HOMOGENEOUS ORTHOGONALLY-ADDITIVE POLYNOMIALS ON BANACH LATTICES

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Abstract. Our main result is a representation theorem for n-homogeneous orthogonally-additive polynomials on Banach lattices.

The representation theorem is used to study the linear span of the set of zeros of homogeneous real-valued orthogonally-additive polynomials. We show that in certain lattices every element can be represented as the sum of two or three zeros or, at least, can be approximated by such sums.

We also indicate how these results can be used to study weak topologies induced by orthogonally-additive polynomials on Banach lattices .

1. Introduction

A continuous scalar-valued map P on a Banach space X is called a homogeneous polynomial of degree n (or a n-homogeneous polynomial) if $P(x) = \Phi(x, \dots, x)$, where Φ is a continuous n-linear form on X. (Vector-valued polynomials are defined similarly.) We only consider continuous polynomials and will therefore usually omit the adjective continuous.

A polynomial is continuous iff it is bounded on the unit ball of X. We denote by $\mathcal{P}(^nX)$ the Banach space of n-homogeneous scalar-valued continuous polynomials equipped with the norm

$$\|P\| = \sup_{\|x\|\leq 1} \|P(x)\|.$$

We shall use standard notation and terminology, see [D] and [M] for notation and results regarding polynomials and [LT] for notation and basic theory of Banach lattices.

Recall that two elements x, y in a Banach lattice are called orthogonal (or disjoint) if $|x| \wedge |y| = 0$.

Definition 1.1. Let X be a Banach lattice. A polynomial P on X is said to be orthogonally-additive if $P(x+y) = P(x) + P(y)$ whenever $x, y \in X$ are orthogonal. The set of all n-homogeneous orthogonally-additive scalar-valued polynomials on X is denoted by $\mathcal{P}_o(^nX)$.

There are various weak topologies induced on a Banach space X by the polynomials on X. Two of the authors studied in [LL] the analogs of these topologies induced by the class of orthogonally-additive polynomials on L_p and l_p. Their

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main tool was a theorem of Sundaresan [S] which gave an explicit representation of n-homogeneous orthogonally-additive polynomials on these spaces.

Our main result, Theorem 2.3, is a representation theorem for polynomials in $\mathcal{P}_o(^nX)$, which generalizes [S]. It turns out that $\mathcal{P}_o(^nX)$ can be identified with the linear functionals on the n-concavification $X_{(n)}$ of X. (In fact, the representation theorem holds for vector-valued polynomials as well.)

In section 4 we generalize the results of [LL] on weak polynomial topologies, induced by the orthogonally-additive polynomials on L_p and l_p, to general Köthe function spaces. To this end we need to study the zero sets of real-valued n-homogeneous orthogonally-additive polynomials. This is done in section 3. Since there has recently been some growing interest in the zero sets of real-valued polynomials (see e.g. [AGZ], [ABRZ]), we analyze these zero sets in some more detail than is really necessary for the study of the weak topologies.

Fix $P \in \mathcal{P}_o(^nX)$ and denote its zero set $P^{-1}(0)$ by Z and the subspace that Z generates by H. Note that the homogeneity of the polynomial implies that Z is a symmetric cone, i.e., $x \in Z$ implies that $\lambda x \in Z$ for all $\lambda \in \mathbb{R}$. Put $D_kZ = \{\sum_{i=1}^k z_i : z_i \in Z\}$ and then $H = \bigcup_{k\geq 1} D_kZ$.

We show that many Banach lattices X (including the Köthe function spaces) have the property that for suitable n's the subspace H is dense in X for every n-homogeneous orthogonally-additive polynomial P on X. In fact, the sets D_3Z or even D_2Z are already either all of X or are dense in it.

2. Representation of Orthogonally-Additive Polynomials

Let X be a Banach lattice. To simplify the presentation we shall assume that X is a lattice of functions on some set (or a lattice of equivalence classes of measurable functions on a measure space (Ω, Σ, μ)) with the usual order. Theorem 2.3 below holds for general lattices without this restriction, but the assumption simplifies the presentation and makes it more intuitive. In particular, working on a function lattice simplifies the functional calculus on X. For example, if $f \in X$ and $\alpha > 0$, then f^α is defined explicitly by $f^\alpha(s) = |f|^\alpha(s)\mathrm{sign} f(s)$. We shall use standard lattice inequalities without further mention (see for example [LT, Proposition 1.d.2]).

