Preface

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Editors

The subject of this book is the solution of polynomial equations, that is, systems of (generally) non-linear algebraic equations. This study is at the heart of several areas of mathematics and has provided the motivation for advances in different branches of mathematics such as algebra, geometry, topology, and numerical analysis. In recent years, an explosive development of algorithms and software has made it possible to solve many problems which had been intractable up to then and greatly expanded the areas of applications to include robotics, machine vision, signal processing, structural molecular biology, computer-aided design and geometric modelling, as well as certain areas of statistics, optimization and game theory, and biological networks. At the same time, symbolic computation has proved to be an invaluable tool for experimentation and conjecture in pure mathematics. As a consequence, the interest in effective algebraic geometry and computer algebra has extended well beyond its original constituency of pure and applied mathematicians and computer scientists, to encompass many other scientists and engineers. While the core of the subject remains algebraic geometry, it also calls upon many other aspects of mathematics and theoretical computer science, ranging from numerical methods, differential equations and number theory to discrete geometry, combinatorics and complexity theory.

The goal of this book is to provide a general introduction to modern mathematical aspects in computing with multivariate polynomials and in solving algebraic systems. It is aimed to upper-level undergraduate and graduate students, and researchers in pure and applied mathematics and engineering, interested in computational algebra and in the connections between computer algebra and numerical mathematics. Most chapters assume a solid grounding in Linear Algebra while for several of them a basic knowledge of Gröbner bases, at the level of [CLO97] is expected. Gröbner bases have become a basic standard tool in Computer Algebra and the reader may consult any other textbook such as [AL94, BW93, CLO98, GP02], or the introductory chapter in [CCS99]. Below we discuss briefly the content of each chapter and some of their prerequisites.

The book describes foundations, recent developments and applications of Gröbner and border bases, residues, multivariate resultants, including toric elimination theory, primary decomposition of ideals, multivariate polynomial factorization, as well as homotopy continuation methods. While some of the chapters are introductory in nature, others present the state-of-the-art in symbolic techniques in polynomial system solving, including effective and algorithmic methods in algebraic geometry and computational algebra, complexity issues and applications. We also discuss several numeric and symbolic-numeric methods. This is not a standard textbook in that each chapter is independent and, largely, self-contained. However, there are strong links between the different chapters as evidenced by the many cross-references. While the reader gains the advantage of being able to access the book at many different places and of seeing the interplay of different views of the same concepts, we should note that, because of the different needs and traditions, some notations inevitably vary between different chapters. We have tried to note this in the text whenever it occurs. The single bibliography and index underline the unity of the subject.

The first chapter gives an introduction to the notions of residues and resultants, and the interplay between them, starting with the univariate case and synthesizing different approaches. The sections on univariate residues and resultants could be used in an undergraduate course on complex analysis, abtract algebra, or computational algebra as an introduction to more advanced topics and to illustrate the interdependence of different areas of mathematics. The multivariate sections, on the other hand, directed to graduate students and researchers, are intended as an introduction to concepts which are widely used in current research and applications.

The second chapter puts the accent on linear algebra methods to deal with polynomial systems: the multiplication maps in the quotient algebra by a polynomial ideal are linear and allow for the use of eigenvalues and eigenvectors, duality, etc. Applications to Galois theory, factoring, and primary decomposition are offered. The first sections require, besides standard linear algebra, some background on computational algebraic geometry (for instance, the first five Chapters of [CLO97]). Some acquaintance with local rings (as in Chapter 4 of [CLO98]) would also be helpful. Known basic facts about field extensions and Galois theory are assumed in the last part.

The third chapter also elaborates on the concepts in the first two chapters, and combines them with numerical methods for polynomial system solving and several applications. The tools and methods developed are used to solve problems arising in implicitization of rational surfaces, determination of the position of a camera or a parallel robot, molecular conformations, and blind identification in signal processing. The required background is very similar to that needed for the first sections of Chapter 2.

Chapter four is devoted to laying the algebraic foundations for border bases of ideals, an extension of the theory of Gröbner bases, yielding more flexible bases of the quotient algebras. Border bases yield a connection between computer algebra and numerical mathematics. An application to design of experiments in statistics is included.

The fifth chapter concentrates on various techniques for computing primary decomposition of ideals. This machinery is applied to study an interesting class of ideals coming from Bayesian networks, establishing an important link between algebraic geometry and and the emerging field of algebraic statistics. Besides Gröbner bases, the readers are expected to have a casual understanding of the algebra-geometry dictionary between ideals in polynomial rings and their zero set. Many propositions that can be found in the literature are stated without proof, but the chapter contains several accesible exercises dealing with the structure and decomposition of polynomial ideals.

Chapter six studies the inherent complexity of polynomial system solving when working with the dense encoding of the input polynomials and under the model of straight-line programs, i.e., when polynomials are not given by their monomials but by evaluation programs. Being a brief survey of algebraic complexity applied to computational algebraic geometry, there is not much background required, though knowledge of basic notions of algebraic geometry and commutative algebra would be helpful. The chapter is mostly self-contained; when necessary, basic bibliography supplements are indicated.

Chapter seven is devoted to the study of sparse systems of polynomial equations, i.e., algebraic equations with a specific monomial structure, presenting a comprehensive state-of-the-art introduction to the field. Combinatorial and discrete geometry, together with matrices of special structure, are ingredients of the presentation of toric (or sparse) elimination theory. The chapter focuses on applications to geometric modelling and computer-aided design. It also provides the tools for exploiting the structure of algebraic systems which may arise in different applications. Some basic knowledge of discrete geometry for polyhedral objects in arbitrary dimension is assumed. This chapter will be of particular interest to graduate students and researchers in theoretical computer science or applied mathematics wishing to combine discrete and algebraic geometry.

Chapter eight deals with numerical algebraic geometry, a term coined some years ago to describe a new field, which bears the same relation to algebraic geometry as numerical linear algebra does to linear algebra. Modern homotopy methods to describe solution components of polynomial systems are presented. The prerequisites include a basic course in numerical analysis, in particular Newton's method for nonlinear systems. Because of the numerical flavor of the proposed methods, this chapter is expected to be particulary appealing to engineers.

Lastly, Chapter nine gives a complete overview of old and recent methods for the important problem of approximate factorization of a multivariate polynomial, in other words, the complex irreducible decomposition of a hypersurface. The main techniques rely on approximate numerical computation but the results are exact and certified. It is addressed to students and researchers with some basic knowledge of commutative algebra, algebraic numbers and holomorphic functions of several variables.

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