Weak-polynomial convergence on spaces ℓ_p and L_p

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Abstract.

This paper is concerned with the study of the set $P^{-1}(0)$, when P varies over all orthogonally additive polynomials on ℓ_p and L_p spaces. We apply our results to obtain characterizations of the weak-polynomial topologies associated to this class of polynomials.

Keywords: Polynomials on Banach spaces, Weak-polynomial topologies, Zeros of Polynomials on lp and Lp.

1. Introduction.

Before going into the problem, we discuss some preliminaries and fix notation. Throughout, X will be a real Banach space and X' its dual. We are going to consider only continuous polynomials and, as usual, denote by $P(^nX)$ the space of all n-homogeneous continuous scalar-valued polynomials with domain X. $P(X)$ will be the space of all continuous scalar-valued polynomials defined on X. $P(^nX)$ is a Banach space endowed with the norm $\|P\| = \sup\{|P(x)| \ : \ \|x\| \leq 1\}$.
We may define various topologies on X, in terms of convergence of nets: the *strong topology* in which a net $x_\alpha \to x$ if and only if $||x_\alpha - x|| \to 0$, the *weak (w) topology* where $x_\alpha \stackrel{w}{\to} x$ if and only if $\varphi(x_\alpha - x) \to 0$, for all $\varphi \in X'$ and the *weak polynomial (wp) topology*, (see (Carne et al., 1989)) with convergence given by $x_\alpha \stackrel{wp}{\to} x$ if and only if $P(x_\alpha - x) \to 0$, for all $P \in P(X)$ or equivalently for all $P \in P(^nX)$, for all $n \in I\!N$. The weak polynomial topology was studied in (Aron et al., 2), (Biström et al., 1989), (Davie and Gamelin, 1989), (González et al., 1997) and (Gutiérrez and Llavona, 1997).
It is easy to see that $x_\alpha \to x \Rightarrow x_\alpha \stackrel{wp}{\to} x \Rightarrow x_\alpha \stackrel{w}{\to} x$. For any infinite dimensional complex Hilbert space H, the wp-topology is not linear

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even when restricted to the unit ball of H (see, (Aron et al., 2)). New examples of both real and complex Banach spaces X such that the wp-topology is not linear are given in (Biström et al.) and (Castillo et al.).
The lack of linearity easily shows that in general these topologies are different, although there are some examples that we will mention later for which $w = wp$ and $wp = norm$. In order to obtain a linear polynomial topology *close* to wp the following topology $\mathcal{T}$ is considered in (Garrido et al.,).

Definition 1.1. (Garrido et al.,) For each $P \in P(^nX)$ define $\tilde{d}_P : X \times X \to I\!R_{\geq 0}$,

$$\tilde{d}_P(x,y) = \inf_{\substack{k-chains \\ k \in Z\!\!Z, k \geq 0}} \left\{ |P(x-z_1)|^{\frac{1}{n}} + |P(z_1-z_2)|^{\frac{1}{n}} + \cdots + |P(z_k-y)|^{\frac{1}{n}} \right\}$$

where by a $k-chain$ we mean the ordered set $\{z_1, \ldots, z_k\}$ if $k \geq 1$ and the empty set if $k = 0$.

We mention some elementary properties:

1. $\tilde{d}_P(x,y) \leq |P(x-y)|^{\frac{1}{n}}$.
2. $\tilde{d}_P(x,y) = \tilde{d}_P(y,x)$, for all $x, y \in X$.
3. $\tilde{d}_P(x,y) \leq \tilde{d}_P(x,z) + \tilde{d}_P(z,y)$, for all $x, y, z \in X$.
4. $\tilde{d}_P(x+h, y+h) = \tilde{d}_P(x,y)$, for all $x, y, h \in X$. *(invariance under translations)*
5. $\tilde{d}_P(\lambda x, \lambda y) = |\lambda| \tilde{d}_P(x,y)$, for all $x, y \in X$, for all $\lambda \in I\!R$.

All these properties follow easily from the definition of $\tilde{d}_P$. For instance, (1) is obtained considering the empty chain and (3) is due to the fact that the infimum is taken over all $k-chains$. The next step is to define $d_P(x) = \tilde{d}_P(x, 0)$, for all $x \in X$.

