K-BOUNDED POLYNOMIALS

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Abstract

For a Banach space E we define the class $\mathcal{P}_K({}^N E)$ of K-bounded N-homogeneous polynomials, where K is a bounded subset of E'. We investigate properties of K which relate the space $\mathcal{P}_K({}^N E)$ with usual subspaces of $\mathcal{P}({}^N E)$. We prove that K-bounded polynomials are approximable when K is a compact set where the identity can be uniformly approximated by finite rank operators. The same is true when K is contained in the absolutely convex hull of a weakly null basic sequence of E'. Moreover, in this case we prove that every K-bounded polynomial is extendible to any larger space.

1. Introduction

If E is a Banach space and K is a bounded subset of its dual, we say that a scalar valued N-homogeneous polynomial P on E is K-bounded if there is a positive constant C such that the inequality $|P(x)| \leq C \sup\{|\gamma(x)|^N : \gamma \in K\}$ holds for all $x \in E$. Note that continuity is equivalent to $B_{E'}$-boundedness, and also (see Proposition 3.2) that finite type polynomials correspond to K-bounded with K finite.

A result of E. Toma [12] (see also [5]) states that a continuous homogeneous polynomial is weakly continuous on bounded sets if and only if it is K-bounded for some compact set K. Our interest in K-bounded polynomials was originally motivated by this result. The closure of the space of finite type polynomials is the space of 'approximable' polynomials and is a subspace of the space of polynomials which are weakly continuous on bounded sets. Thus, we set out to clarify the relationship between 'approximable' and 'K-bounded' (with K something between 'finite' and 'compact'). We obtain several sufficient conditions for approximability of a polynomial and are naturally led to consider also the problem of extendibility (to any larger Banach space) of K-bounded polynomials for several types of subsets K of E'. Note that all finite type polynomials (and even all integral polynomials [6]) are extendible, but the same does not hold for approximable polynomials.

The paper is organised as follows. In section 2, we set our notation and give a few basic properties about K-bounded polynomials, as well as a new (isometric) version of a result of Aron and Galindo regarding the Aron-Berner extension of a K-bounded polynomial. Section 3 is devoted to the search for conditions on K which ensure approximability and extendibility.

2. Basic properties

Throughout, E will be a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and E' will denote its dual space. The space of continuous N-homogeneous polynomials from E into $\mathbb{K}$ will be denoted by $\mathcal{P}(^N E)$. This is a Banach space with the norm $\|P\| = \sup\{|P(x)| : \|x\| \leq 1\}$. If $P \in \mathcal{P}(^N E)$, $\overset{\vee}{P}$ will denote the continuous symmetric N-linear form associated with P.

A polynomial $P \in \mathcal{P}(^N E)$ is said to be of finite type if there exists a finite subset $\{\varphi_i\}_{i=1}^m$ of E' such that

$$P(x) = \sum_{i=1}^{m} \varphi_i^N(x)$$

for complex E. When E is a real Banach space and N is even, the representation must take into account the signs, so $P(x) = \sum_{i=1}^m \varphi_i^N(x) - \sum_{j=1}^n \psi_i^N(x)$. We will denote by $\mathcal{P}_f(^N E)$ the space of N-homogeneous polynomials of finite type and its closure, in $(\mathcal{P}(^N E), \|\cdot\|)$, by $\mathcal{P}_c(^N E)$. Polynomials in $\mathcal{P}_c(^N E)$ will be called approximable.

$\mathcal{P}_w(^N E)$ will denote the space of polynomials which are weakly continuous on bounded sets. This is a closed subspace of $\mathcal{P}(^N E)$, and we have

$$\mathcal{P}_f(^N E) \subset \mathcal{P}_c(^N E) \subset \mathcal{P}_w(^N E) \subset \mathcal{P}(^N E) \tag{1}$$

Let K be a bounded subset of E'. For $x \in E$, we define

$$\|x\|_K = \sup_{\gamma \in K} |\gamma(x)|$$

which is a continuous semi-norm on E.

Definition 2.1. *We say that an N-homogeneous polynomial P is* K**-bounded** *if there exists a positive constant C such that*

$$|P(x)| \leq C \, \|x\|_K^N \tag{2}$$

for all x in E. The smallest constant C that verifies (2) *is called $\|P\|_K$.*

Since $\|\cdot\|_K$ is a continuous semi-norm on E, every K-bounded polynomial is continuous. The space of K-bounded N-homogeneous polynomials will be denoted by $\mathcal{P}_K(^N E)$. On $\mathcal{P}_K(^N E)$, $\|\cdot\|_K$ is a norm and $(\mathcal{P}_K(^N E), \|\cdot\|_K)$ is a Banach space.

We also say that an N-linear form $\Phi : E^N \to \mathbb{K}$ is K-bounded if there exists a positive constant C such that

$$|\Phi(x_1, \ldots, x_N)| \leq C \, \|x_1\|_K \cdots \|x_N\|_K \tag{3}$$

for all $(x_1, \ldots, x_N) \in E^N$ and $\|\Phi\|_K$ will be the smallest constant C verifying (3).

