Isometries between spaces of homogeneous polynomials

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To Seán Dineen on his Sixtieth Birthday

Abstract

We derive Banach-Stone theorems for spaces of homogeneous polynomials. We show that every isometric isomorphism between the spaces of n-homogeneous approximable polynomials on real Banach spaces E and F is induced by an isometric isomorphism of E' onto F'. With an additional geometric condition we obtain the analogous result in the complex case. Isometries between spaces of n-homogeneous integral polynomials and between the spaces of all n-homogeneous polynomials are also investigated.

1 Introduction

Isometries between Banach spaces are those morphisms which preserve the metric structure of the spaces. In 1932 Banach [3] showed that if K and L are compact metric spaces and T is an isometric isomorphism from $C(K)$, the space of continuous real valued functions on K, to $C(L)$, the space of real valued functions on L, then there is a homeomorphism Θ from L into K and a continuous functions h on L with $|h(y)| = 1$ such that

$$(Tf)(y) = h(y)f \circ \Theta(y)$$

for all f in $C(K)$ and all y in L. Later, Stone [40] extended the result to the case where K and L are compact Hausdorff sets, a result which is now refereed to as the Banach-Stone Theorem. In the seventy years since Banach's and Stone's results it has emerged that isometries between a wide range of Banach function spaces have the above form with

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the condition on h relaxed to be an element of the range rather than satisfy $|h(y)| = 1$. Examples of this phenomenon are the isometries of $\mathcal{H}^\infty(\Delta)$ and $\mathcal{H}^1(\Delta)$ characterised by deLeeuw, Rudin and Wermer [15], the isometries of the Hardy spaces $\mathcal{H}^p$, $1 < p < \infty$, $p \neq 2$, by Forelli [22], the isometries of the Bergman spaces, L^p_α, $0 < p < \infty$, and weighted Bergman spaces given by Kolaski [28, 29], the isometries of the Bloch spaces, $\mathcal{B}_o$ and $\mathcal{B}$, due to Cima and Wogen [12], and the isometries of weighted spaces of holomorphic functions, $\mathcal{H}_{v_o}(U)$ and $\mathcal{H}_v(U)$, on bounded subsets U of $\mathbf{C}^n$, [4]. In this paper we will prove results of this type for spaces of homogeneous polynomials on Banach spaces E and F. Our Θ will turn out to be the transpose of an isometric isomorphism from E' into F'.

Before proceeding we introduce some definitions and notation. Throughout the paper E and F are Banach spaces and S_E is the unit sphere of E. A function $P\colon E \to \mathbf{K}$ ($\mathbf{K} = \mathbf{R}, \mathbf{C}$) is said to be a (continuous) n-homogeneous polynomial if there is a (continuous) n-linear map $L_P\colon \underbrace{E \times E \times \ldots \times E}_{\text{n}-\text{times}} \to \mathbf{K}$ such that $P(x) = L_P(x, \ldots, x)$ for all $x \in E$. Continuous n-homogeneous polynomials are bounded on the unit ball and we denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials on E endowed with the norm: $P \to \|P\| := \sup_{\|x\|\leq 1} |P(x)|$.

An n-homogeneous polynomial $P \in \mathcal{P}(^nE)$ is said to be of finite type if there is $\{\phi_j\}_{j=1}^k$ in E' such that $P(x) = \sum_{j=1}^k \pm\phi_j(x)^n$ for all x in E. The closure of the finite type n-homogeneous polynomials in $\mathcal{P}(^nE)$ are called the approximable polynomials. We use $\mathcal{P}_f(^nE)$ to denote the space of finite type n-homogeneous polynomials and $\mathcal{P}_A(^nE)$ to denote the space of all n-homogeneous approximable polynomials.

We say that an n-homogeneous polynomial P on a Banach space E is nuclear if there is bounded sequence $(\phi_j)_{j=1}^\infty \subset E'$ and a sequence $(\lambda_j)_{j=1}^\infty$ in ℓ_1 such that

$$P(x) = \sum_{j=1}^\infty \lambda_j \phi_j(x)^n$$

for every x in E. The space of all nuclear n-homogeneous polynomials on E is denoted by $\mathcal{P}_N(^nE)$ and becomes a Banach space when the norm of P is given as the infimum of $\sum_{j=1}^\infty |\lambda_j|\,\|\phi_j\|^n$ taken over all representations of P of the form described above. This norm is called the nuclear norm of P and is denoted by $\|P\|_N$. When E' has the approximation property $(\mathcal{P}_N(^nE), \|\cdot\|_N)$ is isometrically isomorphic to $\widehat{\bigotimes}_{n,s,\pi} E'$ under the map induced by $\phi^n \to \phi \otimes \phi \otimes \ldots \otimes \phi$.

A polynomial P on E is said to be integral if there is a regular Borel measure μ on

$(B_{E'}, \sigma(E', E))$ such that

$$P(x) = \int_{B_{E'}} \phi(x)^n \, d\mu(\phi) \tag{1}$$

for every x in E. We write $\mathcal{P}_I({}^n E)$ for the space of all n-homogeneous integral polynomials on E. We define the integral norm of an integral polynomial P, $\|P\|_I$, as the infimum of $\|\mu\|$ taken over all regular Borel measures which satisfy (1).