Lemma 1.2. Given $P \in P(^nX)$, $d_P : X \to I\!R_{\geq 0}$ is a continuous seminorm on X.

Definition 1.3. (Garrido et al.) Let $\mathcal{T}$ be the topology defined on X generated by the family of seminorms $(d_P)_P$ when P varies over all n-homogeneous continuous scalar-valued polynomials on X, and n varies in $I\!N$.

This topology $\mathcal{T}$ is convex and linear. Moreover, for any $\varphi \in X'$ we have $d_\varphi(x) = |\varphi(x)|$, so $w \leq \mathcal{T} \leq wp \leq \|.\|$.

For spaces X like c_0, the Tsirelson space T^* and $C(K)$ for scattered compact K, where every polynomial is w-continuous on bounded sets, the topologies w and wp agree on bounded sets (see Pelczyński). Therefore $w = \mathcal{T} = wp$ on bounded sets for these spaces. On the other hand, for spaces X like ℓ_2, which admit a separating polynomial the other equality, $\mathcal{T} = wp = \|.\|$, holds. Recall that a polynomial $Q : X \to I\!R$ is said to be separating if there exists a constant $C > 0$ such that $|Q(x)| \geq C$, for all $x \in X$ with $\|x\| = 1$. Moreover, if a Banach space has a separating polynomial then it has an n-homogeneous separating polynomial P (see Fabián et al.). In this case there exists a constant $C > 0$ such that $|P(x)| \geq C\|x\|^n$, for all $x \in X$ and therefore $d_p(x) \geq C^{\frac{1}{n}} \inf\{\|x - z_1\| + \|z_1 - z_2\| + \cdots + \|z_k\|\} \geq C^{\frac{1}{n}}\|x\|$. So, $C^{\frac{1}{n}}\|x\| \leq d_P(x) \leq K\|x\|$, and d_P is an equivalent norm to the given norm on X. Then, the equality follows.

We consider ℓ_p and $L_p[0,1]$ spaces, $1 \leq p < \infty$, as Banach lattices; each one with the usual order and notion of orthogonality (see Lindenstrauss and Tzafriri). In Banach lattices the space of orthogonally additive functions has a special interest.

Definition 1.4. Let X be a Banach lattice. A function $f : X \to I\!R$ is said to be orthogonally additive if $f(x + y) = f(x) + f(y)$ whenever $x \perp y,\ x, y \in X$.

For a Banach lattice X, we consider the closed subspace of $P(^nX)$ of all orthogonally additive polynomials on X, which we denote by $P_o(^nX)$.

Example 1.5. (Sundaresan, 1991) Let $X = \ell_p,\ 1 \leq p < \infty$. For $n \geq p$, $P_o(^nX)$ is isometrically isomorphic to ℓ_∞ under the map

$$P \leftrightarrow \xi = (a_j = P(e_j))_{j\geq 1}.$$

Note that in this case the set $\bigcup_{n\geq p} P_o(^nX)$ contains nonseparable subspaces, and coincides with the set of all n-homogeneous diagonal polynomials on ℓ_p, with $n \geq p$.

Example 1.6. (Sundaresan) Let $X = L_p[0,1],\ 1 \leq p < \infty$, and let μ be Lebesgue measure.

For $1 \leq n < p$, $P \in P_o(^nX) \Leftrightarrow \exists!\ \xi \in L_{\frac{p}{p-n}}$ such that $P(x) = \int_0^1 \xi x^n d\mu$.

For $n = p$, $P \in P_o(^nX) \Leftrightarrow \exists!\ \xi \in L_\infty$ such that $P(x) = \int_0^1 \xi x^n d\mu$.

For $n > p$, $P \in P_o(^nX) \Leftrightarrow P \equiv 0$.

The map $P \leftrightarrow \xi$ is an isomorphism between the spaces. In particular, the set $\bigcup_{n\leq p} P_o({}^nX)$ contains a nonseparable subspace whenever p is a natural number. Otherwise, it is a finite union of separable spaces.

We define a topology τ and the weak polynomial topology associated to this set.