Clearly, every K-bounded Φ is continuous. From the polarization formula, we have the following inequalities

$$\|P\|_K \leq \| \overset{\vee}{P} \|_K \leq \frac{N^N}{N!} \|P\|_K$$

It follows that there exists a one to one correspondence between K-bounded N-homogeneous polynomials and K-bounded symmetric N-linear forms.

Remark 2.2. For every $x, y \in E$, and every $P \in \mathcal{P}_K({}^N E)$, if $\|x - y\|_K = 0$, then $P(x) = P(y)$.

$$\begin{aligned}
|P(x) - P(y)| &\leq |\overset{\vee}{P}(x, \dots, x) - \overset{\vee}{P}(y, x, \dots, x)| + |\overset{\vee}{P}(y, x, \dots, x) - \overset{\vee}{P}(y, y, x, \dots, x)| \\
&\quad + \cdots + |\overset{\vee}{P}(y, \dots, y, x) - \overset{\vee}{P}(y, \dots, y)| \\
&\leq \|\overset{\vee}{P}\|_K \|x - y\|_K \|x\|_K^{N-1} + \|\overset{\vee}{P}\|_K \|x - y\|_K \|y\|_K \|x\|_K^{N-2} \\
&\quad + \cdots + \|\overset{\vee}{P}\|_K \|x - y\|_K \|y\|_K^{N-1} \\
&\leq N \|\overset{\vee}{P}\|_K \|x - y\|_K \max\{\|x\|_K, \|y\|_K\}^{N-1}.
\end{aligned}$$

Remark 2.3. Since $K_1 \subset K_2 \subset E'$ implies $\|x\|_{K_1} \leq \|x\|_{K_2}$ for all $x \in E$, every K_1-bounded polynomial P is K_2-bounded, with

$$\|P\|_{K_2} \leq \|P\|_{K_1}.$$

Also, if $K \subset E'$ and $\widehat{K} = \overline{\Gamma(K)}$ is its closed, convex, balanced hull, then $\|x\|_K = \|x\|_{\widehat{K}}$ for all $x \in E$. Indeed, let $\gamma_0 \in \Gamma(K)$, say $\gamma_0 = \sum_{i=1}^{n} \alpha_i \gamma_i$, where $\gamma_i \in K$, $\alpha_i \in \mathbb{K}$ for $i = 1, \dots, n$ and $\sum_{i=1}^{n} |\alpha_i| \leq 1$. Then, for all $x \in E$, we have

$$|\gamma_0(x)| = \left| \sum_{i=1}^{n} \alpha_i \gamma_i(x) \right| \leq \sup_{j=1,\dots,n} |\gamma_j(x)| \sum_{i=1}^{n} |\alpha_i| \leq \sup_{\gamma \in K} |\gamma(x)| = \|x\|_K$$

and

$$\|x\|_{\widehat{K}} = \sup_{\gamma \in \widehat{K}} |\gamma(x)| = \sup_{\gamma \in \Gamma(K)} |\gamma(x)| \leq \|x\|_K.$$

From this and the fact that $K \subset \widehat{K}$, it follows that $\|x\|_K = \|x\|_{\widehat{K}}$. Therefore, $\mathcal{P}_K({}^N E) = \mathcal{P}_{\widehat{K}}({}^N E)$ with $\|P\|_K = \|P\|_{\widehat{K}}$.

Since $\| \cdot \|_K$ is a continuous semi-norm on E, then ${}^\circ K = \{x \in E : \ \|x\|_K = 0\}$ is a closed subspace of E. On $E/{}^\circ K$, we can define the following norm

$$|||\Pi(x)||| = \|x\|_K$$

where $\Pi : E \to E/{}^\circ K$ is the quotient projection. The completion of $E/{}^\circ K$, $(E_K, |||\cdot|||)$ is a Banach space.

Lemma 2.4. *Let K be a bounded subset of E'. Then the spaces $(\mathcal{P}_K({}^N E), \|\cdot\|_K)$ and $(\mathcal{P}({}^N E_K), \|\cdot\|)$ are isometrically isomorphic.*

PROOF. For $P \in \mathcal{P}_K({}^N E)$ we define $Q : E/{}^\circ K \to \mathbb{K}$ by

$$Q(\Pi(x)) = P(x) \qquad \forall x \in E.$$

Q is well defined because of remark 2.2. Also, Q is an N-homogeneous polynomial and

$$\begin{aligned}
\|Q\| &= \sup\{|Q(y)| : \ y \in E/{}^\circ K, |||y||| = 1\} = \sup\{|Q(\Pi(x))| : \ x \in E, |||\Pi(x)||| = 1\} \\
&= \sup\{|P(x)| : \ x \in E, \|x\|_K = 1\} = \|P\|_K
\end{aligned}$$

Thus Q is continuous and can be extended to an N-homogeneous polynomial on E_K with the same norm; this extension will still be called Q.