It is shown in [17] (see also [18, Section 2.2]) that $\mathcal{P}_I({}^n E')$ is isometrically isomorphic to $\mathcal{P}_A({}^n E)'$ via the Borel transform B given by $B\Psi(\phi) = \Psi(\phi^n)$ for $\phi \in E'$, $\Psi \in \mathcal{P}_A({}^n E)'$. We use this identification without further reference.

A class of n-homogeneous polynomials on E is a pair consisting of a subspace, $\mathcal{P}_{\mathcal{C}}({}^n E)$, of $\mathcal{P}({}^n E)$ and a norm, $\|\cdot\|_{\mathcal{C}}$, under which $(\mathcal{P}_{\mathcal{C}}({}^n E), \|\cdot\|_{\mathcal{C}})$ is a Banach space.

The spaces of n-homogeneous approximable, nuclear and integral polynomials are all examples of classes of polynomials.

Let us review what is known about isometries and more generally isomorphisms of spaces of homogeneous polynomials.

In [14] Díaz and Dineen posed the following question: *If E and F are Banach spaces and E' is isomorphic to F' does this imply that $\mathcal{P}({}^n E)$ is isomorphic to $\mathcal{P}({}^n F)$?* They obtained a positive solution in the case where E' has both the Schur property and the approximation property. In [7] a positive solution is also obtained in the case where E and F are stable Banach spaces while both [7] and [30] show that Arens regularity of E alone gives us an affirmative answer. (The fact that stability gives a positive solution is actually implicit in [14, Proposition 3].) A positive solution is also obtained in [30] under the assumption that both E and F are symmetrically Arens regular. In addition it is shown that this result is also true for the classes of nuclear, approximable, K-bounded, integral, extendible n-homogeneous polynomials along with the space of n-homogeneous polynomials which are weakly continuous on bounded sets irrespective of further conditions on E or F. In [10] these results are extended to spaces of vector-valued homogeneous polynomials although the techniques required are different. In [30] we are provided with a method of constructing an isometry of spaces of homogeneous polynomials on E and F from an isometry of E' into F' as follows: Given a Banach space E we use J_E to denote the canonical embedding of E into its bidual E''. There is no Hahn-Banach Theorem for homogeneous polynomials of degree 2 or greater. However, Aron and Berner [1] and Davie and Gamelin [13] show that for every $P \in \mathcal{P}({}^n E)$ there is a norm-preserving extension of P to $\overline{P} \in \mathcal{P}({}^n E'')$ such that $\overline{P} \circ J_E(x) = P(x)$ for all $x \in E$. This extension is the key that allows us to lift any morphism s from E' to F' to a morphism $\overline{s}$ from $\mathcal{P}_{\mathcal{C}}({}^nE)$ to $\mathcal{P}_{\mathcal{C}}({}^nF)$, defined by

$$\overline{s}(P) = \overline{P} \circ s' \circ J_F.$$

In the case where s is an (isometric) isomorphism $\bar{s}$ is also an (isometric) isomorphism.

This paper is organised as follows. In Section 2 we characterise 'canonical' isomorphisms between spaces of approximable polynomials in terms of both the algebraic and geometric structures. In the third section we examine the converse of the question of Díaz and Dineen [14] for the case of approximable polynomials. Specifically we show that if E and F are real Banach spaces, n is a positive integer and $T\colon \mathcal{P}_A({}^nE) \to \mathcal{P}_A({}^nF)$ is an isometric isomorphism then there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P}\circ s' \circ J_F$ for all $P \in \mathcal{P}_A({}^nE)$. In Section 4 we show that this result extends to complex Banach spaces when we have additional information on the extreme points of the unit ball of $\mathcal{P}_I({}^nF')$. Isometries between the classes of integral and of all n-homogeneous polynomials are discussed in Section 5.

For further reading on polynomials on Banach spaces we refer the reader to [18] and to [20] for further information on isometries of Banach spaces.

2 Canonical and power-preserving mappings

Let E be a real or complex Banach space and n be a fixed positive integer. We define an equivalence relation $\equiv$ on E' by $\phi \equiv \psi$ if

$$\phi^n = \psi^n.$$

We let $E'/\equiv$ denote the set of all $\equiv$ equivalence classes. Given ϕ in E' we use $[\phi]$ to denote the equivalence class of ϕ in $E'/\equiv$.

Let E and F be real or complex Banach spaces, n be a positive integer and $T\colon \mathcal{P}_A({}^nE) \to \mathcal{P}_A({}^nF)$ be an isomorphism. We define $S^T_{E'}$ by

$$S^T_{E'} = \{\phi \in E' : \|T(\phi^n)\| = 1\}.$$

We use $S^T_{E'}/\equiv$ to denote the set $\{[\phi] : \phi \in S^T_{E'}\}$ and $S_{E'}/\equiv$ to denote the set $\{[\psi] : \psi \in E', \|\psi\| = 1\}$.