The construction of the concavification of X also becomes more direct and intuitive in the case of lattices of functions: Let $q > 1$, then the q-concavification of X is the space $X_{(q)} = \{f^q : f \in X\}$ with the usual algebraic operations and order and with the natural quasi-norm $|||f||| = \|f^{1/q}\|^q$ for $f \in X_{(q)}$. (See [KPR] for information on quasi-norms and quasi-normed spaces.) To see that this is a quasi-norm fix $f, g \in X_{(q)}$. Then

$$\begin{aligned} |||f+g||| &= \|(f+g)^{1/q}\|^q \leq \|(|f|^{1/q} + |g|^{1/q})\|^q \leq (\|f^{1/q}\| + \|g^{1/q}\|)^q \\ &\leq 2^{q-1}(\|f^{1/q}\|^q + \|g^{1/q}\|^q) = 2^{q-1}(|||f||| + |||g|||). \end{aligned}$$

A similar proof (see also [LT, page 54]) shows that if X is q-convex with q-convexity constant M, then

$$|||f_1 + \cdots + f_n||| \leq M^q(|||f_1||| + \cdots + |||f_n|||)$$

for every $f_1, \ldots, f_n \in X_{(q)}$. It follows that when X is q-convex with q-convexity constant $M = 1$, then the quasi-norm $|||\cdot|||$ is actually a norm. When $M > 1$ the quasi-norm $\|\!| \cdot |\!\|$ is equivalent to the norm given by

$$\|\!| f |\!\|_1 = \inf \Big\{ \sum \|\!| f_i |\!\| : f = \sum f_i \Big\}.$$

In what follows we shall not pass to the equivalent norm even when X is q-convex, and we shall only use the quasi-norm $\|\!| \cdot |\!\|$.

Note that the q-concavification of $L_r(\mu)$ is naturally identified with $L_{r/q}(\mu)$. It follows that it is a Banach space for $q \le r$ and a quasi-Banach space if $r < q$.

We shall need some basic facts on Baire-1 functions on compact metric spaces and we refer to Chapter XV of Natanson's book [Nat] for more details. (The book only treats functions on a close interval, but the results and proofs are the same for a general compact metric space.)

Let K be a compact metric space. A function f on K is said to be of Baire class 1 (or a Baire-1 function) if it is the pointwise limit of a sequence of continuous functions. The space of bounded real-valued Baire-1 functions, equipped with the supremum norm, is a Banach lattice, which we denote by $B_1(K)$. In the next two lemmas we shall use Lebesgue's characterization of Baire-1 functions (see [Nat, Theorem 1, page 141]): A real-valued function f on K is Baire-1 iff the sets $\{f > \alpha\}$ and $\{f < \alpha\}$ are F_σ for every $\alpha \in \mathbb{R}$.

The first lemma is well known.

Lemma 2.1. *The simple functions are dense in $B_1(K)$.*

Proof. Fix $f \in B_1(K)$ and assume, without loss of generality, that $0 \le f(k) \le 1$ for every $k \in K$. Fix N and put $A_i = \{k \in K : \frac{i-1}{N} < f(k) < \frac{i+1}{N}\}$ for $0 \le i \le N$. By Lebesgue's theorem the sets A_i are F_σ and clearly $K = \cup A_i$. We now find *disjoint* F_σ sets $B_i \subset A_i$ such that $\cup B_i = K$. (This is just Lemma 2, page 140 in [Nat], which we reproduce for the sake of the reader.) Indeed, since the A_i's are F_σ there are closed sets C_n and disjoint subsets $M_0, \dots, M_N \subset \mathbb{N}$ so that $A_i = \cup_{n \in M_i} C_n$. Put $D_n = C_n \setminus \big(\cup_{j<n} C_j\big)$. Then the D_n's are pairwise disjoint F_σ sets (because closed subsets of K are G_δ). Hence so are the disjoint sets $B_i = \cup_{n \in M_i} D_n$. Clearly $B_i \subset A_i$ and $\cup B_i = K$.