Conversely, let $Q \in \mathcal{P}(^N E_K)$. Clearly, $P(x) = Q(\Pi(x))$ is a K-bounded N-homogeneous polynomial and $\|P\|_K = \|Q\|$. ■

It is known that every $P \in \mathcal{P}(^N E)$ extends to E'' via the Aron-Berner morphism [1], and that this morphism preserves norms [7]. We will denote that extension by $\overline{P}$. Since $K \subset E'$, it can be considered as a subset of E'''. Aron and Galindo [3, corollary 8] proved that the Aron-Berner extension of a K-bounded polynomial is K-bounded, when K is a weakly compact set. Using the construction of the preceding lemma, we will give another proof of this fact. Moreover, we will show that the Aron-Berner morphism is a $\|\cdot\|_K$-isometry.

Proposition 2.5. *Let K be a relatively weakly compact subset of E'. Then the Aron-Berner morphism is an isometry from $(\mathcal{P}_K(^N E),\ \|\cdot\|_K)$ into $(\mathcal{P}_K(^N E''),\ \|\cdot\|_K)$ for every $N \in \mathbb{N}$.*

Proof. For $P \in \mathcal{P}_K(^N E)$, let $Q \in \mathcal{P}(^N E_K)$ as in lemma 2.4, let $\overline{Q} \in \mathcal{P}(^N E''_K)$ be the Aron-Berner extension of Q and $\overline{P} = \overline{Q} \circ \Pi'' : E'' \to \mathbb{K}$ where Π'' is the bitranspose of Π. Using the characterization of the Aron-Berner extension due to Zalduendo [13, theorem 2], it is easy to check that $\overline{P}$ is the Aron-Berner extension of P. Let us see that $\overline{P}$ is K-bounded with $\|\overline{P}\|_K = \|P\|_K$. For $x'' \in E''$,

$$\begin{aligned} |\overline{P}(x'')| &= |\overline{Q}(\Pi''(x''))| \leq \|\overline{Q}\|\ |||\Pi''(x'')|||^N = \|P\|_K\ |||\Pi''(x'')|||^N \\ &= \|P\|_K \sup_{\beta \in B_{E'_K}} |\Pi''(x'')(\beta)|^N = \|P\|_K \sup_{\beta \in B_{E'_K}} |x''(\Pi'(\beta))|^N \end{aligned} \tag{4}$$

We claim that $\Pi'(B_{E'_K})$ is contained in $\overline{\Gamma(K)}$, the closed, convex, balanced hull of K. To see this, let $\beta \in B_{E'_K}$, then

$$|\Pi'(\beta)(x)| = |\beta(\Pi(x))| \leq \|\beta\|\ |||\Pi(x)||| \leq \|x\|_K = \sup_{\gamma \in K} |\gamma(x)| \qquad \forall x \in E$$

By the Hahn-Banach theorem, $\Pi'(\beta)$ belongs to the weak-star closure of $\Gamma(K)$, $\overline{\Gamma(K)}^{w^*}$. Since K is relatively weakly compact, $\overline{\Gamma(K)}$ is weakly compact. Then $\overline{\Gamma(K)}$ is weak-star compact and it follows that $\overline{\Gamma(K)}^{w^*} = \overline{\Gamma(K)}$. Hence, $\Pi'(B_{E'_K}) \subset \overline{\Gamma(K)}$. Returning to (4),

$$|\overline{P}(x'')| \leq \|P\|_K \sup_{\varphi \in \overline{\Gamma(K)}} |x''(\varphi)|^N = \|P\|_K \sup_{\varphi \in K} |x''(\varphi)|^N = \|P\|_K\ \|x''\|_K^N$$

Therefore, $\overline{P}$ is K-bounded and $\|\overline{P}\|_K = \|P\|_K$. ■

3. Main results

We want to describe K-bounded polynomials corresponding to different classes of sets K. We begin by considering finite dimensional subsets of E' which will be related to finite type polynomials as we will see in proposition 3.2. First, we need the following lemma.

Lemma 3.1. *A polynomial $P \in \mathcal{P}(^N E)$ is of finite type if and only if its associated operator $T_P : E \to \mathcal{P}(^{N-1} E)$ has finite rank.*

Proof. If $P(x) = \sum_{i=1}^m \varphi_i^N(x)$, then $T_P(x) = \sum_{i=1}^m \varphi(x)\, \varphi_i^{N-1}$ which is a finite rank operator. Conversely, suppose T_P is a finite rank operator and let $\Pi : E \to E/\ker T_P$ be the quotient projection. We define a polynomial $\tilde{P}$ on $E/\ker T_P$ by $\tilde{P}(\Pi(x)) = P(x)$. To see that $\tilde{P}$ is well defined, let $\Pi(x) = \Pi(y)$. Since $T_P(x) = T_P(y)$,

$$\begin{aligned} P(x) &= T_P(x)(x) = T_P(y)(x) = \overset{\vee}{P}(y, x, \dots, x) = \overset{\vee}{P}(x, y, x \dots, x) \\ &= T_P(x)^\vee(y, x, \dots, x) = T_P(y)^\vee(y, x, \dots, x) = \overset{\vee}{P}(y, y, x \dots, x) \\ &= \cdots\cdots = P(y) \end{aligned}$$

