{\frac{\psi_{m,K'}-\psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi|_K-\psi_{m,K,e}}{\varepsilon}} \, ds}{\frac{1}{\varepsilon} \int_{K'} e^{-\frac{\psi-\psi_{m,K'}}{\varepsilon}} \, dx} \right) \int_e \llbracket \Pi_h v_j \rrbracket \cdot n_K v_i \, ds \end{aligned}$$

and we see again that all the exponentials are computable. Case  $e \subset \Gamma_D$  and the right hand side  $\tilde{b}_m^2$  can be treated analogously, taking

$$\eta_e = \frac{e^{\frac{\psi_{m,K}}{\varepsilon}}}{\frac{1}{\varepsilon} \int_K e^{-\frac{\psi-\psi_{m,K}}{\varepsilon}} \, dx}$$

if  $e \subset K \cap \partial\Omega$ .

*Remark 4.2.1.* In practical computations we will take a different matrix  $C^2$ . The factor

$$\frac{1}{2h_e} \left( e^{\frac{\psi_{m,K}-\psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi|_K-\psi_{m,K,e}}{\varepsilon}} \, ds}{\frac{1}{\varepsilon} \int_K e^{-\frac{\psi-\psi_{m,K}}{\varepsilon}} \, dx} + e^{\frac{\psi_{m,K'}-\psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi|_K-\psi_{m,K,e}}{\varepsilon}} \, ds}{\frac{1}{\varepsilon} \int_{K'} e^{-\frac{\psi-\psi_{m,K'}}{\varepsilon}} \, dx} \right) \quad (4.2.15)$$

that appear in the definition of  $C_{ij}^2$ , may be very small if  $\frac{\psi_{m,K}}{\varepsilon} \ll \frac{\psi_{m,K,e}}{\varepsilon}$  and  $\frac{\psi_{m,K'}}{\varepsilon} \ll \frac{\psi_{m,K,e}}{\varepsilon}$ . In order to assure that the penalization coefficient is large enough (see Section 5), we take then the maximum between the factor (4.2.15) and a fixed constant  $c$ . A similar change is made in the entries corresponding to boundary edges.

### 4.3 The primal formulation

In [8], from the mixed formulation (4.2.8), the authors obtain a primal formulation. For sake of completeness, we present in some detail that deduction, which is valid for a general Discontinuous Galerkin method.

If we add the equations in (4.2.8) over all  $K \in \mathcal{T}_h$ , we obtain

$$\begin{aligned} \int_{\Omega} a_h^{-1} \sigma_h \cdot \tau \, dx + \int_{\Omega} \rho_h \operatorname{div}_h \tau \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{\rho} \tau \cdot n_K \, ds &= 0 \quad \forall \tau \in \Sigma_h \\ \int_{\Omega} \sigma_h \cdot \nabla_h v \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{\sigma} \cdot n_K v \, ds &= \int_{\Omega} f v \, dx \quad \forall v \in V_h \end{aligned}$$

A straightforward computation shows that for all  $q \in T(\Gamma)$  and for all  $\phi \in [T(\Gamma)]^2$ ,

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} q_K \phi_K \cdot n_K \, ds = \int_{\Gamma} \llbracket q \rrbracket \cdot \{\phi\} \, ds + \int_{\Gamma^0} \{q\} \cdot \llbracket \phi \rrbracket \, ds. \quad (4.3.1)$$

After a simple application of this identity, we get

$$\begin{aligned} \int_{\Omega} a_h^{-1} \sigma_h \cdot \tau \, dx + \int_{\Omega} \rho_h \operatorname{div}_h \tau \, dx - \int_{\Gamma} \llbracket \hat{\rho} \rrbracket \cdot \{\tau\} \, ds - \int_{\Gamma^0} \{\hat{\rho}\} \cdot \llbracket \tau \rrbracket \, ds &= 0 \\ \int_{\Omega} \sigma_h \cdot \nabla_h v \, dx - \int_{\Gamma} \llbracket v \rrbracket \cdot \{\hat{\sigma}\} \, ds - \int_{\Gamma^0} \{v\} \cdot \llbracket \hat{\sigma} \rrbracket \, ds &= \int_{\Omega} f v \, dx \end{aligned} \quad (4.3.2)$$

for all  $\tau \in \Sigma_h$  and  $v \in V_h$ . Now, we express  $\sigma_h$  solely in terms of  $\rho_h$ . From the identity (4.3.1) we obtain for all  $\tau \in [H^1(\mathcal{T}_h)]^2$  and  $v \in H^1(\mathcal{T}_h)$ , the integration by parts formula

$$- \int_{\Omega} \operatorname{div}_h \tau v \, dx = \int_{\Omega} \tau \cdot \nabla_h v \, dx - \int_{\Gamma} \{\tau\} \cdot \llbracket v \rrbracket \, ds - \int_{\Gamma^0} \llbracket \tau \rrbracket \{v\} \, ds. \quad (4.3.3)$$

Taking  $v = \rho_h$  in the above identity, and inserting the resulting right hand side into the first equation of (4.3.2), we get that, for all  $\tau \in \Sigma_h$ ,

$$\int_{\Omega} a_h^{-1} \sigma_h \cdot \tau \, dx = \int_{\Omega} \nabla_h \rho_h \cdot \tau \, dx + \int_{\Gamma} \llbracket \hat{\rho} - \rho_h \rrbracket \cdot \{\tau\} \, ds + \int_{\Gamma^0} \{\hat{\rho} - \rho_h\} \llbracket \tau \rrbracket \, ds. \quad (4.3.4)$$

Consider the following lifting operators  $r : [L^2(\Gamma)]^2 \rightarrow \Sigma_h$  and  $l : L^2(\Gamma^0) \rightarrow \Sigma_h$  which are defined by

$$\int_{\Omega} r(\phi) \cdot \tau \, dx = - \int_{\Gamma} \phi \cdot \{\tau\} \, ds, \quad \int_{\Omega} l(q) \cdot \tau \, dx = - \int_{\Gamma^0} q \llbracket \tau \rrbracket \, ds, \quad \forall \tau \in \Sigma_h.$$