The simple function $g_N = \sum \frac{i}{N} \chi_{B_i}$ is then a Baire-1 function which satisfies $\|f - g_N\| \le 1/N$. □

Lemma 2.2. *Let K be a compact metric space and let P be a real-valued polynomial on $C(K)$. Then P extends to a polynomial Q on $B_1(K)$. If P is orthogonally-additive, then so is Q.*

Proof. The extension is given explicitly as follows: fix $g \in B_1(K)$ and a bounded sequence of continuous functions g_j converging pointwise to g. Then put $Q(g) = \lim P(g_j)$. Since $C(K)$ has the Dunford-Pettis property and since the g_j's are a weak Cauchy sequence in $C(K)$, it follows from Pełczyński [Pel1, Corollary 3] that the limit actually exists and is independent of the choice of the sequence g_j.

We now check that when P is orthogonally-additive, then so is Q. Indeed, fix $f, g \in B_1(K)$ with disjoint supports. By Lebesgue's characterization of Baire-1 functions there are two increasing sequence F_n and G_n of closed sets so that $\{|f| > 0\} = \cup F_n$ and $\{|g| > 0\} = \cup G_n$. For each fixed n the two sets F_n and G_n are closed and disjoint, hence there are continuous functions φ_n and ψ_n with disjoint supports, $\|\varphi_n\| = \|\psi_n\| = 1$, so that $\varphi_n \equiv 1$ on F_n and $\psi_n \equiv 1$ on G_n.

Pick now two bounded sequences of continuous functions, f_n and g_n, which converge pointwise to f and g respectively. Then, by the construction, the products

$f_n\varphi_n$ and $g_n\psi_n$ also converge pointwise to f and g respectively, and they clearly satisfy $|f_n\varphi_n| \wedge |g_n\psi_n| = 0$. By the orthogonal-additivity of P

$$Q(f+g) = \lim P(f_n\varphi_n + g_n\psi_n) = \lim \big(P(f_n\varphi_n) + P(g_n\psi_n)\big) = Q(f) + Q(g).$$

□

Remark. In [Pel2] Pełczyński actually shows that weakly compact vector-valued polynomials on $C(K)$ extend to the space of all bounded Baire functions (and not just to $B_1(K)$). We do not need this stronger result.

Although we are mainly interested in the representation of scalar-valued polynomials, Theorem 2.3 applies to vector-valued ones as well, so we introduce the necessary notation.

Let X be a Banach (or quasi-Banach) lattice and let E be a Banach space. We denote the space of bounded linear operators from X to E by $L(X,E)$ and the space of E-valued orthogonally-additive polynomials by $\mathcal{P}_o(^nX,E)$.

For each linear operator $T \in L(X_{(n)},E)$ define a continuous n-homogeneous orthogonally-additive polynomial P_T from X to E by the formula $P_T(f) = T(f^n)$ for $f \in X$. Then P_T is induced by the continuous n-linear map $A(f_1,\dots,f_n) = T(f_1 \cdot \ldots \cdot f_n)$ and it is orthogonally-additive because $(f+g)^n = f^n + g^n$ whenever f and g have disjoint supports. It turns out that this is the general form of such a polynomial.

Theorem 2.3. *Let X be a Banach lattice of functions and let E be a Banach space. Fix $n \in \mathbb{N}$. Then the map $T \to P_T$ is a linear isometry of $L(X_{(n)},E)$ onto $\mathcal{P}_o(^nX,E)$. In particular, when E is the scalar field, the map $\varphi \to P_\varphi$ is a surjective linear isometry between $\big(X_{(n)}, \|\!|\cdot|\!\|\big)^*$ and $\mathcal{P}_o(^nX)$.*

Proof. We denote the quasi-norm of an operator $T \in L(X_{(n)},E)$ by $\|\!|T|\!\|$, i.e., $\|\!|T|\!\| = sup_{\|\!|g|\!\|\le 1}\|Tg\|$.