We need some technical lemmata and definitions.

Lemma 1 *Let E be a real or complex Banach space of dimension at least 3. Then, $S^T_{E'}$ is simply connected.*

Proof: Consider ψ and $-\psi$ in $S_{E'}^T$ and the punctured distorted spheres $U = S_{E'}^T \setminus \{\psi\}$ and $V = S_{E'}^T \setminus \{-\psi\}$. We show that $\{U, V\}$ satisfies the hypothesis of Van Kampen's Theorem (see, for instance, [33]). It is clear that U, V are open sets and cover $S_{E'}^T$ so we have to show that U, V are simply connected and $U \cap V$ is path connected. Moreover, we prove that U, V are contractible. Consider, for instance, V and define $F\colon [0,1] \times V \to V$ by $F(t,\phi) = \dfrac{(1-t)\phi + t\psi}{\|T(((1-t)\phi + t\psi)^n)\|^{1/n}}$, which is continuous since T is an isomorphism and $(1-t)\phi + t\psi = 0$ if and only if $\phi = -\psi$. In addition, $F(0,\phi) = \phi$ and $F(1,\phi) = \psi$, therefore, F is a contraction. In an analogous way it can be shown that U is contractible. To show that $U \cap V$ is path connected, take $\phi_0 \in U \cap V$. If $\phi \in U \cap V$ such that $\{\phi, \phi_0, \psi\}$ is a linear independent set then, $\sigma_{\phi,\phi_0}(t) = \dfrac{(1-t)\phi + t\phi_0}{\|T(((1-t)\phi + t\phi_0)^n)\|^{1/n}}$ defines a path in $U \cap V$ connecting ϕ and ϕ_0. If ϕ belongs to the span of $\{\psi, \phi_0\}$, consider $\eta \in U \cap V$ such that $\{\phi, \phi_0, \eta\}$ is a linear independent set. Now, define $\tilde{\sigma}_{\phi,\phi_0}(t)$ by

$$\tilde{\sigma}_{\phi,\phi_0}(t) \begin{cases} \sigma_{\phi,\eta}(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \sigma_{\eta,\phi_0}(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then $\tilde{\sigma}_{\phi,\phi_0}$ is the mapping required, and this completes the proof. ■

Definition 2 *We say that $T\colon \mathcal{P}_{\mathcal{C}}(^nE) \to \mathcal{P}_{\mathcal{C}}(^nF)$ is power-preserving or T is a power-preserver if for all $\phi \in E'$ with $\phi^n \in \mathcal{P}_{\mathcal{C}}(^nE)$ we have that $T(\phi^n) = \pm\psi^n$ for some ψ in F'.*

Definition 3 *Given a power-preserving isomorphism $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ we call the function $t\colon S_{E'}^T/\equiv \to S_{F'}/\equiv$ the mapping induced by T to be the unique mapping which satisfies the property $t([\phi]) = [\psi]$ where $T(\phi^n) = \psi^n$ for all $\phi \in S_{E'}^T$.*

Lemma 4 *Let E and F be real or complex Banach spaces of dimension at least 3, n be a fixed positive integer and $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be a power-preserving isomorphism. Then, the continuous function $t\colon S_{E'}^T/\equiv \to S_{F'}/\equiv$ induced by T can be lifted to a continuous function $\tilde{t}\colon S_{E'}^T \to S_{F'}$. Further, if ψ_o is such that $t([\phi_o]) = [\psi_o]$ then, there exists a unique isomorphism $s\colon E' \to F'$ so that $s(\phi_o) = \psi_o$ and $[s(\phi)] = t([\phi])$ for all $\phi \in S_{E'}^T$.*

Proof: Fix ϕ_o in $S_{E'}^T$. Consider $t \circ [\,\cdot\,]\colon S_{E'}^T \to S_{F'}/\equiv$. This function is continuous. As $S_{E'}^T$ is simply connected we have that the fundamental group at the point ϕ_o, $\pi(S_{E'}^T, \phi_o)$, is trivial. Choose ψ_o in $S_{F'}$ so that $t([\phi_o]) = [\psi_o]$. Then by [31, Theorem 5.1] there is a unique continuous mapping $\tilde{t}\colon S_{E'}^T \to S_{F'}$ so that $\tilde{t}(\phi_o) = \psi_o$ and $[\tilde{t}(\phi)] = t([\phi])$ for all

