ORTHOGONALLY ADDITIVE POLYNOMIALS OVER $C(K)$ ARE MEASURES —A SHORT PROOF

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Abstract. We give a simple proof of the fact that orthogonally additive polynomials on $C(K)$ are represented by regular Borel measures over K. We also prove that the Aron-Berner extension preserves this class of polynomials.

Introduction.

The main result of this paper is due to Benyamini, Lassalle and Llavona [3], and also to Pérez-García and Villanueva [9]. It is the following.

Theorem. *For any orthogonally additive polynomial P over $C(K)$ —i.e., those for which $P(u+v) = P(u) + P(v)$ when $u \perp v$, (that is, $uv = 0$)—, there is a regular Borel measure μ over K such that*

$$P(u) = \int_K u^k \, d\mu \qquad \textit{for all} \quad u \in C(K).$$

We viewed their result as a linearization result: k-homogeneous orthogonally additive polynomials are linearized through the map $C(K) \to C(K)$ given by $u \mapsto u^k$. In [5], a general linearization procedure had been recently constructed and we set out to prove the theorem as an application of that construction, for the space of orthogonally additive polynomials. The resulting proof is short and simple, and therefore we believe that this presentation of the Benyamini-Lassalle-Llavona-Pérez-García-Villanueva theorem may be of interest.

The tools we use are standard functional analysis and topology, plus the linearization theorem mentioned above and the Aron-Berner extension of a homogeneous polynomial to the bidual [2]. We also study the Aron-Berner extension of orthogonally additive polynomials and show that this class is preserved by the extension morphism.

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We refer to [7, 8] for notation and general results regarding polynomials on infinite dimensional spaces.

1. The Proof.

Let K be a compact Haussdorff space. We define, for each closed or open $A \subset K$,

$$r_A : C(K) \to C(K)'' \text{ given by } r_A(u) = u1_A$$

where 1_A is the indicator function of A. Clearly r_A is a continuous linear transformation, and $\|r_A\| \leq 1$. Now for each $P \in \mathcal{P}^k(C(K))$ and $A \subset K$ closed or open, define

$$P_A = \overline{P} \circ r_A,$$

where $\overline{P}$ is the Aron-Berner extension of P. Note that $(P_A)_B = \overline{P_A} \circ r_B = \overline{P} \circ r_{A\cap B} = P_{A\cap B}$.

Lemma 1.1. *Given a closed subset A of K and $u, v \in C(K)$, there are nets $(u_i), (v_i) \subset C(K)$ w^*-converging to $u1_A$ and $v1_{A^c}$ respectively, with $u_j \perp v_i$ for all $j \geq i$.*

Proof. We construct such nets converging to 1_A and 1_{A^c}. The nets in the statement of the lemma are then (uu_i) and (vv_i). Let $M(K)$ be the space of regular Borel measures on K. We will use, as an indexing set

$$I = \{(\mu_1, \dots, \mu_n; \frac{1}{m}) : n, m \in N, \mu_1, \dots, \mu_n \in M(K)\}$$

ordered by $(\nu_1, \dots, \nu_k; \frac{1}{r}) \geq (\mu_1, \dots, \mu_n; \frac{1}{m})$ if $\{\mu_1, \dots, \mu_n\} \subset \{\nu_1, \dots, \nu_k\}$ and $m \leq r$. Given $i = (\mu_1, \dots, \mu_n; \frac{1}{m})$, by regularity of the measures there is an open set $V \subset K$ such that $|\mu_r(V - A)| < \frac{1}{m}$ for $r = 1, \dots, n$. Since K is normal, there is an open U such that $A \subset U \subset \overline{U} \subset V$. Also, take $u_i : K \to [0, 1]$ and $v_i : K \to [0, 1]$ continuous and such that $u_i = 1$ on A and $\operatorname{supp}(u_i) \subset U$, $v_i = 1$ on V^c and $\operatorname{supp}(v_i) \subset \overline{U}^c$. Note that $u_i \perp v_i$, and also that having defined this for all $i \leq j$ (and there are finitely many such i's!), the definition for j can be made with a smaller U, so that $u_j \perp v_i$. We have for $r = 1, \dots, n$

$$\left| \int_K u_i - 1_A \, d\mu_r \right| \leq \int_{U-A} |u_i - 1_A| \, d|\mu_r| \leq |\mu_r(U - A)| \leq |\mu_r(V - A)| < \varepsilon,$$

$$\left| \int_K v_i - 1_{A^c} \, d\mu_r \right| \leq \int_{V-A} |v_i - 1_{A^c}| \, d|\mu_r| \leq |\mu_r(V - A)| < \varepsilon.$$