The map $T \to P_T$ is clearly linear. It is an isometry by the definition of $\|\!|\cdot|\!\|$:

$$\|P_T\| = \sup_{\|f\|\le 1} \|P_T(f)\| = \sup_{\|f\|\le 1} \|T(f^n)\| = \sup_{\|\!|g|\!\|\le 1} \|T(g)\| = \|\!|T|\!\|.$$

To show that the map is surjective fix $P \in \mathcal{P}_0(^nX,E)$ and put $T(f) = P(f^{1/n})$. It is clear that T is 1-homogeneous and continuous. We only need to check that T linear. It is then clear that $P = P_T$.

Thus fix $f_1, f_2 \in X_{(n)}$ and assume first that they are simple functions. Passing to the algebra of sets generated by the atoms of f_1 and f_2 we may assume that the two functions are actually linear combinations of the same disjointly supported characteristic functions, i.e., $f_1 = \sum a_i\chi_{E_i}$ and $f_2 = \sum b_i\chi_{E_i}$, where the sets E_i are disjoint. Then $(f_1+f_2)^{1/n} = \sum (a_i+b_i)^{1/n}\chi_{E_i}$ and the orthogonal-additivity and n-homogeneity of P yield

$$\begin{aligned} T(f_1+f_2) &= P((f_1+f_2)^{1/n}) = P\big(\sum (a_i+b_i)^{1/n}\chi_{E_i}\big) \\ &= \sum P\big((a_i+b_i)^{1/n}\chi_{E_i}\big) = \sum (a_i+b_i)P(\chi_{E_i}) \\ &= P\big(\sum a_i^{1/n}\chi_{E_i}\big) + P\big(\sum b_i^{1/n}\chi_{E_i}\big) \\ &= P(f_1^{1/n}) + P(f_2^{1/n}) = T(f_1) + T(f_2). \end{aligned}$$

If the lattice X is such that every element of X is a limit of simple functions, then the theorem follows by approximating f_1 and f_2. To prove the additivity for a general lattice we may assume, by composing P with linear functionals on E, that P is scalar-valued. We shall assume that the scalar field is $\mathbb{R}$. Only simple modifications are needed in the complex case.

The following standard construction enables us to pass from X to a $C(K)$ space, with K compact and metrizable: Fixing $f_1, f_2 \in X$, put $h = |f_1| + |f_2|$ and let

$$N = \big\{ f \in X : |f| \le \lambda h \ \text{ for some } \lambda > 0 \big\}.$$

For $f \in N$ put $\|f\|_N = \inf\{\lambda > 0 : |f| \le \lambda h\}$. Then $\|f\| \le \|f\|_N$ for every $f \in N$. One checks easily that $(N, \|\cdot\|_N)$ is complete, hence a Banach lattice under the order induced from X. Also h is a strong unit in N and N is an abstract M-space (see Lindenstrauss and Tzafriri [LT, Definition 1.b.1]). The same is true for the closed sublattice $(M, \|\cdot\|_N)$ of N generated by f_1 and f_2. By Kakutani's representation theorem [LT, Theorem 1.b.6] M is isometric and order-isomorphic to a $C(K)$-space, and since it is separable, it follows that K is compact and metrizable.

Let P_1 be the composition of P with the formal identity from M to X. Then P_1 is a continuous n-homogeneous polynomial which is orthogonally-additive because the orders on X and M coincide. By lemma 2.2 P_1 extends to an orthogonally-additive polynomial Q on $B_1(K)$. Since by lemma 2.1 every function in $B_1(K)$ is a limit of simple functions, the argument in the beginning of the proof gives that $Q\big((g_1+g_2)^{1/n}\big) = Q(g_1^{1/n}) + Q(g_2^{1/n})$ for every $g_1, g_2 \in B_1(K)$. In particular $P_1\big((f_1+f_2)^{1/n}\big) = P_1(f_1^{1/n}) + P_1(f_2^{1/n})$ for the given functions f_1 and f_2, and then the same identity holds also for P. □

Remarks. 1. Sundaresan [S] proved that for $1 \le n \le p < \infty$ the space $\mathcal{P}_o({}^nL_p)$ can be identified with $L_{p/(p-n)}$, and similarly for l_p. For $p > n$ he showed that $\mathcal{P}_o({}^nL_p) = \{0\}$ and $\mathcal{P}_o({}^nl_p) = l_\infty$. These are special cases of Theorem 2.3 and the known representations of the duals of L_r and l_r for $0 < r < \infty$.