$\phi \in S^T_{E'}$. Set $s(\phi) = \|T(\phi^n)\|^{1/n}\tilde{t}\left(\phi/\|T(\phi^n)\|^{1/n}\right)$ when $\phi \neq 0$ and $s(0) = 0$ to get a unique continuous homogeneous function $s\colon E' \to F'$ so that $s(\phi_o) = \psi_o$ and $s(\phi) = \tilde{t}(\phi)$ for all $\phi \in S^T_{E'}$. Let X be a finite dimensional subspace of E. Then X' is a subspace of E'. Consider $T|_{\widehat{\bigotimes}_{n,s,\epsilon}X'}\colon \widehat{\bigotimes}_{n,s,\epsilon}X' \to \widehat{\bigotimes}_{n,s,\epsilon}F'$. The proof of [39, Exercise 4.5.5] (see also [39, Theorem 4.5.5]) gives us a linear operator $t_{X'}\colon X' \to F'$ so that $T(\phi^n) = (t_{X'}(\phi))^n$ for all $\phi \in X'$. Then $(t_{X'}(\phi))^n = T(\phi_o^n) = \psi_o^n$ and so $t_{X'}(\phi) \equiv \psi_o$. By uniqueness of lifting we get that $s|_{X'} = \lambda t_{X'}$ for some λ with $|\lambda| = 1$. Since this holds for all finite dimensional subspaces of E', s is linear.

Clearly, as s is linear it must be injective. We claim that it is also surjective. Suppose that this is not the case. Let $Y \subset F'$ denote the range of s. It follows by [38, Lemma 1.2] that Y is a closed subspace of F'. As $T(\phi^n) = (s(\phi))^n$ for all $\phi \in E'$ and $\{\phi^n : \phi \in E'\}$ spans $\mathcal{P}_f(^nE) = \bigotimes_{n,s}E'$ we have that $T(\mathcal{P}_f(^nE)) = \bigotimes_{n,s}Y$. Since T is both continuous and open it follows that $T(\mathcal{P}_A(^nE)) = \widehat{\bigotimes}_{n,s,\epsilon}Y$ which is strictly contained in $\mathcal{P}_A(^nF)$. Thus, s is a bijection. ■

Definition 5 *An isomorphism $T\colon \mathcal{P}_{\mathcal{C}}(^nE) \to \mathcal{P}_{\mathcal{C}}(^nF)$ is said to be canonical if $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_{\mathcal{C}}(^nE)$ and some isomorphism $s\colon E' \to F'$.*

Theorem 6 *Let E and F be real or complex Banach spaces of dimension at least 3, n be a positive integer and $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be an isomorphism. Then the following are equivalent:*

(a) T is canonical,

(b) T is power preserving,

(c) there is an isomorphism $S\colon \mathcal{P}_A(^{2n}E) \to \mathcal{P}_A(^{2n}F)$ such that $S(PQ) = T(P)T(Q)$ for all $P, Q \in \mathcal{P}_A(^nE)$,

(d) If P_1, Q_1, P_2, $Q_2 \in \mathcal{P}_A(^nE)$ satisfy

$$P_1Q_1 = P_2Q_2$$

then,

$$T(P_1)T(Q_1) = T(P_2)T(Q_2).$$

Proof: Clearly we have that (a) implies (b). Now suppose that (b) holds. Then, Lemma 4 gives us an isomorphism $s\colon E' \to F'$ so that

$$T(\phi^n)(y) = \pm(s(\phi))^n(y) = \pm\phi^n \circ s' \circ J_F(y)$$

for all $\phi \in E'$ and $y \in F$. By linearity we get that

$$T(P)(y) = \pm\overline{P} \circ s' \circ J_F(y)$$

for all finite type polynomials P on E. Continuity extends the result to $\mathcal{P}_A(^nE)$ which shows that T is canonical.

Suppose that (a) holds and that $T(P) = \overline{P} \circ s' \circ J_F$ for some isomorphism $s\colon E' \to F'$. Then define $S\colon \mathcal{P}_A(^{2n}E) \to \mathcal{P}_A(^{2n}F)$ by $S(R) = \overline{R} \circ s' \circ J_F$ for $R \in \mathcal{P}_A(^{2n}E)$. As the Aron-Berner extension is multiplicative (see [13]) we have that $S(PQ) = T(P)T(Q)$ for all $P, Q \in \mathcal{P}(^nE)$. Statement (d) follows from (c).

Finally, let us show that (d) implies (b). Fix $\phi \in E'$ and choose $\psi \in E'$ which is linearly independent of ϕ. For $k = 0, 1, \dots, n$ let $Q_k \in \mathcal{P}_A(^nF)$ be defined by $Q_k = T(\phi^k\psi^{n-k})$. As (d) holds we have that $Q_k^2 = Q_{k+1}Q_{k-1}$ for $k = 1, \dots, n-1$. We can rewrite this as

$$Q_0(y)/Q_1(y) = Q_1(y)/Q_2(y) = \cdots = Q_{n-1}(y)/Q_n(y)$$

for all y in F. An old result of Mazur and Orlicz [32] allows us to show that we have unique factorization of polynomials on infinite dimensional spaces. Let us write the rational function

$$Q_0/Q_1 = Q_1/Q_2 = \cdots = Q_{n-1}/Q_n$$

in its lowest possible form as R/S. We have that $\deg R = \deg S = p$. As ψ is not a linear multiple of ϕ we have that $p > 0$. However,

$$Q_o(y)/Q_n(y) = (Q_0/Q_1)(y)\,(Q_1/Q_2)(y)\dots(Q_{n-1}/Q_n)(y) = (R(y)/S(y))^n$$

for all $y \in F$ and therefore we have that $p \leq 1$ and so both R and S are linear. In particular, we have that

$$T(\phi^n)(y) = Q_o(y) = \lambda\,(R(y))^n$$

for some constant λ and therefore T is a power-preserver. ■

The above result illustrates that there is a connection between the isometric properties of spaces of homogeneous polynomials and their 'algebraic' structure. This phenomenon can be observed in other function spaces and algebras, see for example, [9], [15], [23], [34] and [35].