Then, taking into account that  $\nabla_h V_h \subset \Sigma_h$  and that  $a_{h,K}$  is a non-zero constant for all  $K \in \mathcal{T}_h$ , we may rewrite (4.3.4) as

$$\sigma_h = \sigma_h(\rho_h) := a_h \nabla_h \rho_h - a_h r(\llbracket \hat{\rho} - \rho_h \rrbracket) - a_h l(\{\hat{\rho} - \rho_h\}). \quad (4.3.5)$$

Now, if we take  $\tau = a_h \nabla_h v$  in (4.3.4) then we can rewrite the second equation in (4.3.2) as follows

$$A_h(\rho_h, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_h, \quad (4.3.6)$$

where

$$\begin{aligned} A_h(\rho_h, v) := & \int_{\Omega} a_h \nabla_h \rho_h \cdot \nabla_h v \, dx + \int_{\Gamma} (\llbracket \hat{\rho} - \rho_h \rrbracket \cdot \{a_h \nabla_h v\} - \{\hat{\sigma}\} \cdot \llbracket v \rrbracket) \, ds \\ & + \int_{\Gamma^0} (\{\hat{\rho} - \rho_h\} \llbracket a_h \nabla_h v \rrbracket - \llbracket \hat{\sigma} \rrbracket \cdot \{v\}) \, ds. \end{aligned} \quad (4.3.7)$$

For any functions  $\rho_h, v \in H^2(\mathcal{T}_h)$ , (4.3.7) defines  $A_h(\rho_h, v)$ , with the understanding that  $\hat{\rho} = \hat{\rho}(\rho_h)$  and  $\hat{\sigma} = \hat{\sigma}(\rho_h, \sigma_h(\rho_h))$ , where the map  $\rho_h \rightarrow \sigma_h(\rho_h)$  is given by (4.3.5). If we have homogeneous Dirichlet datum (and then  $\chi_h \equiv 0$ ), the form  $B_h : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \rightarrow \mathbb{R}$ ,  $B_h(w, v) = A_h(w, v)$  is bilinear and, if  $(\rho_h, \sigma_h) \in V_h \times \Sigma_h$  solves (4.2.8), then  $\rho_h$  solves (4.3.6) and  $\sigma_h$  is given by (4.3.5). If we have a non-homogeneous Dirichlet datum, then we can write  $A_h(\rho_h, v)$  as  $B_h(\rho_h, v) + L_h(v)$ , with  $B_h : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \rightarrow \mathbb{R}$  a bilinear form, and  $L_h : H^2(\mathcal{T}_h) \rightarrow \mathbb{R}$  a linear operator. Following [8], we call (4.3.6) the *primal formulation* of the method and call the bilinear form  $B_h(\cdot, \cdot)$  the *primal form*, and in either case, the method becomes: Find  $\rho_h \in V_h$  such that

$$B_h(\rho_h, v) = \int_{\Omega} f v \, dx - L_h(v) \quad \forall v \in V_h. \quad (4.3.8)$$

Now we explicit  $B_h$  and  $L_h$  for our particular choices of the numerical fluxes (4.2.9) and (4.2.10). We have

$$\llbracket \hat{\rho} \rrbracket|_e = \begin{cases} 0 & \text{if } e \in \mathcal{E}^0 \\ \chi_h n & \text{if } e \subset \Gamma_D \\ \Pi_h \rho_h n & \text{if } e \subset \Gamma_N \end{cases} \quad \llbracket \hat{\sigma} \rrbracket|_e = 0 \quad \text{if } e \in \mathcal{E}^0 \quad (4.3.9)$$

$$\{\hat{\rho}\}|_e = \{\Pi_h \rho_h\} \text{ if } e \in \mathcal{E}^0 \quad \{\hat{\sigma}\}|_e = \begin{cases} \{a_h \nabla_h \rho_h\} - \mu \llbracket \Pi_h \rho_h \rrbracket & \text{if } e \in \mathcal{E}^0 \\ a_h \nabla_h \rho_h - \mu(\Pi_h \rho_h - \chi_h) n & \text{if } e \subset \Gamma_D \\ 0 & \text{if } e \subset \Gamma_N \end{cases} \quad (4.3.10)$$

Then the form  $A_h$  becomes

$$\begin{aligned}
A_h(\rho_h, v) &= \int_{\Omega} a_h \nabla_h \rho_h \cdot \nabla_h v \, dx - \int_{\Gamma^0} (\llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\} + \llbracket v \rrbracket \cdot (\{a_h \nabla_h \rho_h\} - \mu \llbracket \Pi_h \rho_h \rrbracket)) \, ds \\
&\quad + \int_{\Gamma_D} ((\chi_h - \rho_h)n \cdot \{a_h \nabla_h v\} - \llbracket v \rrbracket \cdot (a_h \nabla_h \rho_h - \mu(\Pi_h u_h - \chi_h)n)) \, ds \\
&\quad + \int_{\Gamma_N} (\Pi_h \rho_h - \rho_h)n \cdot \{a_h \nabla_h v\} \, ds + \int_{\Gamma^0} \{\Pi_h \rho_h - \rho_h\} \llbracket a_h \nabla_h v \rrbracket \, ds \quad (4.3.11)
\end{aligned}$$