2. The case $X = C(K)$ of Theorem 2.3 was recently proved independently by Pérez-García and Villanueva [PV].

3. It should be noted that the representation theorem is trivial for discrete lattices, i.e., for spaces with an unconditional basis $\{e_j\}$. Indeed, if $f = \sum a_j e_j$, then $P(f^{1/n}) = P\big(\sum a_j^{1/n} e_j\big) = \sum a_j P(e_j)$, which is clearly linear.

4. For earlier results on representation of orthogonally-additive functions on certain classes of Banach lattices see, e.g., [DO, FK, MM, MS, Pi, S] and their references.

From now on we shall turn to real-valued polynomials, i.e., to $\mathcal{P}_o({}^nX)$. The successful application of the theorem depends on a good description of the dual of $X_{(n)}$. To this end we shall restrict ourselves from now on to Köthe function spaces.

Definition 2.4. A Banach lattice X of equivalent classes of locally integrable measurable functions on a complete σ-finite measure space (Ω, Σ, μ) is called a Köthe function space if

- If $g \in X$ and if f is measurable and $|f(\omega)| \le |g(\omega)|$ a.e., then $f \in X$ and $\|f\| \le \|g\|$.
- $\chi_E \in X$ for every $E \in \Sigma$ with finite measure.

The class of Köthe function spaces contains many of the common Banach lattices. Moreover, by [LT, Theorem 1.b.14] every order continuous Banach lattice with a weak unit is isomorphic as a Banach space and as a lattice to a Köthe function space. We shall use the easy fact that order continuous Köthe function spaces satisfy the Dominated Convergence Theorem: if $f_n \to 0$ in measure and if there is a $g \in X$ so that $|f_n| \le g$ for every n, then $\|f_n\| \to 0$.

By the discussion in [LT, page 29] it follows that when X is an order continuous Köthe function space, then its dual is given by integrals. More precisely, every continuous linear functional on X is given by $\varphi(f) = \int f\xi d\mu$, where ξ is a measurable function such that $f\xi \in L_1(\mu)$ for every $f \in X$. This representation also holds for quasi-Banach lattice, hence for functionals on $(X_{(n)}, \|\!| \cdot \|\!|)$ (which is order continuous whenever X is). Note, however, that when X is not n-convex it may very well happen that $(X_{(n)}, \|\!| \cdot \|\!|)^* = \{0\}$. This happens, for example, when $X = L_p$ and $p < n$.

We summarize the results of this section in the way they will be used in the next sections:

Corollary 2.5. *Let X be an order continuous Köthe function space. Then every n-homogeneous orthogonally-additive polynomial $P \in \mathcal{P}_o(^nX)$ can be represented as*

$$P(f) = \int f^n \xi d\mu$$

for some measurable function ξ on Ω.

3. Sums of zeros

In this section we study the zero sets of n-homogeneous orthogonally-additive polynomials on order continuous Köthe function spaces and the subspaces that these zero sets generate. Recall that the zero set of the polynomial P is denoted by Z and that $D_kZ = \{\sum_{j=1}^k z_j : z_j \in Z\}$. We shall always assume that μ is a nonnegative measure.

Theorem 3.1. *Let X be an order continuous Köthe function space and let $P(f) = \int f^n \xi d\mu$ be a n-homogeneous orthogonally-additive polynomial on X.*

1. If n is even and ξ does not have a constant sign, then $X = D_2Z$.

2. If $n > 1$ is odd and $\int_A \xi d\mu \neq 0$ for at least three disjoint measurable subsets A, then $X = D_3Z$.

Proof. Fix $f \in X$, and we first prove (1). Denote the restrictions of f to the disjoint sets $\{\xi > 0\}$ and $\{\xi \le 0\}$ by f_1 and f_2 respectively and put $a_j = P(f_j)$. Then $a_1 \ge 0 \ge a_2$ (because n is even). If $a_1 \neq 0$ and $a_2 \neq 0$ choose $\lambda \in \mathbb{R}$ such that $a_1 + \lambda^n a_2 = 0$. It then follows that both $z_1 = f_1 + \lambda f_2$ and $z_2 = f_1 - \lambda f_2$ are in Z and $f = \frac{\lambda+1}{2\lambda} z_1 + \frac{\lambda-1}{2\lambda} z_2$.