3 Isometries between spaces of approximable polynomials on real Banach spaces

The isometric properties of Banach spaces are those properties which are intrinsically connected with the shape of the unit ball. To understand these properties we can naturally lead to examine certain subsets of points in the unit sphere which are invariant under isometries. These include extreme points, exposed points and denting points.

An extreme point of the (closed) unit ball of E, $\overline{B_E}$ is a point x with the property that whenever $x = \lambda y + (1-\lambda)z$ for y, z in $\overline{B_E}$ and $0 < \lambda < 1$, then, $x = y = z$.

As we will also use the notation given below in Section 4 we state it for both the real and complex cases.

Let X be a Banach space and n be a positive integer. It is shown in [5, Proposition 1] that the set of (real) extreme points of the unit ball of $\mathcal{P}_I({}^nX)$ is contained in $\{\pm\varphi^n : \varphi \in X', \|\varphi\| = 1\}$. Hence, given a Banach space X and a positive integer n we use $\mathcal{E}_n(X')$ to denote the set

$$\{\varphi \in X' : \varphi^n \text{ is an extreme point of } B_{\mathcal{P}_I({}^nX)}\}.$$

Theorem 7 *Let E and F be real Banach spaces and $T\colon \mathcal{P}_A({}^nE) \to \mathcal{P}_A({}^nF)$ be an isometric isomorphism. Then, there is a isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A({}^nE)$.*

Proof: Since T is an isometry we have that T' maps extreme points of $B_{\mathcal{P}_I({}^nF')}$ to extreme points of $B_{\mathcal{P}_I({}^nE')}$. Therefore, by [5, Propositions 1], for each $y \in \mathcal{E}_n(F'')$ we can find $x \in \mathcal{E}_n(E'')$ so that $T'(y^n) = \pm x^n$. Let us see that this equality extends to give us that T' is a power-preserver. Given $y \in F''$, $\|y\| = 1$, by [5, Proposition 5] and the Bishops-Phelps Theorem we can find a sequence $\{y_k\} \subset \mathcal{E}_n(F'')$ so that $y_k \to y$ in norm. By choosing a subsequence, if necessary, we may suppose that either $T'(y_k^n) = x_k^n$ or $T'(y_k^n) = -x_k^n$ for all k. By [38, Lemma 1.2 (a)] we have that $\{x^n : x \in E'', \|x\| = 1\}$ is closed in $\mathcal{P}_I({}^nE')$. Thus, there is $x \in E''$ so that $T'(y^n) = \pm x^n$. Since it is bijective, T' maps $\{\pm y^n : y \in F'', \|y\| = 1\}$ onto $\{\pm x^n : x \in E'', \|x\| = 1\}$. By homogeneity it follows that T' is a power-preserver.

We claim that either $T'(y^n) = x^n$ or $T'(y^n) = -x^n$ for all $y \in S_{F''}$ and hence for all y in F''. Suppose this not the case and assume that n is even. (The odd case is immediate.) Consider the disjoint sets $A = \{y \in S_{F''} : T'(y) = x^n\}$ and $B = \{y \in S_{F''} : T'(y) = -x^n\}$ which have union equal to the sphere of F''. Let us prove that A is open. Suppose that

$u \in A$ is the limit of a sequence $(u_k)_k$ in B. Then $T'(u_k^n) = -v_k^n$ converges to $T'(u^n)$. Choose $\phi \in F'$ so that $T'(u^n)(\phi) = \delta > 0$. Then we have that $-v_k(\phi)^n$ converges to the positive number δ, which is impossible. Similarly, B is open and as $S_{F''}$ is connected A or B must be empty.

Without loss of generality we assume that $T'(y^n) = x^n$ for all $y \in F''$. Let $i\colon F'' \to \mathcal{P}_I({}^nF')$ denote the n-homogeneous polynomial $i(y) = y^n$. Given ϕ and ψ in E' and $0 \leq k \leq n$ the Borel Transform gives us that $\phi^k\psi^{n-k}$ may be regarded as a continuous linear functional on $\mathcal{P}_I({}^nE')$ with $\phi^k\psi^{n-k}(x^n) = \phi^k(x)\psi^{n-k}(x)$ for all $x \in E''$. Hence, for any ϕ, ψ in E' the function $\phi^k\psi^{n-k} \circ T' \circ i$ belongs to $\mathcal{P}_A({}^nF')$. Fix ϕ in E' and consider $\psi \in E'$ which is not a linear multiple of ϕ. Let us use Q_k to denote the n-homogeneous approximable polynomial given by $Q_k = T(\phi^{n-k}\psi^k) = \phi^{n-k}\psi^k \circ T' \circ i$. For $y \in F''$ we have that

$$Q_0(y)/Q_1(y) = Q_1(y)/Q_2(y) = \cdots = Q_{n-1}(y)/Q_n(y) = \phi(x)/\psi(x)$$

where $x^n = T(y^n)$. Therefore we have that

$$Q_0(y)/Q_1(y) = Q_1(y)/Q_2(y) = \cdots = Q_{n-1}(y)/Q_n(y)$$

for all y in F''.