Thus, u_i converges $w*$ to 1_A and v_i to 1_{A^c}. □

Lemma 1.2. *Let $P \in \mathcal{P}^k(C(K))$ be orthogonally additive, and $A \subset K$ closed (or open). Then $P = P_A + P_{A^c}$ and $\|P\| = \|P_A\| + \|P_{A^c}\|$. Also, P_A and P_{A^c} are orthogonally additive.*

Proof. For the first equality, it will be enough to see that, given $u \in C(K)$, $\overline{P}(u) = \overline{P}(u1_A) + \overline{P}(u1_{A^c})$, for then

$$P(u) = \overline{P}(u) = \overline{P}(u1_A) + \overline{P}(u1_{A^c}) = P_A(u) + P_{A^c}(u) = (P_A + P_{A^c})(u).$$

Let ϕ be the symmetric k-linear form associated to P. Since $C(K)$ is Arens regular, its Aron-Berner extension $\overline{\phi}$ is symmetric. Thus

$$\begin{aligned}\overline{P}(u) &= \overline{P}(u1_A + u1_{A^c}) = \overline{\phi}(u1_A + u1_{A^c}, \dots, u1_A + u1_{A^c}) \\ &= \overline{P}(u1_A) + \sum_{r=1}^{k-1} \binom{k}{r} \overline{\phi}(\underbrace{u1_A, \dots, u1_A}_{k-r}, \underbrace{u1_{A^c}, \dots, u1_{A^c}}_{r}) + \overline{P}(u1_{A^c}).\end{aligned}$$

We will show that $\overline{\phi}(\underbrace{u1_A, \dots, u1_A}_{k-r}, \underbrace{u1_{A^c}, \dots, u1_{A^c}}_{r}) = 0$ for $1 \le r \le k-1$.

Consider nets $(u_i), (v_i)$ w^*-converging to $u1_A$ and $u1_{A^c}$, respectively, as in Lemma 1.1. Note that, again by Arens-regularity, we may write

$$\begin{aligned}\overline{\phi}(u1_A, &\dots, u1_A, u1_{A^c}, \dots, u1_{A^c}) \\ &= \lim_{i_1} \dots \lim_{i_r} \lim_{j_1} \dots \lim_{j_{k-r}} \phi(u_{j_1}, \dots, u_{j_{k-r}}, v_{i_1}, \dots, v_{i_r})\end{aligned}$$

so that we may suppose all u_j's to be orthogonal to all v_i's. Hence, using the polarization formula and the fact that P is orthogonally additive,

$$\begin{aligned}&\phi(u_{j_1}, \dots, u_{j_{k-r}}, v_{i_1}, \dots, v_{i_r}) = \\ &= \frac{1}{k!2^k} \sum_{\varepsilon_j' s, \varepsilon_i' s} \varepsilon_{j_1} \cdots \varepsilon_{j_{k-r}} \varepsilon_{i_1} \cdots \varepsilon_{i_r} P(\varepsilon_{j_1} u_{j_1} + \cdots + \varepsilon_{j_{k-r}} u_{j_{k-r}} + \varepsilon_{i_1} v_{i_1} + \cdots + \varepsilon_{i_r} v_{i_r}) \\ &= \frac{1}{k!2^k} \sum_{\varepsilon_i' s} \varepsilon_{i_1} \cdots \varepsilon_{i_r} \sum_{\varepsilon_j' s} \varepsilon_{j_1} \cdots \varepsilon_{j_{k-r}} P(\varepsilon_{j_1} u_{j_1} + \cdots + \varepsilon_{j_{k-r}} u_{j_{k-r}}) \\ &+ \frac{1}{k!2^k} \sum_{\varepsilon_j' s} \varepsilon_{j_1} \cdots \varepsilon_{j_{k-r}} \sum_{\varepsilon_i' s} \varepsilon_{i_1} \cdots \varepsilon_{i_r} P(\varepsilon_{i_1} v_{i_1} + \cdots + \varepsilon_{i_r} v_{i_r}) = 0 + 0 = 0.\end{aligned}$$