Taking into account that if  $v \in \mathcal{P}_1(\mathcal{T}_h)$  then

$$\begin{aligned}
\int_{\Gamma^0} \{\Pi_h \rho_h - \rho_h\} \llbracket a_h \nabla_h v \rrbracket \, ds &= 0 \\
\int_{\Gamma_N} (\Pi_h \rho_h - \rho_h)n \cdot \{a_h \nabla_h v\} \, ds &= 0 \\
\int_{\Gamma} \mu \llbracket v \rrbracket \cdot \llbracket \Pi_h \rho_h \rrbracket \, ds &= \int_{\Gamma} \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket \, ds
\end{aligned}$$

we have for all  $v \in V_h$ ,  $A_h(\rho_h, v) = B_h(\rho_h, v) + L_h(v)$  with

$$\begin{aligned}
B_h(\rho_h, v) &= \int_{\Omega} a_h \nabla_h \rho_h \cdot \nabla_h v \, dx \\
&\quad - \int_{\Gamma^0 \cup \Gamma_D} (\llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\} + \llbracket v \rrbracket \cdot \{a_h \nabla_h \rho_h\}) \, ds + \int_{\Gamma^0} \mu \llbracket \Pi_h \rho_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket \, ds \quad (4.3.12)
\end{aligned}$$

and

$$L_h(v) = \int_{\Gamma_D} (\chi_h n \cdot \{a_h \nabla_h v\} - \mu \chi_h n \cdot \llbracket v \rrbracket) \, ds.$$

Note that in this case, the formulation (4.3.8) is equivalent to use the mid-point-quadrature formula in the edge integrals in the following primal formulation of the standard interior penalty method: Find  $\rho_h \in V_h$  such that for all  $v \in V_h$  it holds

$$\begin{aligned}
&\int_{\Omega} a_h \nabla_h \rho_h \cdot \nabla_h v \, dx - \\
&\int_{\Gamma^0 \cup \Gamma_D} (\llbracket \rho_h \rrbracket \cdot \{a_h \nabla_h v\} + \llbracket v \rrbracket \cdot \{a_h \nabla_h \rho_h\}) \, ds + \int_{\Gamma^0 \cup \Gamma_D} \mu \llbracket \rho_h \rrbracket \cdot \llbracket v \rrbracket \, ds = \\
&\int_{\Omega} f v \, dx - \int_{\Gamma_D} (\chi_h n \cdot \{a_h \nabla_h v\} - \mu \chi_h n \cdot \llbracket v \rrbracket) \, ds.
\end{aligned}$$

The scheme (4.3.8) leads to a linear system

$$\widetilde{M}_p \rho_h = b_p$$

with  $M_p \in \mathbb{R}^{3Nel \times 3Nel}$ . Using the inverse transformation (4.2.11) we obtain the following linear system for  $u_h$

$$M_p u_h = b_p \quad (4.3.13)$$

where the entries of  $M_p$  are given by

$$M_{p,ij} = \int_e e^{-\frac{\psi|_K}{\varepsilon}} ds \widetilde{M}_{p,ij}$$

if  $K = \text{supp}(v_j)$  and  $e$  is the edge of  $K$  where  $v_j$  is identically 1 (recall that  $\{v_i\}_{i=1}^{3Nel}$  is the basis of  $V_h$  defined in the previous Section).

Now, we perform an analysis of the entries of the matrix  $M_p$ , similar to that made in the previous Section for the matrix  $M_m$ .

In order to obtain computable coefficients, we take  $a_h$  defined by (4.2.12).

The bilinear form  $B_h$  of equation (4.3.12) leads us to decompose  $\widetilde{M}_p$  as  $\widetilde{M}_p = \widetilde{M}_p^1 + \widetilde{M}_p^2 + \widetilde{M}_p^3$  where  $\widetilde{M}_p^k$  corresponds to the  $k$ -term of that bilinear form. Analogously we write  $M_p = M_p^1 + M_p^2 + M_p^3$ .

It is easy to see that, using the notation introduced above,

$$M_{p,ij}^1 = \varepsilon e^{\frac{\psi_{m,K} - \psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi_{K-\psi_{m,K,e}}}{\varepsilon}} ds}{\int_K e^{-\frac{\psi - \psi_{m,K}}{\varepsilon}} ds} \int_K \nabla_h v_i \cdot \nabla_h v_j dx.$$

Also if  $K' = \text{supp} v_i$  and  $e'$  is the edge of  $K'$  where  $v_i$  is identically 1, we see that  $M_{p,ij}^2 \neq 0$  only if either  $K = K'$  or  $K \cap K' = e$  or  $K \cap K' = e'$  (since  $V_h = \mathcal{P}^1(\mathcal{T}_h)$  and  $a_h \in \mathcal{P}^0(\mathcal{T}_h)$ ).

Then it results

$$\begin{aligned} M_{p,ij}^2 = \varepsilon e^{\frac{\psi_{m,K} - \psi_{m,K,e}}{\varepsilon}} & \frac{\int_e e^{-\frac{\psi_{K-\psi_{m,K,e}}}{\varepsilon}} ds}{\int_K e^{-\frac{\psi - \psi_{m,K}}{\varepsilon}} ds} \int_e \llbracket v_j \rrbracket \cdot \{\nabla_h v_i\} ds \\ & + \varepsilon e^{\frac{\psi_{m,K'} - \psi_{m,K,e}}{\varepsilon}} \frac{\int_e e^{-\frac{\psi_{K-\psi_{m,K,e}}}{\varepsilon}} ds}{\int_{K'} e^{-\frac{\psi - \psi_{m,K'}}{\varepsilon}} ds} \int_{e'} \llbracket v_i \rrbracket \cdot \{\nabla_h v_j\} ds. \end{aligned}$$

Finally, we note that  $\widetilde{M}_{p,ij}^3 \neq 0$  only if  $e = e'$ . Then, as in the mixed formulation, if we define  $\eta_e$  by (4.2.14) we obtain  $M_{p,ij}^3 = C_{ij}^2$ .

By our previous observations we see that all the exponentials appearing in  $M_p^k$ ,  $k = 1, 2, 3$  are computable.

In view of the definition (4.2.13) of  $\chi_h$ , the edge integrals in  $L_h$  can be treated in similar way.