If $a_2 = 0$ (say), then take any g supported in $\{\xi < 0\}$ such that $P(g) = -a_1$, and then $P(f \pm g) = 0$ and $f = (f+g)/2 + (f-g)/2$.

To prove (2) choose three disjointly supported functions f_1, f_2, f_3 with $P(f_i) \neq 0$ such that f is a linear combination of the f_i's. (This is possible by the assumption on the measure $\xi d\mu$.) Put $P(f_i) = a_i^{-n}$ and let

$$\begin{aligned} z_1 &= a_1 f_1 - a_2 f_2 \\ z_2 &= a_2 f_2 - a_3 f_3 \\ z_3 &= a_1 f_1 + a_2 f_2 - 2^{1/n} a_3 f_3. \end{aligned}$$

Clearly $z_i \in Z$, and since the matrix $\begin{pmatrix} a_1 & -a_2 & 0 \\ 0 & a_2 & -a_3 \\ a_1 & a_2 & -2^{1/n}a_3 \end{pmatrix}$ is invertible it follows that the z_i's and the f_i's span the same three dimensional subspace of X. In particular f is a linear combination of the z_i's, i.e., $f \in D_3Z$. □

Remark. The conditions of the theorem are necessary:

1. If n is even and ξ has a constant sign, then clearly $Z = \{f \in X : f\xi = 0 \text{ a.e. } d\mu\}$, i.e., $f \in Z$ iff the support of f is disjoint from the support of ξ.

2. If $n > 1$ is odd and the measure $\xi d\mu$ consists of just two atoms, then Z is a one-codimensional subspace. Indeed, every $f \in X$ is constant on the two atoms and we denote these values by f_1, f_2 respectively. Also denote by α_1, α_2 the $\xi d\mu$ measures of the atoms. Then $Z = \{f \in X : f_1\alpha_1^{1/n} = -f_2\alpha_2^{1/n}\}$.

3. If $n > 1$ is odd we really need to pass to D_3Z and it is no longer true that $X = D_2Z$. A simple example is $P(f) = \int f^3 d\mu$ (i.e., $\xi \equiv 1$) on $L_p[0,1]$ for $p \geq 3$. In this case $\chi_E \notin D_2Z$ for any measurable set of positive measure. Indeed, assume for contradiction that $\chi_E = f_1 + f_2$ with $f_i \in Z$ and choose g such that $f_1 = \frac{1}{2}\chi_E + g$ and $f_2 = \frac{1}{2}\chi_E - g$. Then

$$0 = P(f_1) + P(f_2) = \int (\frac{1}{2}\chi_E + g)^3 + \int (\frac{1}{2}\chi_E - g)^3 = \int \chi_E(\frac{1}{4} + 3g^2)$$

which is impossible because E has positive measure.

The next result shows that when $n > 1$ and the measure space is non-atomic, then D_2Z, which (as we saw above) is not necessarily equal to X, is at least dense in it.

Theorem 3.2. *Let X be an order continuous Köthe function space on a non-atomic measure space. Let $n > 1$ be an odd integer and let $P(f) = \int_0^1 f^n \xi d\mu$. Then the set D_2Z is dense in X.*

Proof. Fix $f \in X$ and we start with a few reductions. We may assume that f and ξ are nonnegative. Indeed, Ω decomposes as the disjoint union of four sets on each of which f and ξ have constant sign, and it suffices to approximate f on each of these sets separately. We may also assume by approximation that the support of f, which we denote by E, has finite measure and then, by normalizing μ and ξ, that $\mu(E) = P(\chi_E) = 1$.

Fix $m \in \mathbb{N}$ and put $t = t_m = \frac{(m+1)^n}{2m+1}$.