Since $E/\ker T_P$ is a finite dimensional space, $\tilde{P}$ becomes a finite type polynomial and so does P. ■

Proposition 3.2. *Let $K \subset E'$ be a bounded set. Then every K-bounded N-homogeneous polynomial is of finite type if and only if the subspace spanned by K is finite dimensional.*

Proof. Suppose span(K) is finite dimensional and let $\{\gamma_1, \dots, \gamma_m\} \subset E'$ be a basis of span(K) such that $K \subset \Gamma(\{\gamma_1, \dots, \gamma_m\})$. If $u : E \to \mathbb{K}^m$ is defined by $u(x) = (\gamma_1(x), \dots, \gamma_m(x))$, then u is a continuous linear map satisfying $\|u(x)\|_\infty \geq \|x\|_K$. Given $P \in \mathcal{P}_K({}^N E)$, we define $Q : \mathrm{Im}(u) \to \mathbb{K}$ by $Q(u(x)) = P(x)$, which is well defined by remark 2.2. Since Q is a continuous N-homogeneous polynomial from a subspace of $\mathbb{K}^m$ into $\mathbb{K}$, we can write

$$Q(z) = \sum_{|\alpha|=N} a_\alpha\, z_1^{\alpha_1} \cdots z_m^{\alpha_m} \qquad \forall z = (z_1, \dots, z_m) \in \mathrm{Im}(u)$$

where $a_\alpha \in \mathbb{K}$, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Then,

$$P(x) = Q(u(x)) = \sum_{|\alpha|=N} a_\alpha\, \gamma_1(x)^{\alpha_1} \cdots \gamma_m(x)^{\alpha_m} \qquad \forall x \in E.$$

In particular, P is a polynomial of finite type.

To see the converse we will use the identification given in lemma 2.4. Let $P \in \mathcal{P}_K({}^N E)$ of finite type; then the corresponding polynomial $Q \in \mathcal{P}({}^N E_K)$ is of finite type too. Indeed, P being of finite type, its associated operator $T_P : E \to \mathcal{P}_K({}^{N-1}E)$ has finite rank. Since $T_P = T_Q \circ \Pi$, where $T_Q : E_K \to \mathcal{P}({}^{N-1}E_K)$ is the operator associated to Q and $\Pi : E \to E_K$ is the natural projection, then T_Q has finite rank. By lemma 3.1, Q is a polynomial of finite type and then every continuous N-homogeneous polynomial on E_K is of finite type. We conclude that E_K is finite dimensional. Thus the subspace spanned by K has finite dimension. ■

As a corollary, we have that a K-bounded polynomial of finite type can be written in terms of K-bounded functionals (just compose with Π the functionals representing Q as a finite type polynomial).

It is clear that every polynomial of finite type is K-bounded for a finite set K, so we have

$$\mathcal{P}_f({}^N E) = \bigcup_{\substack{K \subset E' \\ K \text{ finite}}} \mathcal{P}_K({}^N E).$$

In [12, 5] it is shown that

$$\mathcal{P}_w({}^N E) = \bigcup_{\substack{K \subset E' \\ K \text{ compact}}} \mathcal{P}_K({}^N E).$$

and clearly

$$\mathcal{P}(^N E) = \mathcal{P}_{B_{E'}}(^N E)$$

where $B_{E'}$ denotes the closed unit ball of E'.

Taking into account the inclusions given in (1), we will try to find sets $K \subset E'$ for which K-bounded polynomials are approximable.

Since approximable polynomials are w-continuous on bounded sets, we start by considering compact subsets K of E'. In addition, w-continuous polynomials on bounded sets are approximable when E' has the approximation property. This suggests the following proposition:

Proposition 3.3. *Let $K \subset E'$ be a compact set such that the identity* Id $: E' \to E'$ *can be uniformly approximated on K by finite rank operators. Then every K-bounded N-homogeneous polynomial is approximable.*

Proof. Without loss of generality, we suppose $K \subset B_{E'}$. Let $P \in \mathcal{P}_K(^N E)$ and $dP \in \mathcal{P}(^{N-1}E; E')$ its derivative. We have

$$\begin{array}{ccc} E & \xrightarrow{dP} & E' \\ \Pi \downarrow & & \uparrow \Pi' \\ E_K & \xrightarrow{dQ} & E'_K \end{array}$$