*Remark 4.3.1.* Given a triangle  $K \in \mathcal{T}_h$  we set  $M_K$  be the  $3 \times 3$  matrix such that  $M_{K,ij} = B_h(v_i^K, v_j^K)$ . According with the above notation we have  $M_{K,ij} = M_{K,ij}^1 + M_{K,ij}^2 + M_{K,ij}^3$ . Now we will to analyze the matrix  $N_K = M_K^1 + M_K^2$ . Let  $e_1, e_2$  and  $e_3$  be the edges of  $K$ . Suppose that  $\nabla\psi|_K \sim (c_1, c_2)$ , and that  $\psi$  take different values on the vertexes of  $K$ . Also we suppose that the lengths of the sides of  $K$  are of the order of  $h$ . If  $e_j$  is such that  $\psi_{m,K,e_j} = \psi_{m,K}$  then we have

$$\int_{e_j} e^{-\frac{\psi|_K - \psi_{m,K,e_j}}{\varepsilon}} ds \sim \frac{1}{h} \int_0^h e^{-\frac{cs}{\varepsilon}} ds \sim \frac{\varepsilon}{h},$$

while if  $\psi_{m,K,e_j} \neq \psi_{m,K}$  we have

$$\int_{e_j} e^{-\frac{\psi|_K - \psi_{m,K,e_j}}{\varepsilon}} ds \sim 0.$$

Also, we have

$$\int_K e^{-\frac{\psi|_K - \psi_{m,K}}{\varepsilon}} dx \sim \frac{1}{h^2} \int_0^h r e^{-\frac{cr}{\varepsilon}} dr \sim \frac{\varepsilon^2}{h^2}.$$

It follows that

$$\varepsilon e^{\frac{\psi_{m,K} - \psi_{m,K,e_j}}{\varepsilon}} \frac{\int_{e_j} e^{-\frac{\psi|_K - \psi_{m,K,e_j}}{\varepsilon}} ds}{\int_K e^{-\frac{\psi|_K - \psi_{m,K}}{\varepsilon}} ds} \sim \begin{cases} h & \text{if } \psi_{m,K,e_j} = \psi_{m,K} \\ 0 & \text{if } \psi_{m,K,e_j} \neq \psi_{m,K} \end{cases}$$

So we see that

$$N_{K,ij} \sim \begin{cases} O(h) & \text{if } \psi_{m,K,e_j} = \psi_{m,K} \\ 0 & \text{if } \psi_{m,K,e_j} \neq \psi_{m,K} \end{cases}$$

That is, the coefficients corresponding to the node on the edge where  $\psi$  does not reach its minimum are zero (with respect to the machine precision). Such a node can be regarded as a downwind node (wind =  $\nabla\psi$ ), and the scheme as an upwind scheme. This behavior is called ‘‘automatic upwinding’’ in [13].

## 4.4 Numerical examples

We present some examples of solutions obtained using the method introduced in the previous Section.

We recall that the penalization term is treated as in Remark 4.2.1. In this way, we assure that it is large enough on each edge, that, as we will see in the next Section, it is a sufficient condition in order to have convergence (and it seems to be also necessary).

We want also to remark that the pictures that we present in Figure 4.2, show a continuous interpolation of each solution. As we are using the basis functions previously introduced, on each interior edge  $e$  we have two corresponding values,  $u_e^+$  and  $u_e^-$ . Then if  $\mathbf{v}$  is an interior vertex we define the interpolation  $u_{\mathbf{v}}$  as an average of the values  $\{u_e^+, u_e^- : \mathbf{v} \text{ is a vertex of } e\}$ . If  $\mathbf{v}$  is a boundary vertex, we define  $u_{\mathbf{v}}$  as an average of the values  $\{u_e : \mathbf{v} \text{ is a vertex of the boundary edge } e\}$ .

For all the examples in Figure 4.2 we have considered  $\Omega = [-1, 1]^2$ . We have taken  $\varepsilon = 10^{-6}$  and the following data: (a)  $\Gamma_D = \partial\Omega$ ,  $g = 0$ ,  $\psi(x, y) = \frac{1}{2}x + y$ , and  $f = 1$ ; (b)  $\Gamma_D = \{-1, 1\} \times [0, 1]$ ,  $g(-1, y) = \frac{1}{2}(y + |y|)$ ,  $g(1, y) = 0$ ,  $\psi(x, y) = x$  and  $f = 0$ ; (c)  $\Gamma_D$  and  $\psi$  as in (b),  $g = 0$  and  $f = 1$ ; (d)  $\Gamma_D = \{-1\} \times [-1, -0.6] \cup [-1, -0.6] \times \{-1\} \cup \{1\} \times [0.6, 1] \cup [0.6, 1] \times \{1\}$ ,  $g(x, y) = 10^3$  if  $x < 0$  and  $g(x, y) = 10^{17}$  if  $x > 0$ ,  $\psi = \psi(r)$  where  $r^2 = (x + 1)^2 + (y + 1)^2$  and

$$\psi(r) = \begin{cases} 0 & \text{if } r < 1.5 \\ 2(r - 1.5) & \text{if } 1.5 \leq r < 1.9 \\ 0.8 & \text{if } 1.9 \leq r \leq 2, \end{cases}$$

and  $f = f(r)$  with

$$f(r) = \begin{cases} 0 & \text{if } r < 1.6 \\ \frac{1}{2}10^{16} & \text{if } 1.6 \leq r < 1.8 \\ 0 & \text{if } 1.8 \leq r \leq 2. \end{cases}$$

The last example is taken from [13].

We have used unstructured meshes like that of Figure 4.3 (the mesh corresponding to the example (d) was refined once more). Notice the effectiveness of the scheme in capturing the internal and boundary layers without any spurious oscillations. We point out that the undershooting about the arc  $r = 1.5$ , in example (d), is an actual feature of the solution and it is not due to the numerical pollution.