Repeating the argument of Theorem 6 we obtain a continuous linear functional R on F'' such that $Q_o(y) = \pm\,(R(y))^n$.

Whence, we have

$$T(\phi^n)(y) = Q_o(y) = \pm R(y)^n$$

for all $y \in F''$. Restricting to F, we see that $T(\phi^n) = \pm(R|_F)^n$ proving that T is a power-preserver. Theorem 6 now gives us that there is an isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A({}^nE)$. Since T is an isometry it follows that s must also be an isometry. ■

The following Theorem may be regarded as a converse to the observation in [30, Section 3] which states that $\mathcal{P}_A({}^nE)$ and $\mathcal{P}_A({}^nF)$ are isomorphically isomorphic when E' and F' are isometrically isomorphic.

Theorem 8 *Let E and F be real Banach spaces such that $\mathcal{P}_A({}^nE)$ and $\mathcal{P}_A({}^nF)$ are isometrically isomorphic for some integer n. Then, E' and F' are isometrically isomorphic.*

Under the additional assumption that E' and F' have the approximation property from [2] we obtain:

Corollary 9 *Let E and F be real Banach spaces with duals which have the approximation property. Let $T\colon \mathcal{P}_w(^nE) \to \mathcal{P}_w(^nF)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_w(^nE)$.*

4 Isometries of spaces of approximable polynomials on complex Banach spaces

Let us now turn to the complex case.

Theorem 10 *Let E and F be complex Banach spaces and n be a positive integer with $\overline{\mathcal{E}_n(F'')}^{w^*} = \overline{\mathcal{E}_{2n}(F'')}^{w^*}$. Let $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A(^nE)$.*

Proof: Fix ϕ in E' and consider $\psi \in E'$ which is not a linear multiple of ϕ. As in Theorem 6 let $Q_k = T(\phi^{n-k}\psi^k) = \phi^{n-k}\psi^k \circ T' \circ i$. For $y \in \mathcal{E}_n(F'')$ we get that

$$Q_0(y)/Q_1(y) = Q_1(y)/Q_2(y) = \cdots = Q_{n-1}(y)/Q_n(y) = \phi(x)/\psi(x)$$

where $x^n = T'(y^n)$. As each Q_k is weak*-continuous we get that

$$Q_k(y)Q_{k+2}(y) = Q_{k+1}(y)^2$$

for all $y \in \overline{\mathcal{E}_n(F'')}^{w^*} = \overline{\mathcal{E}_{2n}(F'')}^{w^*}$, $k = 0, \ldots, k-2$. As the extreme points of $\mathcal{P}_I(^{2n}F')$ separate $\mathcal{P}_A(^{2n}F)$, [26, page 75], we have that

$$Q_k(y)Q_{k+2}(y) = Q_{k+1}(y)^2$$

for all $y \in F''$. Hence, we get that

$$Q_0(y)/Q_1(y) = Q_1(y)/Q_2(y) = \cdots = Q_{n-1}(y)/Q_n(y)$$

for all $y \in F''$.

Proceeding as in Theorem 6 we obtain a continuous linear functional R on F'' such that

$$T(\phi^n)(y) = R(y)^n$$

for all $y \in F''$. Therefore, T is a power-preserver and it follows from Theorem 6 that T is canonical. Moreover, it is of the form $T(P) = \overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A(^nE)$ with s an isometric isomorphism from E' into F'. ∎

To obtain examples of complex Banach spaces where the equality $\overline{\mathcal{E}_n(F'')}^{w^*} = \overline{\mathcal{E}_{2n}(F'')}^{w^*}$ holds we need the concepts of complex extreme points and weak*-exposed points.

A point x is said to be a complex extreme point of the (closed) unit ball of E if $\|x + \lambda y\| \leq 1$ for all $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ implies $y = 0$. Every real extreme point of $\overline{B_E}$ is a complex extreme point. To distinguish between real and complex extreme points we use $\mathrm{Ext}_{\mathbf{R}}(E)$ and $\mathrm{Ext}_{\mathbf{C}}(E)$.

We recall that a unit vector x in a Banach space E is exposed if there is a unit vector $\phi \in E'$ so that $\phi(x) = 1$ and $\phi(y) < 1$ for $y \in B_E \setminus \{x\}$. If $E = X'$ is a dual space and the vector ϕ which exposes x is in X we say that x is weak*-exposed.