Definition 1.7. Let τ be the topology defined on X by the family of seminorms $(d_P)_P$ when P varies over $P_o({}^nX)$ and n varies in $\mathbb{N}$.

Definition 1.8. Let X be a Banach lattice. We say that a net $(x_\alpha) \subset X$ converges to $x \in X$ in the wp_o-topology if and only if for all $n \in \mathbb{N}$, for all $P \in P_o({}^nX)$, $P(x_\alpha - x) \to 0$.

In this paper, our main problem concerns the characterization of the τ-topology on bounded sets, and its relation with the wp_o-topology, for ℓ_p and L_p spaces. The set $P^{-1}(0)$ plays a fundamental role in the estimates of seminorms $d_P's$ as we will see in the following section. Other people studied the set $P^{-1}(0)$, P homogeneous polynomial on X, using a different approach. For instance, in (Aron et al., 1), (Aron et al., 3) and Aron and Rueda the size of $P^{-1}(0)$ is considered.

2. τ on the spaces ℓ_p.

We give a characterization of those bounded nets $(x_\alpha)_\alpha \subset \ell_p$ verifying that $x_\alpha \xrightarrow{\tau} x$. For this purpose we want to describe d_P when P is an orthogonally additive polynomial on ℓ_p. It is clear that $x_\alpha \xrightarrow{\tau} x$ implies $x_\alpha \xrightarrow{w} x$ because of the equality $|\varphi(a)| = d_\varphi(a)$. Taking into account that $d_P(x) \leq |P(x)|^{\frac{1}{n}}$, those polynomials which are w-continuous on bounded sets do not give different information about τ-convergence from the elements of X'. Indeed, suppose we know, for a bounded net, that $d_\varphi(x_\alpha - x) \to 0$ for all φ in X'. Then,$d_P(x_\alpha - x) \to 0$ for all P n-homogeneous polynomial w-continuous on bounded sets. This fact allows us to pay attention only to orthogonally additive polynomials which are not w-continuous on bounded sets. Since every polynomial of degree less than p on ℓ_p is w-continuous on bounded sets (see Bonic and Frampton or Llavona, Thm. 4.4.7 and Thm. 4.5.9), when it is opportune, we restrict our description of d_P to the set of n-homogeneous orthogonally additive polynomials with $n \geq p$, in other words, the set of n-homogeneous diagonal polynomials with $n \geq p$.

Proposition 2.1. Let $1 \leq p < \infty$, n an odd integer, $n \geq p$ and $P(x) = \sum_{j=1}^\infty a_j x_j^n$ with $(a_j)_{j\geq 1} \in \ell_\infty$. If P is not w-continuous on bounded sets on ℓ_p, then $d_P(x) = 0$ for all $x \in \ell_p$.

Table I.

Coord:	1	2	$\cdots$	k	$k+1$	$\cdots$	$2k-1$	$2k$	$2k+1$	$\cdots$
z_1^k :	$\frac{2k-1}{2k}$	$\frac{2k-3}{2k}$	$\cdots$	$\frac{1}{2k}$	$-\frac{1}{2k}$	$\cdots$	$-\frac{2k-3}{2k}$	$-\frac{2k-1}{2k}$	0	$\cdots$
z_2^k :	0	$-\frac{2k-2}{2k}$	$\cdots$	$-\frac{2}{2k}$	0	$\cdots$	$\frac{2k-4}{2k}$	$\frac{2k-2}{2k}$	0	$\cdots$