## 4.5 Analysis of the modified interior penalty method

In this Section we study the convergence properties of the discontinuous Galerkin method proposed in the Sections 2 and 3, when it is used to solve a general elliptic equation. We call this method Modified Interior Penalty Method. To simplify the notation we consider

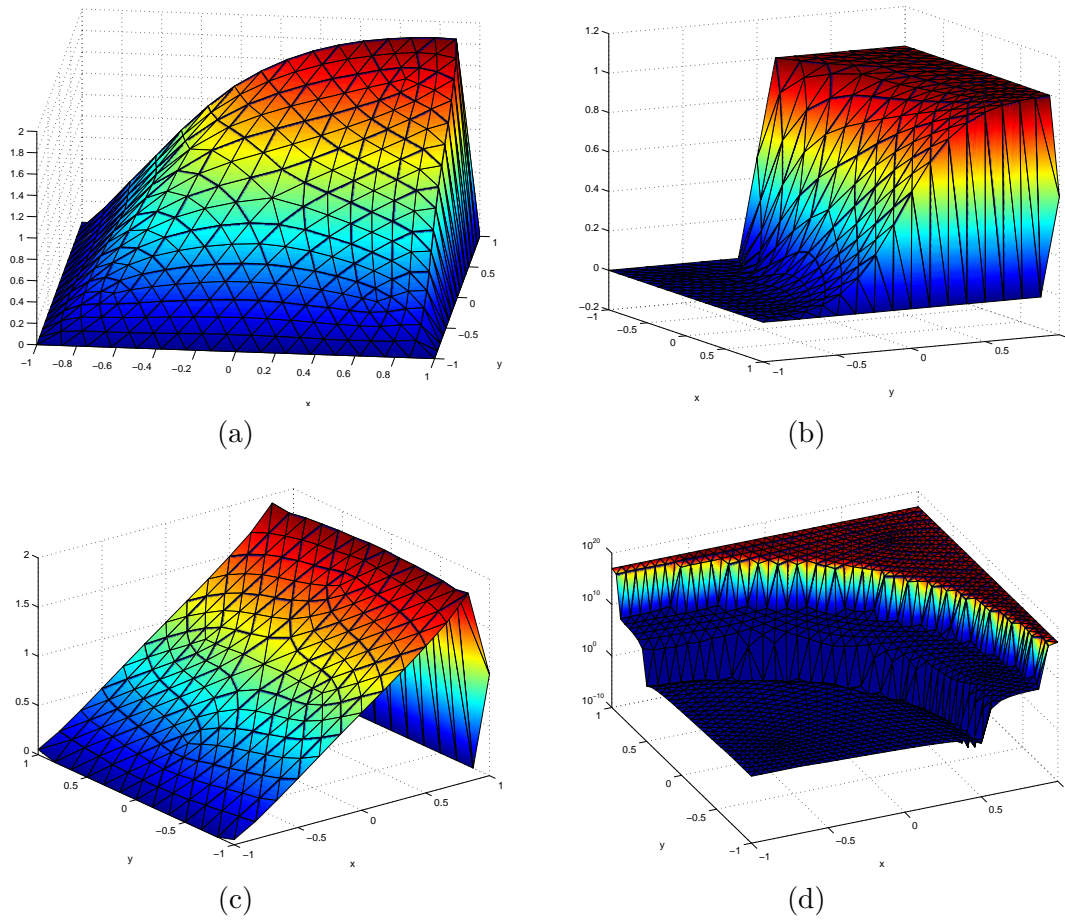


Figure 4.2: Numerical examples with exponential fitting

the model problem

$$\begin{aligned} -\Delta\rho &= f && \text{in } \Omega \\ \rho &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.5.1}$$

on a convex polygonal domain  $\Omega$ . Then the solution  $\rho$  verifies  $\rho \in H^2(\Omega)$ . Since  $\chi_h \equiv 0$ , using the notation of Section 3, it follows that  $L_h(v) = 0$  for all  $v \in H^2(\mathcal{T}_h)$ , and then  $A_h = B_h$  is a bilinear form. So, the primal formulation of our method is given by: Find  $\rho_h \in V_h$  such that

$$B_h(\rho_h, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_h$$



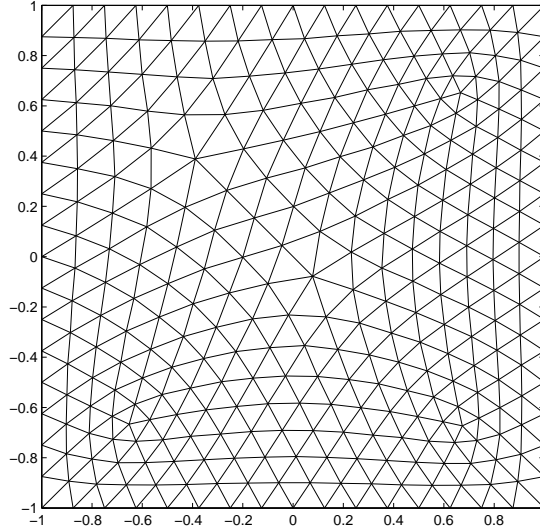


Figure 4.3: An unstructured mesh

where  $B_h$  is given by the r.h.s. of (4.3.7), that is,

$$B_h(\rho_h, v) = \int_{\Omega} \nabla_h \rho_h \cdot \nabla_h v \, dx + \int_{\Gamma} (\llbracket \hat{\rho} - \rho_h \rrbracket \cdot \{\nabla_h v\} - \{\hat{\sigma}\} \cdot \llbracket v \rrbracket) \, ds + \int_{\Gamma^0} (\{\hat{\rho} - \rho_h\} \llbracket \nabla_h v \rrbracket - \llbracket \hat{\sigma} \rrbracket \{v\}) \, ds$$

independently of the choice of the numerical fluxes  $\hat{\rho}$  and  $\hat{\sigma}$ . As in [8], Section 3.3, by using the integration by parts formula (4.3.3), if  $\rho$  is the solution of (4.5.1), we have for any  $v \in H^2(\mathcal{T}_h)$  that

$$\int_{\Omega} \nabla_h \rho \cdot \nabla_h v \, dx = - \int_{\Omega} \Delta \rho v \, dx + \int_{\Gamma} \{\nabla_h \rho\} \cdot \llbracket v \rrbracket \, ds + \int_{\Gamma^0} \llbracket \nabla_h \rho \rrbracket \{v\} \, ds.$$