Claim. *There is a partition of E to $2m+2$ disjoint sets B and $\{A_j\}_{j=-m}^m$ such that $\mu(A_j) = \frac{1}{(2m+1)(t+1)} = P(f\chi_{A_j})$ for every j and such that $\mu(B) = \frac{t}{t+1} = P(f\chi_B)$.*

Indeed, note that $\nu(A) = P(f\chi_A) = \int_A f^n \xi d\mu$ is a non-atomic probability measure on E (by our normalization that $P(\chi_E) = 1$). By Liapounoff's theorem (see [R, Theorem 5.5]) the range of the vector measure (μ, ν) is convex. Since $\mu(\emptyset) = \nu(\emptyset) = 0$ and $\mu(E) = \nu(E) = 1$, it follows that there is a subset $A_{-m} \subset E$ such that $\mu(A_{-m}) = \nu(A_{-m}) = \frac{1}{(2m+1)(t+1)}$. The set A_{-m+1} is obtained similarly by applying Liapounoff's theorem to $E \setminus A_{-m}$. We continue inductively to obtain the other A_j's and then take $B = E \setminus \cup_{-m}^m A_j$.
This proves the claim.

Define two functions by

$$g_m(\omega) = \begin{cases} f(\omega) & \omega \in B \\ jf(\omega) & \omega \in A_j,\ j \geq 1 \\ (j-1)f(\omega) & \omega \in A_j,\ j \leq 0 \end{cases}$$

and

$$h_m(\omega) = \begin{cases} 0 & \omega \in B \\ (j+1)f(\omega) & \omega \in A_j,\ j \geq 1 \\ 0 & \omega \in A_0 \\ (j-1)f(\omega) & \omega \in A_j,\ j \leq -1 \end{cases}$$

and we check that they are in Z. To show that $P(g_m) = 0$ write (by the orthogonal-additivity of P)

$$P(g_m) = \sum_{j=1}^{m} \big(P(g_m\chi_{A_j}) + P(g_m\chi_{A_{1-j}})\big) + \big(P(g_m\chi_{A_{-m}}) + P(g_m\chi_B)\big).$$

For each $1 \leq j \leq m$ the j'th term in the sum vanishes because

$$P(g_m\chi_{A_j}) = j^n P(f\chi_{A_j}) = \frac{j^n}{(2m+1)(t+1)} = -P(g_m\chi_{A_{1-j}}).$$

The remaining term also vanishes. Indeed, $P(g_m\chi_B) = P(f\chi_B) = \frac{t}{(t+1)}$ and the choice of t and of A_{-m} gives

$$P(g_m\chi_{A_{-m}}) = -(m+1)^n P(f\chi_{A_{-m}}) = \frac{-(m+1)^n}{(2m+1)(t+1)} = \frac{-t}{1+t}.$$

We omit the computation, similar to the first computation above, which shows that $P(h_m) = 0$.

Since

$$(g_m - h_m)(\omega) = \begin{cases} f(\omega) & \omega \in B \\ -f(\omega) & \omega \in A_j,\ j \geq 0 \\ 0 & \omega \in A_j,\ j \leq -1 \end{cases},$$

it follows that $f - (g_m - h_m) = f \cdot \big(2\chi_{\cup_{j\geq 0} A_j} + \chi_{\cup_{j\leq -1} A_j}\big)$. But $\mu(\cup A_j) = \frac{1}{t+1} < \frac{2m+1}{(m+1)^n} \to 0$ as $m \to \infty$. Thus $g_m - h_m \to f$ by the Dominated Convergence Theorem in the order continuous lattice X.

□

4. Weak polynomial topologies

In analogy with the weak topology ω on a Banach space, it is natural to define the weak polynomial topology wp, where a net x_α converges to x iff $P(x_\alpha) \to P(x)$ for every polynomial P on X (see Carne, Cole and Gamelin [CCG]). It turns out that wp is not a vector space topology: addition, although clearly continuous in each variable separately, is not necessarily continuous as a function of two variables. This led Garrido, Jaramillo and Llavona [GJL] to introduce and study the maximal locally convex topology, τ_p, weaker than wp. It is given by the seminorms

$$d_P(x) = \inf\big\{|P(y_1 - y_0)|^{\frac{1}{n}} + |P(y_2 - y_1)|^{\frac{1}{n}} + \ldots + |P(x - y_k)|^{\frac{1}{n}}\big\}$$

where P is a n-homogeneous polynomial, and the infimum is taken over all k-chain $\{0 = y_0, y_1, \ldots, y_k = x\}$.

