7/26/11

ELGA

Exercises for Deformation theory (Roughly two per day)

1. Twisted cubic curves.

a) Show that the twisted cubic curves in $\mathbb{P}^3$ form a family of dimension 12.

b) If Y is a twisted cubic curve, compute the normal bundle $\mathcal{N}_{Y/\mathbb{P}^3}$ and show that $h^0(Y, \mathcal{N}_{Y/\mathbb{P}^3}) = 12$. Hence the Hilbert scheme is smooth at that point.

2. Zero-schemes of length 4.

a) Show that sets of four distinct points in $\mathbb{P}^3$ form an open subset U of dimension 12 of the Hilbert scheme X of zero-schemes of length 4 in $\mathbb{P}^3$.

b) Let Y be the length 4 zero-scheme concentrated at a point whose local form is $\operatorname{Spec} k[x,y,z]/\mathfrak{m}^2$, where $\mathfrak{m} = (x,y,z)$. Show that $h^0(Y, \mathcal{N}_{Y/\mathbb{P}^3}) = 18$, so Y corresponds to a singular point of the Hilbert scheme. To prove this, you need to show also

c) Y is in the closure of the open set U of a).

3. A rigid scheme.

Let X be the affine scheme defined as $X = \operatorname{Spec} k[x,y,z,w]/(x,y) \cap (z,w)$. Show that X is a rigid scheme, even though X is singular. (Geometrically, it is the union of two planes meeting at a single point.

4. A node.

Let $B = k[x,y]/(xy)$. Compute $T^1(B/k, B)$. Show that the family defined by $k[x,y,t]/(xy-t)$ gives a first-order deformation that corresponds to a non-zero element of T^1.

5. Twisted cubic curves.

If Y is a twisted cubic curve in $\mathbb{P}^3$, verify that $H^1(Y, \mathcal{N}_{Y/\mathbb{P}^3}) = 0$, giving another proof (cf. #1 above) that the Hilbert scheme is smooth at that point.

6. Quartic surfaces in $\mathbb{P}^3$.

a) Show that the Hilbert scheme of quartic (degree 4) surfaces in $\mathbb{P}^3$ is smooth of dimension 34.

b) If $X_1, X_2 \subseteq \mathbb{P}^3$ are two quartic surfaces, and if ~~[illegible] as abstract surfaces~~, $\sigma \in \operatorname{Aut}\mathbb{P}^3$ is an automorphism of $\mathbb{P}^3$ that sends X_1 to X_2, then $X_1 \cong X_2$ are isomorphic as abstract surfaces. Since the group $\operatorname{Aut}\mathbb{P}^3$ has dimension 15, we expect ~~a~~ the family of abstract nonsingular quartic surfaces to have dimension $\leq 34-15=19$.

c) Show however that for X ~~a~~ ~~[illegible]~~ a nonsingular quartic surface in $\mathbb{P}^3$, $H^0(X, T_X) = 0$, $h^1(X, T_X) = 20$, $h^2(X, T_X) = 0$, so we expect a family of abstract nonsingular surfaces of dimension 20.

d) Prove that if X_0 is a nonsingular quartic surface in $\mathbb{P}^3$, then there exists a deformation of X_0, as an abstract surface, over the dual numbers D, that does not arise from any ~~[illegible]~~ ~~[illegible]~~ flat family $X \subseteq \mathbb{P}^3_D$, flat over D.

7. Elliptic curves.

We define an elliptic curve to be a nonsingular projective curve X of genus 1 over an algebraically closed field k together with a fixed point P.

a) Show that if to each elliptic curve (X, P) we associate its j-invariant [AG, IV, §4], then the j-line $\operatorname{Spec} k[j]$ acts as a coarse moduli space for elliptic curves.

b) However, show that there does not exist a tautological family $\mathcal{X}$ of elliptic curves over $\operatorname{Spec} k[j]$, with the property that for each j, the fibre $\mathcal{X}_j$ is an elliptic curve with invariant j.

8. Invertible sheaves on an integral curve.

Let X be an integral projective curve over k algebraically closed.

a) Show that the family of all invertible sheaves L on X of fixed degree d is a bounded family, i.e. there exists a scheme of finite type T/k and a family $\mathcal{L}$ of invertible sheaves on $X \times T$, whose fibres $\mathcal{L}_t$ for $t \in T$ include all possible invertible sheaves on X.

b) Show that the family of invertible sheaves is separated, i.e. if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two families on $X \times T$, where T is a nonsingular curve, if $0 \in T$ is a point, and if $(\mathcal{L}_1)_t \cong (\mathcal{L}_2)_t$ for all $t \neq 0$, then also $(\mathcal{L}_1)_0 \cong (\mathcal{L}_2)_0$.

c) Show however that the family of all L's may not be complete, i.e. it is possible to have a family $\mathcal{L}$ defined on $X \times (T \setminus \{0\})$, where T is a nonsingular curve, $0 \in T$, but there is no family $\overline{\mathcal{L}}$ on $X \times T$ extending $\mathcal{L}$. (If X is nonsingular, then the family is complete.)

9. <u>Rational curves of degree 5 in $\mathbb{P}^3$</u>

a) Show that the Hilbert scheme of smooth, rational curves in $\mathbb{P}^3$ of degree 5 is smooth and irreducible of dimension 20.

b) Show that a general such curve (meaning a general point on that Hilbert scheme) is contained in some cubic surface, but is not contained in any quadric surface.

10. <u>Stable curves.</u>

A <u>stable curve</u> is a reduced, connected, projective curve having ~~any~~ at most nodes as singularities, of arithmetic genus $p_a \geq 2$ with the additional property that if any irreducible component has $p_a = 0$, then that component meets the rest of the curve in at least three points. (Example: two rational curves meeting each other in three points:  .)

a) If X_0 is a stable curve, show that there are no obstructions to deformations of X_0 (as an abstract curve).

b) Show that deformations of X_0 over the dual numbers D are classified by a vector space of dimension $3p_a - 3$.

c) Conclude that the modular family of stable curves (assuming it exists) is smooth of dimension $3p_a - 3$.