Note that

$$dP(B_E) = \Pi'(dQ(\Pi(B_E))) \subset \Pi'(dQ(B_{E_K})) \subset \|dQ\|\Pi'(B_{E'_K}) \subset \|dQ\|\overline{\Gamma(K)}$$

(the last inclusion was explained in proposition 2.5). Furthermore, $K_1 = \|dQ\|\overline{\Gamma(K)}$ is a compact subset of E' on which the identity can also be uniformly approximated by finite rank operators. Thus, for each $n \in \mathbb{N}$, there exists a finite rank operator $I_n : E' \to E'$ verifying

$$\|I_n(\gamma) - \gamma\| \leq \frac{1}{n} \quad \forall \gamma \in K_1$$

so

$$\|I_n(dP(x)) - dP(x)\| \leq \frac{1}{n} \quad \forall x \in B_E.$$

We define $P_n(x) = \frac{1}{N} I_n(dP(x))(x)$. Since $dP_n = I_n \circ dP$ and $T_{P_n}(x)(y) = \frac{1}{N}\left(dP_n(y)\right)(x)$ then T_{P_n} has finite rank which implies, by lemma 3.1, that P_n is a polynomial of finite type. We also have

$$|P_n(x) - P(x)| = \left|\frac{1}{N}(I_n(dP(x))(x) - \frac{1}{N} dP(x)(x)\right| \leq \frac{1}{N}\frac{1}{n} \quad \forall x \in B_E.$$

Therefore, P is approximable. ∎

It is known (see, for example, [11]) that a subset K of E' is compact if and only if it is contained in the closed convex balanced hull of a null sequence. By remark 2.3,

$$\bigcup_{\substack{K \subset E' \\ K \text{ compact}}} \mathcal{P}_K(^N E) = \bigcup_{\substack{K = \{\gamma_n\}_n \subset E' \\ \|\gamma_n\| \longrightarrow 0}} \mathcal{P}_K(^N E).$$

Let us consider $K = \{\gamma_n\}_{n \in \mathbb{N}} \subset E'$ where $\|\gamma_n\| \overset{n\to\infty}{\longrightarrow} 0$. We can define a linear operator $u : E \to c_0$ by

$$u(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_n(x), \ldots) \tag{5}$$

which is easily seen to be compact. This fact will enable us to prove that K-bounded polynomials are approximable, given some further assumption on the image of u.

Proposition 3.4. *Let $K = \{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ where $\|\gamma_n\| \overset{n\to\infty}{\longrightarrow} 0$ and let u as in (5). If $\overline{\mathrm{Im}(u)}$ has the approximation property, then K-bounded homogeneous polynomials are approximable.*

PROOF. Let $N \in \mathbb{N}$, and $P \in \mathcal{P}_K({}^N E)$. We define $Q : \mathrm{Im}(u) \to \mathbb{K}$ by $Q(u(x)) = P(x)$, which again is well defined by remark 2.2 and is a continuous N-homogeneous polynomial with

$$\|Q\| = \sup\{|Q(u(x))| : \|u(x)\|_\infty \leq 1\} = \sup\{|P(x)| : \|x\|_K \leq 1\} = \|P\|_K$$

We can extend Q to a continuous N-homogeneous polynomial on $\overline{\mathrm{Im}(u)}$ with the same norm, which will still be called Q. This gives us the following diagram

$$\begin{array}{ccc} & & \overline{\mathrm{Im}(u)} \\ & \overset{u}{\nearrow} & \downarrow \mathrm{Q} \\ E & \overset{P}{\to} & \mathbb{K} \end{array}$$

Since $\overline{u(B_E)} \subset \overline{\mathrm{Im}(u)}$, u is compact and $\overline{\mathrm{Im}(u)}$ has the approximation property, there exist finite rank operators $T_n : \overline{\mathrm{Im}(u)} \to \overline{\mathrm{Im}(u)}$ such that

$$\|T_n(u(x)) - u(x)\|_\infty < \frac{1}{n} \qquad \forall x \in B_E$$

In this way we have a sequence of finite type polynomials $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_f({}^N E)$, given by $P_n(x) = (Q \circ T_n \circ u)(x)$, approximating P. Indeed,

$$|P(x) - P_n(x)| = |Q(u(x)) - Q(T_n(u(x)))| \leq M\|u(x) - T_n(u(x))\|_\infty \leq M\frac{1}{n} \qquad \forall x \in B_E$$

where the constant M can be chosen independent of $x \in B_E$ and $n \in \mathbb{N}$ (see remark 2.2) ∎

As a consequence of this proposition we derive the following result that Grothendieck proved (see [11]) while studying the existence of spaces without the approximation property:

If there exists a Banach space without the approximation property then there exists a subspace of c_0 without the approximation property.