Taking into account the regularity of  $\rho$  it follows that  $\{\rho\} = \rho$ ,  $\llbracket \rho \rrbracket = 0$ ,  $\{\nabla_h \rho\} = \nabla \rho$ ,  $\llbracket \nabla_h \rho \rrbracket = 0$ , then we have (see also eq. (3.12) of [8])

$$B_h(\rho, v) = \int_{\Omega} f v \, dx + \int_{\Gamma} (\llbracket \hat{\rho} \rrbracket \cdot \{\nabla_h v\} + (\nabla \rho - \{\hat{\sigma}\}) \cdot \llbracket v \rrbracket) \, ds + \int_{\Gamma^0} ((\{\hat{\rho}\} - \rho) \llbracket \nabla_h v \rrbracket - \llbracket \hat{\sigma} \rrbracket \{v\}) \, ds.$$

It is understood  $\hat{\rho} = \hat{\rho}(\rho)$  and  $\hat{\sigma} = \hat{\sigma}(\rho, \sigma_h(\rho))$  with

$$\sigma_h(\rho) = \nabla_h \rho - r(\llbracket \hat{\rho} - \rho \rrbracket) - l(\{\hat{\rho} - \rho\}).$$

We have

$$\hat{\rho}(\rho) = \Pi_0^e \rho \quad \forall e \text{ edge.}$$

It follows that  $[[\hat{\rho}]] = [[\rho]] = 0$ ,  $\{\hat{\rho}\} = \Pi_0^e \rho$  and  $\{\rho\} = \rho$ . Therefore

$$\sigma_h(\rho) = \nabla_h \rho - l(\Pi_0^e \rho - \rho).$$

Also we have

$$\hat{\sigma}(\rho, \sigma_h(\rho)) = \hat{\sigma}(\rho) = \{\nabla_h \rho\} - \mu [[\Pi_h \rho]] = \nabla_h \rho.$$

Then  $[[\hat{\sigma}]] = 0$  and  $\{\hat{\sigma}\} = \nabla_h \rho$ . We arrive at

$$B_h(\rho, v) = \int_{\Omega} f v \, dx + \int_{\Gamma^0} (\Pi_0^e \rho - \rho) [[\nabla_h v]] \, ds.$$

It follows that

$$B_h(\rho, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_h \tag{4.5.2}$$

and then, following [8], the primal formulation is consistent. Note that the equation (4.5.2) is not valid  $\forall v \in H^2(\mathcal{T}_h)$ . Being  $\hat{\rho}(\rho)|_e = \Pi_0^e \rho \neq \rho$  and  $\hat{\sigma}(\rho, \nabla \rho) = \hat{\sigma}(\rho) = \nabla \rho$  it results that the flux  $\hat{\rho}$  is not consistent and the flux  $\hat{\sigma}$  is consistent. Finally, taking into account that  $[[\hat{\rho}]] = 0$  and  $[[\hat{\sigma}]] = 0$  the primal formulation is also adjoint consistent [8].

Now we deal with the convergence analysis of our method. We introduce a functional setting. Let  $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\mathcal{T}_h)$ . On  $V(h)$  we consider the norm

$$|||v|||^2 = |v|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_*^2$$

where

$$|v|_*^2 = \sum_{e \in \mathcal{E}} h_e^{-1} ||[\Pi_h v]||_{0,e}^2 \quad [[\Pi_h v]]|_e = \Pi_0^e v^+ n^+ + \Pi_0^e v^- n^-.$$

It follows from the discrete Poincaré inequality (4.5.5) (proved in Lemma 4.5.3) that

$$|||v|||_0 \leq C (|v|_{1,h}^2 + |v|_*^2)^{\frac{1}{2}} \leq C |||v||| \quad \forall v \in V(h).$$

Also, by an inverse inequality

$$|||v||| \leq C (|v|_{1,h}^2 + |v|_*^2)^{\frac{1}{2}} \quad \forall v \in V_h.$$

The proof of the next Proposition is postponed to the end of the section.

**Proposition 4.5.1.** *The bilinear form  $B_h$  verifies*

$$|B_h(w, v)| \leq C_b \|w\| \|v\| \quad \forall w \in V(h), v \in V_h, \quad (4.5.3)$$

and if  $\eta_e \geq \eta_0$  for all  $e \in \mathcal{E}$  with  $\eta_0$  large enough then

$$B_h(v, v) \geq C_s \|v\|^2 \quad \forall v \in V_h. \quad (4.5.4)$$

The *a priori* error estimates can now be proven in the standard way [8]. Indeed, let  $\rho_I \in V_h$  be the usual continuous interpolant of  $\rho$  which satisfies

$$\|\rho - \rho_I\|^2 = |\rho - \rho_I|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\rho - \rho_I|_{2,K}^2 \leq C_a h^2 |\rho|_{2,\Omega}^2.$$

Note that the jumps of  $\rho - \rho_I$  are zero at the interelement. Then, using the stability (4.5.4), consistency (4.5.2), boundedness (4.5.3) and the above approximation property, we have

$$\begin{aligned} C_s \|\rho_I - \rho_h\|^2 &\leq B_h(\rho_I - \rho_h, \rho_I - \rho_h) \\ &= B_h(\rho_I - \rho, \rho_I - \rho_h) \\ &\leq C_b \|\rho_I - \rho\| \|\rho_I - \rho_h\| \\ &\leq C_b C_a h |\rho|_{2,\Omega} \|\rho_I - \rho_h\| \end{aligned}$$

that is

$$\|\rho_I - \rho_h\| \leq \frac{C_b C_a}{C_s} h |\rho|_{2,\Omega}.$$

Thus, the following Theorem is a consequence of the triangle inequality.