Since $P = P_A + P_{A^c}$, we have $\|P\| \le \|P_A\| + \|P_{A^c}\|$. To conclude the second equality, set $\varepsilon > 0$ and let u, v be norm-one functions such that $\|P_A\| - \frac{\varepsilon}{2} < P_A(u)$ and $\|P_{A^c}\| - \frac{\varepsilon}{2} < P_{A^c}(v)$. Then

$$\begin{aligned}\|P_A\| + \|P_{A^c}\| - \varepsilon &< P_A(u) + P_{A^c}(v) = \overline{P}(u1_A) + \overline{P}(v1_{A^c}) = \overline{P}(u1_A + v1_{A^c}) \\ &\le \|\overline{P}\| = \|P\|,\end{aligned}$$

this last equality by the Davie-Gamelin theorem [6]. Thus $\|P_A\| + \|P_{A^c}\| = \|P\|$.

Finally, if $u \perp v$, set $F = \operatorname{supp}(u)$. Considering nets as above such that $u_i \to 1_F$ and $v_i \to 1_{F^c}$, we obtain

$$P_A(u+v) = \overline{P}(u1_A1_F + v1_A1_{F^c}) = \overline{P}(u1_A1_F) + \overline{P}(v1_A1_{F^c}) = P_A(u) + P_A(v).$$

The same for P_{A^c}. □

We denote by $\mathcal{P}^k_{OA}(C(K))$ the space of orthogonally additive continuous k-homogeneous polynomials. This is a closed subspace of $\mathcal{P}^k(C(K))$.

Lemma 1.3. *The extremal elements of the unit ball of $\mathcal{P}^k_{OA}(C(K))$ are of the form λP_x, where $P_x(u) = u(x)^k$ and $|\lambda| = 1$.*

Proof. Consider an extremal P. For any closed or open $A \subset K$, $P = P_A + P_{A^c}$. If P were neither P_A nor P_{A^c} both would be non-zero and

$$P = \|P_A\| \frac{P_A}{\|P_A\|} + \|P_{A^c}\| \frac{P_{A^c}}{\|P_{A^c}\|}$$

would, by Lemma 1.2, be a non-trivial convex combination, contradicting the extremality of P. Thus $P = P_A$ or $P = P_{A^c}$. Now consider

$$\mathcal{S} = \{F \subset K : F \text{ is closed and } P = P_F\}.$$

$K \in \mathcal{S}$, so $\mathcal{S}$ is non-empty. Let $F_0 = \bigcap_{\mathcal{S}} F$. F_0 is non-empty: otherwise, by compactness the intersection of finitely-many F's would be empty, but this cannot be, for we would have $P = (\ldots(P_{F_1})_{F_2}\ldots)_{F_n} = P_{F_1\cap\ldots\cap F_n} = P_\emptyset = 0$. Let $x \in F_0$ and let V be any open neighborhood of x. $P = P_V$ or $P = P_{V^c}$. But P cannot be P_{V^c}, for we would have $x \in F_0 \subset V^c$. So $P = P_V$ for every open neighborhood V of x. If x_1 and x_2 were two different points of F_0, take non-intersecting open neighborhoods V_1 and V_2 of each. Then $P = P_{V_1} = P_{V_2}$, and thus $P = P_{V_1\cap V_2} = P_\emptyset = 0$, absurd. Thus $F_0 = \{x\}$, and $P = P_{\{x\}}$. For any u, $P(u) = \overline{P}(u 1_{\{x\}}) = \overline{P}(u(x)1_{\{x\}}) = u(x)^k\overline{P}(1_{\{x\}})$. Thus, setting $\lambda = \overline{P}(1_{\{x\}})$, $P = \lambda P_x$. Since P_x and P have norm one, $|\lambda| = 1$. □

We can now prove the main result.