To see this, let X be a Banach space without the approximation property. We proceed as in [2]. There is a Banach space Z and a compact operator $T : Z \to X$ which is not approximable by finite rank operators. If $Y = Z \oplus X'$, we define $S : Y \to Y'$ by $S(z, x') = (T'x', Tz)$, where $z \in Z$, $x' \in X'$ and T' is the transpose of T. Thus S is a compact operator that cannot be approximated by finite rank operators. Indeed, the existence of finite rank operators approximating S from Y into Y' would imply the existence of finite rank operators from Z into X'' approximating T. T being compact, it would be possible to construct finite rank operators from Z into X approximating T (see [11, lemma 1.e.6]) and that is absurd. Now, by means of the compacity of S, the 2-homogeneous polynomial $P \in \mathcal{P}({}^2Y)$, $P(y) = S(y)(y)$, is w-continuous on bounded sets [4] but is not approximable. Therefore, P is K-bounded, for some $K = \{\gamma_n\}_{n\in\mathbb{N}} \subset Y'$, where $\|\gamma_n\| \to 0$, and defining u as in (5), the subspace $\overline{\mathrm{Im}(u)}$ of c_0 fails to have the approximation property.

From the proof above, we can conclude that the existence of a Banach space without the approximation property is equivalent to the existence of a homogeneous non-approximable polynomial which is w-continuous on bounded sets.

There is no general method to decide whether a Banach space has the approximation property or not. However, every Hilbert space has it. Let $K = \{\gamma_n\}_{n\in\mathbb{N}} \subset E'$, where $\sum_{n=1}^{\infty} \|\gamma_n\|^2 < \infty$. Now, we can modify the construction (5) by putting $u : E \to \ell_2$, $u(x) = (\gamma_1(x), \ldots, \gamma_n(x), \ldots)$, which is also a compact operator. Defining the polynomial $Q : \mathrm{Im}(u) \subset \ell_2 \to \mathbb{K}$ by $Q(u(x)) = P(x)$ we note that in this case

$$|Q(u(x))| \leq \|P\|_K \|u(x)\|_\infty^N \leq \|P\|_K \|u(x)\|_2^N$$

Since $\overline{\mathrm{Im}(u)}$ is a Hilbert space, we can proceed as in proposition 3.4 to conclude that every K- bounded polynomial is approximable. Moreover, working on a Hilbert space we can state an extension result. We have:

Proposition 3.5. *Let $K = \{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ such that $\sum_{n=1}^{\infty} \|\gamma_n\|^2 < \infty$. Then every K-bounded polynomial $P \in \mathcal{P}_K({}^N E)$ is approximable. Moreover, if G is a Banach space containing E, there exists $\widetilde{P} \in \mathcal{P}({}^N G)$ which is an extension of P.*

PROOF. We prove the second statement, the first one having been explained in the previous paragraph. If $E \subset G$, for each $n \in \mathbb{N}$, we have an extension of γ_n, $\widetilde{\gamma}_n \in G'$, with the same norm. As $\sum_{n=1}^{\infty} \|\widetilde{\gamma}_n\|^2 < \infty$, the operator $\widetilde{u} : G \to \ell_2$, $\widetilde{u}(x) = (\widetilde{\gamma}_1(x), \ldots, \widetilde{\gamma}_n(x), \ldots)$ is an extension of u. Let $\widetilde{Q} : \overline{\mathrm{Im}(\widetilde{u})} \to \mathbb{K}$, $\widetilde{Q}(y) = Q(\Pi(y))$, where Π is the orthogonal projection onto $\overline{\mathrm{Im}(u)}$. This completes the following diagram:

$$\begin{array}{ccccc} G & \xrightarrow{\widetilde{u}} & \overline{\mathrm{Im}(\widetilde{u})} & & \\ & & & \widetilde{Q} & \\ i\uparrow & & \downarrow \Pi & \searrow & \\ E & \xrightarrow{u} & \overline{\mathrm{Im}(u)} & \xrightarrow{Q} & \mathbb{K} \end{array}$$

We may define $\widetilde{P} : G \to \mathbb{K}$, $\widetilde{P}(x) = \widetilde{Q}(\widetilde{u}(x))$. Thus $\widetilde{P} \in \mathcal{P}({}^N G)$ and becomes an extension of P. ∎

In [10], Kirwan and Ryan gave the following definition:

Definition 3.6. *A polynomial $P \in \mathcal{P}({}^N E)$ will be called* **extendible** *if, for every Banach space $G \supset E$, there exists $\widetilde{P} \in \mathcal{P}({}^N G)$ which is an extension of P.*

Proposition 3.5 states that if we consider $K = \{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ with $\sum_{n=1}^{\infty} \|\gamma_n\|^2 < \infty$, then every K-bounded polynomial is extendible. Moreover, for each $G \supset E$, there exists an extension morphism

$$\begin{array}{rcl} \Lambda : \mathcal{P}_K({}^N E) & \longrightarrow & \mathcal{P}({}^N G) \\ P & \longmapsto & Q \circ \Pi \circ \widetilde{u} \end{array}$$

We will now investigate those bounded subsets K of E' for which every K-bounded polynomial is extendible and whether the resulting extension becomes $\widetilde{K}$-bounded, where $\widetilde{K} = \{\widetilde{\gamma} : \gamma \in K\}$ and $\widetilde{\gamma}$ is a norm preserving extension of γ to G. Moreover, we want to establish conditions for the existence of an extension morphism. Still, we keep in mind the problem of approximability of K-bounded polynomials.