**Theorem 4.5.2.** *If  $\eta_e \geq \eta_0$  for all  $e \in \mathcal{E}$  with  $\eta_0$  large enough, then there exists a constant  $C$  such that*

$$\|\rho - \rho_h\| \leq Ch |\rho|_{2,\Omega}.$$

Now we prove the following discrete Poincaré inequality, showing that the norm  $\|\cdot\|$  is stronger than the  $L^2$  norm  $\|\cdot\|_0$ . We introduce the following notation. For an edge  $e \in \mathcal{E}$  we choose an unit normal vector  $n = n_e$ , such that  $n$  is pointing outward if  $e \subset \partial\Omega$ . For a function  $\psi \in H^2(\Omega)$ , on the edge  $e$  we put  $\frac{\partial\psi}{\partial n} = \nabla\psi \cdot n$ , and for a function  $\phi \in H^1(\mathcal{T}_h)$  we define  $[\phi]$  on  $e$  as  $[[\phi]] \cdot n$ .

**Lemma 4.5.3.** For all  $\phi \in H^1(\mathcal{T}_h)$  we have

$$\|\phi\|_0 \leq C \left( |\phi|_{1,h}^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \| [\![\Pi_h]\!] \|_{0,e}^2 \right)^{\frac{1}{2}}. \quad (4.5.5)$$

where  $\mathcal{E}$  denotes the set of edges of  $\mathcal{T}_h$ , and  $h_e$  is the length of the edge  $e$ .

*Proof.* Let  $\phi \in H^1(\mathcal{T}_h)$ . Consider  $\psi$  defined by  $-\Delta\psi = \phi$  in  $\Omega$  and  $\psi = 0$  on  $\partial\Omega$ . Then  $\|\psi\|_2 \leq C\|\phi\|_0$ . We have

$$\begin{aligned} \|\phi\|_0^2 &= -(\phi, \Delta\psi) = -(\phi, \operatorname{div}(\nabla\psi)) \\ &= (\nabla_h\psi, \nabla_h\phi) - \sum_{e \in \mathcal{E}} (\{\nabla\psi\}, [\![\phi]\!] )_e - \sum_{e \in \mathcal{E}^0} ([\![\nabla\psi]\!] , \{\phi\})_e \\ &= (\nabla_h\psi, \nabla_h\phi) - \sum_{e \in \mathcal{E}} \int_e [\phi] \frac{\partial\psi}{\partial n} ds \\ &= (\nabla_h\psi, \nabla_h\phi) - \sum_{e \in \mathcal{E}} \int_e [\Pi_h\phi] \frac{\partial\psi}{\partial n} ds - \sum_{e \in \mathcal{E}} \int_e [\phi - \Pi_h\phi] \frac{\partial\psi}{\partial n} ds \\ &\leq \left( \|\nabla\phi\|_{0,h}^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \| [\phi - \Pi_h\phi] \|_{0,e}^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \| [\Pi_h\phi] \|_{0,e}^2 \right)^{\frac{1}{2}} \\ &\quad \left( \|\nabla\psi\|_{0,h}^2 + 2 \sum_{e \in \mathcal{E}} h_e \| \frac{\partial\psi}{\partial n} \|_{0,e}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.5.6)$$

For all  $T \in \mathcal{T}_h$  and  $e$  edge of  $T$  the following trace inequality holds (see [7] ineq. (2.4))

$$\|v\|_{0,e}^2 \leq C (h_e^{-1} \|v\|_{0,T}^2 + h_e |v|_{1,T}^2) \quad \forall v \in H^1(T), \quad (4.5.7)$$

then we obtain

$$\|\phi - \Pi_0^e\phi\|_{0,e}^2 \leq C (h_e^{-1} \|\phi - \Pi_0^e\phi\|_{0,T}^2 + h_e |\phi|_{1,T}^2). \quad (4.5.8)$$

Applying Theorem 1 of [11], the following Poincaré inequality holds if  $\hat{T}$  is a reference triangle and  $\hat{e}$  is one of its edges

$$\|\hat{v} - \Pi_0^{\hat{e}}\hat{v}\|_{0,\hat{T}} \leq \hat{C} |\hat{v}|_{1,\hat{T}} \quad \forall \hat{v} \in H^1(\hat{T})$$

and then by standard scaling arguments we obtain

$$\|v - \Pi_0^e v\|_{0,T} \leq \hat{C} h_T |v|_{1,T} \quad \forall v \in H^1(T)$$

where  $h_T$  is the diameter of the triangle  $T$ . So it follows from (4.5.8) that

$$\|\phi - \Pi_0^e \phi\|_{0,e}^2 \leq C (h_e^{-1} h_T^2 + h_e) |\phi|_{1,T}^2.$$

Because the mesh is regular we have

$$\|[\phi - \Pi_h \phi]\|_{0,e}^2 \leq C h_e \sum_{T/e \subset T} |\phi|_{1,T}^2. \quad (4.5.9)$$

From (4.5.6) we have

$$\|\phi\|_0^2 \leq C \left( |\phi|_{1,h}^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \|[\Pi_h \phi]\|_{0,e}^2 \right)^{\frac{1}{2}} \left( |\psi|_1^2 + 2 \sum_{e \in \mathcal{E}} h_e \left\| \frac{\partial \psi}{\partial n} \right\|_{0,e}^2 \right)^{\frac{1}{2}}.$$

As in [7] we have

$$|\psi|_1^2 + 2 \sum_{e \in \mathcal{E}} h_e \left\| \frac{\partial \psi}{\partial n} \right\|_{0,e}^2 \leq \|\phi\|_0^2$$

from which we obtain (4.5.5).  $\square$

*Proof of the Proposition 4.5.1.* We begin with the boundedness (4.5.3). Notice that in our case, from equation (4.3.12) we have for all  $v \in V_h$

$$\begin{aligned} B_h(\rho_h, v) &= \int_{\Omega} \nabla_h \rho_h \cdot \nabla_h v \, dx - \\ &\quad \int_{\Gamma} ([\rho_h] \cdot \{\nabla_h v\} + [v] \cdot \{\nabla_h \rho_h\}) \, ds + \int_{\Gamma_0} \mu [\Pi_h \rho_h] \cdot [\Pi_h v] \, ds \end{aligned}$$