Note that the seminorm associated with a linear functional $\varphi \in X^*$ is given by $d_\varphi(x) = |\varphi(x)|$, hence the weak topology ω satisfies $\omega \subset \tau_p$. One also checks easily that $\tau_p \subset wp \subset \|\cdot\|$, where $\|\cdot\|$ denotes the norm topology on X.

Lassalle and LLavona [LL] introduced analogous topologies on Banach lattices. These topologies are defined similarly by using only the orthogonally-additive polynomials and they are denoted by wp_o and τ respectively. As before $\tau \subset wp_o$. When X is an order continuous Köthe function space every linear functional is given by an integral, hence orthogonally-additive. It follows that $\omega \subset \tau$.

The paper [LL] is devoted to the study of the special case of l_p and L_p. Their main result in this direction is

Theorem 4.1. *A net $\{x_\alpha\}$ in L_p (respectively l_p) is τ-convergent to x iff it is weakly convergent to x and $\|x_\alpha - x\|_{2k} \to 0$, where k is the largest integer with $2k \le p$ (respectively the smallest with $2k \ge p$).*

The main tool in [LL] is Sundaresan's identification [S] of orthogonally-additive n-homogeneous polynomials on L_p as the dual of $L_{p/n}$ (respectively $l_{p/n}$). By Theorem 2.3 this tool is now available in general lattices.

The representation of orthogonally-additive polynomials gives explicit formulas for the seminorms d_P. This makes it possible to analyze their zero sets and the subspaces generated by these zeros, as we did in section 3. This is important for the analysis of the τ topology: It was observed in [GJL] that if P is a homogeneous polynomial on X and $z \in H = \text{span}\{\ker(P)\}$, then $d_P(x) = d_P(x+z)$ for every $x \in X$. Indeed, assume that $z = \sum_{j=1}^m z_j$ with $P(z_j) = 0$ and put $y_i = \sum_{j=1}^i z_j$. Extending any chain from 0 to x by adjoining successively the $x + y_i$'s at the end of the given chain gives a new chain from 0 to $x+z$ so that P is zero on the new differences. Hence $d_P(x+z) \le d_P(x)$, and $d_P(x) \le d_P(x+z)$ similarly. It follows in particular that

$$d_P \equiv 0 \text{ for every polynomial } P \text{ for which } H \text{ is dense in } X.$$

It follows that in the analysis of τ we need only consider polynomials whose zero sets do not span a dense subspace of X.

Corollary 4.2. *Let X be an order continuous Köthe function space on a measure space (Ω, μ) and let x_α be a net in X which converges weakly to x. Then $x_\alpha \xrightarrow{\tau} x$ iff $d_P(x_\alpha - x) \to 0$ for every n-homogeneous orthogonally-additive polynomial $P(f) = \int f^n \xi d\mu$ on X with even n and nonnegative ξ.*

Proof. By Theorem 3.1 $d_P \equiv 0$ unless P is as above, or when $n > 1$ is odd and $\xi d\mu$ has just two nonzero atoms. In the latter case, write $P(f) = f_1^n \alpha_1 + f_2^n \alpha_2$ and consider the linear functional $\varphi(f) = f_1\alpha_1 + f_2\alpha_2$. Then one checks directly that $d_P(f_\alpha - f) \to 0$ iff $\varphi(f_\alpha - f) \to 0$. □

Theorem 4.1 for L_p, say, follows immediately, because by Remark 1 after Theorem 2.3 the only non-zero continuous homogeneous orthogonally-additive polynomials on L_p are of degree $n \le p$, and by Corollary 4.2 only the even degree ones give nontrivial seminorms and influence the topology. The fact that L_p is rearrangement invariant and the density of simple functions in $L^*_{p/n} = L_{p/(n-p)}$ yield easily that we may assume that $\xi \equiv 1$. Finally, Hölder's inequality implies that we only need to consider the largest admissible even n.

The same procedure can, of course, be used for other lattices, and we shall not give detailed examples. We just mention that one can identify explicitly the concavifications of, say, Orlicz or Lorentz spaces, and then use them to give analogous results on τ for these lattices.

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