Note that there are non-extendible approximable polynomials. For instance, $P \in \mathcal{P}({}^2\ell_2)$, given by $P(x) = \sum_{n=1}^{\infty} \frac{x_n^2}{n}$, is approximable but not nuclear, so it cannot be extendible [10, proposition 8].

It is known that every homogeneous polynomial on c_0 is approximable. Moreover, if $Q \in \mathcal{P}(^N c_0)$, there exists a sequence of scalars $\{a_{i_1,\ldots,i_N}\}_{(i_1,\ldots,i_N)\in D}$ such that

$$Q = \sum_{(i_1,\ldots,i_N)\in D} a_{i_1,\ldots,i_N}\, e'_{i_1} \ldots e'_{i_N} \tag{6}$$

where $D = \{(i_1,\ldots,i_N) \in \mathbb{N}^N : i_1 \geq \ldots \geq i_N\}$ is ordered by the square ordering, $\{e'_n\}_{n\in\mathbb{N}}$ is the unit vector basis of $c'_0 = \ell_1$, and the convergence of the series (6) is in the norm of $\mathcal{P}(^N c_0)$ (see [8]).

For certain sets K, we will be able to factor K-bounded homogeneous polynomials on E by homogeneous polynomials on c_0. This will enable us to lift the property of being approximable from polynomials on c_0 to K-bounded polynomials on E and also to obtain an extension result, as in proposition 3.5.

Proposition 3.7. *Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ be a basic sequence such that $\gamma_n \xrightarrow{w} 0$ and let $K = \{\gamma_n\}_{n\in\mathbb{N}}$. If $P \in \mathcal{P}_K(^N E)$ then P is approximable and extendible. Furthermore, if $E \subset G$, then there exists an extension morphism $\Lambda : \mathcal{P}_K(^N E) \to \mathcal{P}_{\widetilde{K}}(^N G)$, where $\widetilde{K} = \{\widetilde{\gamma}_n\}_{n\in\mathbb{N}}$, with $\widetilde{\gamma}_n$ a norm preserving extension of γ_n. The morphism is an isometry if we consider the $\|\cdot\|_K$ and $\|\cdot\|_{\widetilde{K}}$ norms.*

PROOF. As in (5), let $u : E \to c_0$, with $u(x) = (\gamma_1(x), \ldots, \gamma_n(x), \ldots)$. Since $\{\|\gamma_n\|\}_{n\in\mathbb{N}}$ is bounded, $u \in \mathcal{L}(E; c_0)$ and $\|u(x)\| = \|x\|_K$. Let $\overline{u} : E'' \to c_0$, $\overline{u}(z) = (z(\gamma_1), \ldots, z(\gamma_n), \ldots)$. Again, $\overline{u} \in \mathcal{L}(E''; c_0)$ and $\|\overline{u}(z)\| = \|z\|_K$, for all $z \in E''$.

Since $\{\gamma_n\}_{n\in\mathbb{N}}$ is a basic sequence in E', there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset E''$ such that $z_n(\gamma_m) = \delta_{nm}$ and so $\overline{u}(z_n) = e_n$, where $\{e_n\}_{n\in\mathbb{N}}$ is the unit vector basis of c_0. It follows that $\mathrm{Im}(\overline{u})$ is a dense subspace of c_0.

Let $P \in \mathcal{P}_K(^N E)$. As stated in proposition 2.5, its Aron-Berner extension $\overline{P}$ belongs to $\mathcal{P}_K(^N E'')$. Consider $Q : \mathrm{Im}(\overline{u}) \to \mathbb{K}$ defined by $Q(\overline{u}(z)) = \overline{P}(z)$. Q is a continuous N-homogeneous polynomial with $\|Q\| = \|\overline{P}\|_K = \|P\|_K$. We may extend Q to a continuous N-homogeneous polynomial on c_0, which will still be called Q, with the same norm. This gives us the following diagram:

$$\begin{array}{ccc} E'' & \xrightarrow{\overline{u}} & c_0 \\ i\uparrow & \overset{\overline{P}}{\searrow} & \downarrow Q \\ E & \xrightarrow{P} & \mathbb{K} \end{array}$$

Being a continuous N-homogeneous polynomial on c_0, Q admits a representation as in (6). Thus

$$P = \sum_{(i_1,\ldots,i_N)\in D} a_{i_1,\ldots,i_N}\, \gamma_{i_1} \ldots \gamma_{i_N} \tag{7}$$

It can be seen that the series (7) converges in the norms of both $\mathcal{P}(^N E)$ and $\mathcal{P}_K(^N E)$. Moreover, from the isometric isomorphism between $\mathcal{P}_K(^N E)$ and $\mathcal{P}(^N c_0)$ it follows that the sequence $\{\gamma_{i_1}, \ldots, \gamma_{i_N}\}_{(i_1,\ldots,i_N)\in D}$ (with the square ordering) is a Schauder basis of $(\mathcal{P}_K(^N E), \|\cdot\|_K)$. Now that we have proved that P is approximable, we turn to the extension morphism.