Therefore it is sufficient to bound only terms like

$$\int_{\Gamma} \{\nabla_h w\} \cdot [v] \, ds = \sum_{e \in \mathcal{E}} \int_e \left( \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n} \right) (v^+ - v^-) \, ds$$

for  $w, v \in V(h)$ , where if  $e$  is an internal edge,  $e = K \cap K'$ , and  $v \in V(h)$ , we put  $v^+ = v|_K$  and  $v^- = v|_{K'}$ . We have

$$\int_e \frac{\partial w}{\partial n} (v^+ - v^-) \, ds = \int_e \frac{\partial w}{\partial n} \Pi_0^e (v^+ - v^-) \, ds + \int_e \frac{\partial w}{\partial n} ((v^+ - v^-) - \Pi_0^e (v^+ - v^-)) \, ds.$$

Since  $\Pi_0^e (v^+ - v^-) = [\Pi_h v] \cdot n^+$  and  $v^+ - v^- = [v] \cdot n^+$  it follows that

$$\left| \int_e \frac{\partial w}{\partial n} (v^+ - v^-) \, ds \right| \leq h_e^{\frac{1}{2}} \left\| \frac{\partial w}{\partial n} \right\|_{0,e} \left( h_e^{-\frac{1}{2}} \|[\Pi_h v]\|_{0,e} + h_e^{-\frac{1}{2}} \|[v - \Pi_h v]\|_{0,e} \right).$$

From (4.5.9) we have

$$h_e^{-1} \| [v - \Pi_h v] \|_{0,e}^2 \leq C \sum_{T/e \subset T} |v|_{1,T}^2$$

and, using the trace inequality (4.5.7), we have

$$\left| \int_e \frac{\partial w}{\partial n} (v^+ - v^-) ds \right| \leq C \sum_{T/e \subset T} (|w|_{1,T}^2 + h_T^2 |w|_{2,T}^2)^{\frac{1}{2}} \left( h_e^{-1} \| [\Pi_h v] \|_{0,e}^2 + \sum_{T/e \subset T} |v|_{1,T}^2 \right)^{\frac{1}{2}}$$

Adding on all the  $e \in \mathcal{E}$  it results

$$\begin{aligned} \left| \int_{\Gamma} \{\nabla_h w\} \cdot [v] ds \right| &\leq C \left( |w|_{1,h}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 |w|_{2,T}^2 \right)^{\frac{1}{2}} \left( |v|_1^2 + \sum_{e \in \mathcal{E}} h_e^{-1} \| [\Pi_h v] \|_{0,e}^2 \right)^{\frac{1}{2}} \\ &\leq C \| |w| \| (|v|_{1,h} + |v|_*). \end{aligned}$$

Then we have

$$\left| \int_{\Gamma} \{\nabla_h w\} \cdot [v] ds \right| \leq C \| |w| \| \| |v| \| \quad \forall v, w \in V(h)$$

and (4.5.3) easily follows.

In order to prove the stability (4.5.4) of  $B_h$ , note that if  $v \in V_h$  we have

$$\int_{\Gamma} \{\nabla_h v\} \cdot [v] ds = \int_{\Gamma} \{\nabla_h v\} \cdot [\Pi_h v] ds = \int_{\Gamma} h_e^{\frac{1}{2}} \left( \frac{\partial v^+}{\partial n} - \frac{\partial v^-}{\partial n} \right) h_e^{-\frac{1}{2}} (\Pi_0^e v^+ - \Pi_0^e v^-) ds$$

and then

$$\begin{aligned} \left| \int_{\Gamma} \{\nabla_h v\} \cdot [v] ds \right| &\leq \sum_e \left( \sum_{T/e \subset T} (|v|_{1,T}^2 + h_T^2 |v|_{2,T}^2) \right)^{\frac{1}{2}} h_e^{-\frac{1}{2}} \| [\Pi_h v] \|_{0,e} \\ &\leq C \left( |v|_{1,h}^2 + \sum_T h_T^2 |v|_{2,T}^2 \right)^{\frac{1}{2}} \left( \sum_e h_e^{-1} \| [\Pi_h v] \|_{0,e}^2 \right)^{\frac{1}{2}} \\ &\leq C \| |v| \| |v|_*. \end{aligned}$$

Now, we impose that  $\eta_e$  is taken on each edge  $e$  greater than a fixed positive constant  $\eta_0$ , chosen later on. Thus, using that  $(|v|_{1,h}^2 + |v|_*^2) \leq \| |v| \|^2 \leq M(|v|_{1,h}^2 + |v|_*^2)$  for all  $v \in V_h$  we

have

$$\begin{aligned}
B_h(v, v) &\geq |v|_{1,h}^2 + \eta_0 |v|_*^2 - 2C \|v\| |v|_* \\
&\geq |v|_{1,h}^2 + \eta_0 |v|_*^2 - \theta C \|v\|^2 - \frac{1}{\theta} C |v|_*^2 \\
&= (|v|_{1,h}^2 + |v|_*^2 - \frac{1}{2M} \|v\|^2) + (\eta_0 - 1) |v|_*^2 - \frac{1}{\theta} C |v|_*^2 \\
&\geq \frac{1}{2} (|v|_{1,h}^2 + |v|_*^2) \\
&\geq \frac{1}{2M} \|v\|^2
\end{aligned}$$

if we take  $\theta C = \frac{1}{2M}$  and  $\eta_0 - 1 - \frac{C}{\theta} \geq 0$ . Thus, we obtain (4.5.4).  $\square$

## 4.6 A final remark

As we have seen in Section 2, the exponential fitting is not a suitable stabilization procedure to be applied to the well known Interior Penalty method in the mixed formulation. For which we have introduced a modification in the definition of the numerical fluxes, and we have obtained a new method, whose primal formulation seem to have the same good properties as the original method.

However, it is possible to stabilize the Interior Penalty method in the primal formulation with the exponential fitting. In fact, in some numerical experiments, this approach seem also to work well, with respect to the stability, the only difference is that it seems to be more diffusive.

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