Theorem 1.4. *For any $P \in \mathcal{P}^k_{OA}(C(K))$, there is a regular Borel measure μ over K such that*

$$P(u) = \int_K u^k \, d\mu \qquad \textit{for all} \quad u \in C(K).$$

Proof. Consider the (incomplete) linearizing predual X_α of $\mathcal{P}^k_{OA}(C(K))$ constructed in [5]. Define

$$X_\alpha \to C(K) \text{ given by } s = \sum_u a_u e_u \mapsto \sum_u a_u u^k = \widetilde{s}.$$

This is linear, one-to-one, and open. Linearity is clear. If $\widetilde{s} = 0$, then $0 = \sum_u a_u u(x)^k = \sum_u a_u P_x(u) = s(P_x)$, where we consider s as a linear form over $\mathcal{P}^k_{OA}(C(K))$. This linear form s is therefore zero on all extremal points of the unit ball of $\mathcal{P}^k_{OA}(C(K))$ by Lemma 1.3, and must be null by the Krein-Milman theorem. Similarly, the condition $\|\widetilde{s}\| < \varepsilon$ guarantees that $|s(P_x)| < \varepsilon$ for all $x \in K$, thus $|s| < \varepsilon$ on the unit ball B of $\mathcal{P}^k_{OA}(C(K))$. The topology on X_α is given by uniform convergence over equicontinuous pointwise-compact disks of $\mathcal{P}^k_{OA}(C(K))$. But any equicontinuous subset D

of $\mathcal{P}^k_{OA}(C(K))$ is norm-bounded. Say $D \subset rB$. Then $|s| < r\varepsilon$ on D. Thus $\widetilde{s} \mapsto s$ is continuous.

Now take $P \in \mathcal{P}^k_{OA}(C(K))$, and consider it as continuous linear form L_P over X_α, that is, $P(u) = L_P(e_u) = L(u^k)$, where $L(\widetilde{s}) = L_P(s)$ is a continuous linear form on the subspace of $C(K)$ spanned by $\{u^k : u \in C(K)\}$. Extend L to the closure by continuity and then to all of $C(K)$ by Hahn-Banach. Thus L corresponds to a regular Borel measure μ over K such that

$$P(u) = L_P(e_u) = L(u^k) = \int_K u^k \, d\mu. \tag{1}$$

□

2. The Aron-Berner extension of an orthogonally additive polynomial

The algebra structure of $C(K)$ induces an algebra structure on $C(K)''$ via the Arens product (see [1]). Thus, $\varphi, \psi \in C(K)''$ are said to be orthogonal ($\varphi \perp \psi$) whenever their Arens product is zero: $\varphi \cdot \psi = 0$. In this context, the Aron-Berner extension preserves the class of orthogonally additive polynomials on $C(K)$. We obtain this result as a corollary to Theorem 1.4. It can also be obtained by proceeding as in Lemma 1.2.

Corollary 2.1. *If $P \in \mathcal{P}^k_{OA}(C(K))$ then $\overline{P} \in \mathcal{P}^k_{OA}(C(K)'')$.*

Proof. First, recall that for any k-homogeneous polynomial $Q : X \to X$ and for any $x' \in X'$, the Aron-Berner extension of the scalar valued k-homogeneous polynomial $x' \circ Q$ is given in terms of $\overline{Q} : X'' \to X''$ by $(\overline{x' \circ Q})(x'') = \overline{Q}(x'')(x')$, for every $x'' \in X''$.

On the other hand, if we consider the polynomial $Q : C(K) \to C(K)$ given by $Q(u) = u^k$, then $\overline{Q}(\varphi) = \varphi^k$, the kth power of φ in the Arens product.

Now take $P \in \mathcal{P}^k_{OA}(C(K))$, by the representation theorem there is a regular Borel measure μ over K such that $P = \mu \circ Q$. Then,

$$\overline{P}(\varphi) = (\overline{\mu \circ Q})(\varphi) = \overline{Q}(\varphi)(\mu) = \varphi^k(\mu). \tag{2}$$

Therefore, $\overline{P}$ is orthogonally additive. □

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