Let $G \supset E$ and $\widetilde{u} : G \to \ell_\infty$ given by $\widetilde{u}(y) = (\widetilde{\gamma}_1(y), \ldots, \widetilde{\gamma}_n(y), \ldots)$, with $\widetilde{\gamma}_n \in G'$ a norm preserving extension of γ_n. Let us consider $\overline{Q} \in \mathcal{P}(^N \ell_\infty)$ the Aron-Berner extension of $Q \in \mathcal{P}(^N c_0)$. Then we have the following diagram:

$$\begin{array}{ccccc} G & \xrightarrow{\widetilde{u}} & \ell_\infty & & \\ i\uparrow & & \uparrow J & \overset{\overline{Q}}{\searrow} & \\ E & \xrightarrow{u} & c_0 & \xrightarrow{Q} & \mathbb{K} \end{array}$$

where J is the canonical inclusion from c_0 into ℓ_∞.

If we define $\widetilde{P} : G \to \mathbb{K}$ by $\widetilde{P}(y) = \overline{Q}(\widetilde{u}(y))$ we obtain an extension of P to G with

$$|\widetilde{P}(y)| = |\overline{Q}(\widetilde{u}(y))| \leq \|\overline{Q}\| \ \|\widetilde{u}(y)\|^N = \|P\|_K \ \|y\|_{\widetilde{K}}^N \quad \forall y \in G$$

This implies that $\widetilde{P}$ is $\widetilde{K}$-bounded and $\|\widetilde{P}\|_{\widetilde{K}} = \|P\|_K$. Hence, the extension morphism

$$\begin{array}{rcl} \Lambda : \mathcal{P}_K(^N E) & \longrightarrow & \mathcal{P}_{\widetilde{K}}(^N G) \\ P & \longmapsto & \overline{Q} \circ \widetilde{u} \end{array}$$

is an isometry. ∎

Remark 3.8. Note that, in the previous proposition, K is a weakly null sequence, although we are mainly concerned with norm null sequences. In the norm null case, the result is stronger in the sense that $\widetilde{K}$ turns out to be a norm null sequence too. In the general case, when $\gamma_n \overset{w}{\to} 0$, it is possible to choose $\widetilde{\gamma}_n$ converging weakly to zero but, unfortunately, the extension morphism fails to be an isometry.

The assumption that $\{\gamma_n\}_{n\in\mathbb{N}}$ is a w-null basic sequence can be replaced by other conditions which will enable us to proceed as in the previous proof.

Ovsepian and Pelczynski proved (see [LT], for example) that every infinite dimensional separable Banach space verifies that, for each $\varepsilon > 0$, there exist two sequences $\{x_n\}_{n\in\mathbb{N}} \subset E$ and $\{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ such that

(i) $\gamma_n(x_m) = \delta_{nm} \quad \forall n, m \in \mathbb{N}$.

(ii) $\|x_n\| = 1; \quad \|\gamma_n\| \leq 1 + \varepsilon \ \ \forall n \in \mathbb{N}$.

(iii) $E = \overline{[\{x_n\}_{n\in\mathbb{N}}]}$.

(iv) $\{\gamma_n\}_{n\in\mathbb{N}}$ is a total system over E, i.e. $\gamma_n(x) = 0 \ \forall n \in \mathbb{N}$ implies that $x = 0$.

A pair of sequences in these conditions will be called an O-P system. Note that $\{\gamma_n\}_{n\in\mathbb{N}}$ is a weak-star null sequence.

Proposition 3.9. *Let E be an infinite dimensional separable Banach space. Let $\{x_n\}_{n\in\mathbb{N}} \subset E$ and $\{\gamma_n\}_{n\in\mathbb{N}} \subset E'$ be an O-P system. If $K = \{\gamma_n\}_{n\in\mathbb{N}}$ then every K-bounded N-homogeneous polynomial P is approximable and extendible by an extension morphism.*

Proof. Let $u : E \to c_0$ be as in (5), which is well defined because $\{\gamma_n\}_{n\in\mathbb{N}}$ is a weak-star null sequence. Since $u(x_n) = e_n$, $\mathrm{Im}(u)$ is a dense subspace of c_0 and we can define $Q : c_0 \to \mathbb{K}$ such that $Q(u(x)) = P(x)$ for all $x \in E$. Thus P is approximable with a representation as in (7). The extension result is derived just as in the previous proposition. ∎

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