Corollary 11 *Let E and F be complex separable Banach spaces with $\ell_1 \not\hookrightarrow F'$. Suppose that $\mathrm{Ext}_{\mathbf{R}}(F'') = \mathrm{Ext}_{\mathbf{C}}(F'')$. Let $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A(^nE)$.*

Proof: Using [25, Theorem 3.3] we observe that the unit ball of F'' is the weak*-closed convex hull of its extreme points. This in turn is equal to the weak*-closed convex hull of $\mathrm{Exp}_{w^*}(F'')$, the set of weak*-exposed points of the unit ball of F'', (see [21, Page 640]). Applying [26, Theorem II.13.B] we see that $\mathrm{Ext}_{\mathbf{R}}(F'') \subseteq \overline{\mathrm{Exp}_{w^*}(F'')}^{w^*}$. It follows from [19, Propositions 3 and 5] that $\mathrm{Exp}_{w^*}(F'') \subseteq \mathcal{E}_n(F'') \subseteq \mathrm{Ext}_{\mathbf{C}}(F'')$ for all n. Therefore, we have that $\overline{\mathcal{E}_n(F'')}^{w^*} = \overline{\mathcal{E}_{2n}(F'')}^{w^*}$ and an application of Theorem 10 completes the proof. ■

We also get:

Corollary 12 *Let E and F be complex reflexive Banach spaces with $\mathrm{Ext}_{\mathbf{R}}(F) = \mathrm{Ext}_{\mathbf{C}}(F)$. Let $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = P \circ s' \circ J_F$ for all $P \in \mathcal{P}_A(^nE)$.*

Proof: The proof is similar to that of Corollary 11 but we use [21, Proposition 4.18] instead of the result on [21, Page 640]. ■

In particular we get

Corollary 13 *Let E be a reflexive JB*-triple and n be a positive integer. Suppose that $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ is an isometric isomorphism. Then there is a continuous linear isometry $s\colon E' \to F'$ such that $T(P) = \overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_A(^nE)$.*

Proof: It follows from [27] (see also [6]) that $\mathrm{Ext}_{\mathbf{R}}(E) = \mathrm{Ext}_{\mathbf{C}}(E)$. Now apply Corollary 11. ∎

The reflexive JB*-triples are listed in [11].

From the proofs of Theorem 7 and Theorem 10 we obtain the following proposition.

Proposition 14 *Let E and F be real or complex Banach spaces and $T\colon \mathcal{P}_A(^nE) \to \mathcal{P}_A(^nF)$ be an isomorphism such that T' is a power-preserver then, T is also a power-preserver.*

We know of no complex Banach space E or positive integer n where we do not have $\overline{\mathcal{E}_n(E'')}^{w^*} = \overline{\mathcal{E}_{2n}(E'')}^{w^*}$. It follows from [19] that $\mathcal{E}_n(E) = \mathcal{E}_{2n}(E)$ whenever the real and complex extreme points of the unit ball of a finite dimensional Banach space E coincide or whenever each point of the unit ball of E'' is a weak*-exposed point. By Corollary 11 we have that each isometry of the space of n-homogeneous approximable polynomials on the complex Banach spaces $L^p(\mu)$ and ℓ_p, $1 \le p < \infty$ are canonical. From [19, Example 4] it also follows that every isometry of $\mathcal{P}(^nc_o^m)$ is canonical.

5 Isometries between other spaces of homogeneous polynomials

Let us begin by considering spaces of integral polynomials.

Theorem 15 *Let E and F be real Banach spaces and n be a positive integer. Suppose that $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon}E$ and that $T\colon \mathcal{P}_I(^nE) \to \mathcal{P}_I(^nF)$ is an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_I(^nE)$.*

Proof: Since T is an isometry T maps the extreme points of the unit ball of $\mathcal{P}_I(^nE)$ onto the extreme points of the unit ball of $\mathcal{P}_I(^nF)$. Arguing as in Theorem 7 we get that T maps $\{\phi^n : \phi \in E'\}$ bijectively onto $\{\psi^n : \psi \in F'\}$ or $\{-\psi^n : \psi \in F'\}$. As in the proof of Theorem 7 we obtain an isometry $s\colon E' \to F'$ so that $T(\phi^n) = s(\phi)^n$ or $-s(\phi)^n$ for all $\phi \in E'$. Without loss of generality we assume that $T(\phi^n) = s(\phi)^n$ for all $\phi \in E'$. Since $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon}E$ [5, Theorem 2] also [8, Theorem 1.5] tells us that we have that $\mathcal{P}_I(^nE)$ is isometrically isomorphic to $\mathcal{P}_N({}^nE)$. Therefore, we have that

$$\begin{aligned} T(P) =& T(\sum_{k=1}^{\infty} \lambda_k \phi_k^n) \\ =& \sum_{k=1}^{\infty} \lambda_k T(\phi_k^n) \\ =& \sum_{k=1}^{\infty} \lambda_k s(\phi_k)^n \\ =& \overline{P} \circ s' \circ J_F. \end{aligned}$$

■

Corollary 16 *Let E and F be real Banach spaces and n be a positive integer. Suppose that E' has the Radon-Nikodým property (RNP) and that $T\colon \mathcal{P}_I({}^nE) \to \mathcal{P}_I({}^nF)$ is an isometric isomorphism. Then, there is an isometric isomorphism $s\colon E' \to F'$ such that $T(P) = \pm\overline{P} \circ s' \circ J_F$ for all $P \in \mathcal{P}_I({}^nE)$.*

Theorem 15 does not cover the case of real Banach space $E = F = \ell_1$. In this case we have the following result.

Theorem 17 *Let $T\colon \mathcal{P}_I({}^n\ell_1) \to \mathcal{P}_I({}^n\ell_1)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon \ell_1 \to \ell_1$ such that $T(P) = \pm P \circ s$, for all $P \in \mathcal{P}_I({}^n\ell_1)$.*

Proof: Let us first observe that the n-fold injective tensor product of ℓ_1, $\widehat{\bigotimes}_{n,\epsilon}\ell_1$, has the Radon-Nikodým property. To see this we use induction on n. Suppose that we have proved that $\widehat{\bigotimes}_{k,\epsilon}\ell_1$ has RNP. We note that $\widehat{\bigotimes}_{k+1,\epsilon}\ell_1$ may be regarded as the space of unconditionally convergent series in $\widehat{\bigotimes}_{k,\epsilon}\ell_1$. It follows from [16, Page 219] that $\widehat{\bigotimes}_{k+1,\epsilon}\ell_1$ has RNP and our claim is proved.

We therefore have that $\widehat{\bigotimes}_{n,s,\epsilon}\ell_1$ has RNP. Applying [24, 4), Page 103] we conclude that T is the transpose of an isometry $S\colon \mathcal{P}({}^nc_o) \to \mathcal{P}({}^nc_o)$. The result now follows from Theorem 7. ■

Let us now turn our attention to isometries between spaces of homogeneous polynomials.

Theorem 18 *Let E and F be real Banach spaces and n be a positive integer. Suppose that E is Asplund and E' has the approximation property. Let $T\colon \mathcal{P}({}^nE') \to \mathcal{P}({}^nF')$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon F' \to E'$ such that $T(P) = \pm P \circ s$, for all $P \in \mathcal{P}({}^nE')$.*

Proof: Since E is Asplund it follows from [5, Theorem 3] or [8, Theorem 1.4] that $\mathcal{P}_I({}^nE)$ is isometrically isomorphic to $\mathcal{P}_N({}^nE)$ while [37, Theorem 1.9] gives us that $\mathcal{P}_N({}^nE)$ has RNP. Since E' has the approximation property we have that $\mathcal{P}_N({}^nE)' = \mathcal{P}({}^nE')$. By [24, Theorem 10] we have that $\mathcal{P}_N({}^nE)$ is isometrically isomorphic to $\mathcal{P}_N({}^nF)$, while [24, 4), Page 103] implies that T is the transpose of an isometry $S\colon \mathcal{P}_N({}^nF) \to \mathcal{P}_N({}^nE)$. The result now follows from Corollary 16. ∎

Corollary 19 *Let E and F be reflexive Banach spaces with the approximation property and $T\colon \mathcal{P}({}^nE) \to \mathcal{P}({}^nE)$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon F \to E$ such that $T(P) = P \circ s$ for all $P \in \mathcal{P}({}^nE)$.*

Theorem 20 *Let E and F be real Banach spaces and n be a positive integer. Suppose that E' has the approximation property and that $\ell_1 \not\hookrightarrow \mathcal{P}({}^nE')$. Let $T\colon \mathcal{P}({}^nE') \to \mathcal{P}({}^nF')$ be an isometric isomorphism. Then, there is an isometric isomorphism $s\colon F' \to E'$ such that $T(P) = \pm P \circ s$, for all $P \in \mathcal{P}({}^nE')$.*

Proof: Since $\ell_1 \not\hookrightarrow \mathcal{P}({}^nE')$ we have that $\ell_1 \not\hookrightarrow \mathcal{P}_A({}^nE')$ i.e. $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon}E''$. Since symmetric tensor products respect subspaces we have $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon}E$. Applying [5, Theorem 1] we have that $\mathcal{P}_I({}^nE)$ is isometrically isomorphic to $\mathcal{P}_N({}^nE)$ which is in turn isometrically isomorphic to $\widehat{\bigotimes}_{n,s,\pi}E'$, as E' has the approximation property.

From [24, Theorem 10] we conclude can that $\mathcal{P}_N({}^nE)$ is isometrically isomorphic to $\mathcal{P}_N({}^nF)$. This time [24, Corollary 13] implies that T is the transpose of an isometry $S\colon \mathcal{P}_N({}^nF) \to \mathcal{P}_N({}^nE)$. Again, the result follows from Theorem 15. ∎

The above results for real Banach spaces extend to complex Banach spaces under the additional assumption that $\overline{\mathcal{E}_n(F')}^{w^*} = \overline{\mathcal{E}_{2n}(F')}^{w^*}$.

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