Proof. Assume that every $a_j \neq 0$. We first show that $d_P(e_1) = 0$ by constructing a suitable $2-chain$ sequence, that allows us to estimate that value.
For every fixed $k \in I\!N$, consider the vectors defined as in Table I.
With $b_j = a_j^{-\frac{1}{n}}$ define the sequences $\tilde{z}_1^k = (b_j(z_1^k)_j),\ \tilde{z}_2^k = (b_j(z_2^k)_j)$. In order to estimate $d_P(b_1e_1)$ consider the $2-chain$ sequence given by $w^k = \tilde{z}_1^k + \tilde{z}_2^k$ and $\tilde{z}_2^k$.
Since $d_P(b_1e_1) \leq |P(b_1e_1 - w^k)|^{\frac{1}{n}} + |P(w^k - \tilde{z}_2^k)|^{\frac{1}{n}} + |P(\tilde{z}_2^k)|^{\frac{1}{n}}$, $w^k - \tilde{z}_2^k = \tilde{z}_1^k$ and the weights $(b_j)_j$ were chosen in order to produce zeros of P, we only have to compute $|P(b_1e_1 - w^k)|$. A straightforward calculation shows that its value is $\frac{2k}{(2k)^n}$. Therefore, $d_P(b_1e_1) \leq \frac{\sqrt[n]{2k}}{2k}$, for any $k \in I\!N$ whence, $d_P(b_1e_1) = 0$ and consequently $d_P(e_1) = 0$.
Now, the sequences can be placed to the j-th coordinate to show that $d_P(e_j) = 0$ for all $j \in I\!N$. Since d_P is a seminorm it follows that d_P is 0 on every element of finite support. By continuity and density $d_P(x) = 0$ for all $x \in \ell_p$.
For any polynomial P not w-continuous on bounded sets we have that $n > 1$ and there are infinitely many values of j such that $a_j \neq 0$. If $P(x) = \sum_{j=1}^{\infty} a_{m_j} x_{m_j}^n$, with $a_{m_j} \neq 0$, a slight modification of the above sequences gives the result; it is enough to place the non zero coordinates at the $m_j{}'s$ indices to obtain that $d_P(e_{m_j}) = 0$ for all m_j. For the other $j's$, $P(e_j) = 0$ and so is $d_P(e_j)$. Therefore, $d_P(e_j) = 0$ for all $j \in I\!N$ and the result follows. □

Remark 2.2. The hypothesis "P is not w-continuous on bounded sets of ℓ_p" cannot be avoided in the previous proposition. Indeed, if P is an n-homogeneous orthogonally additive polynomial on ℓ_p which is w-continuous on bounded sets, n an odd integer $n \geq p$, then $d_P \not\equiv 0$ for many cases. For instance, any linear function φ produces the seminorm $|\varphi|$. Also, consider any projection to one coordinate $\varphi \in \ell_p', \varphi \neq 0$. The seminorm d_P associated to the orthogonally additive polynomial $P = \varphi^n$ satisfies $d_P = d_{\varphi^n} = |\varphi|$ for any n.

Proposition 2.6 will provide us with examples of n-homogeneous w-continuous orthogonally additive polynomials P with infinitely many coefficients different from zero associated to non zero seminorms, if n is an even integer. However, the proof of Proposition 2.1 shows that this situation is not possible when n is an odd integer, $n > 1$.

The following lemma shows the main role that zeros of polynomials play in this topology and will allow us to determine d_P for $P \in P(^nX)$ a diagonal polynomial when n is an even integer.

Lemma 2.3. Let X be a real Banach space and P any n-homogeneous polynomial on X. Then,

$$d_P(x) = d_P(x+z) \quad \textit{for all } z \textit{ such that } P(z) = 0, \quad \textit{for all } x \in X.$$

Proof. Whenever $P(z) = 0,\ d_P(z) = 0$. Since d_P is a seminorm, for any $x \in X$ we have that $d_P(x+z) \leq d_P(x)$ and $d_P(x) = d_P((x+z)+(-z)) \leq d_P(x+z)$ so, the equality holds. □

Lemma 2.4. Let X be a real Banach space and P an n-homogeneous polynomial on X. If $x \in \overline{\operatorname{span}[P^{-1}(0)]}$, then $d_P(x) = 0$. Therefore, if $\operatorname{span}[P^{-1}(0)]$ is dense in X, then $d_P \equiv 0$ in X.

Proof. Fix $x \in \overline{\operatorname{span}[P^{-1}(0)]}$. Given $\varepsilon > 0$ there exists $w = \sum_{j=1}^{M} z_j$ such that $P(z_j) = 0$, for all $j = 1, \ldots, M$, and $\|x - w\| < \varepsilon$. By Lemma 2.3 it is enough to note that

$$d_P(x) = d_P(x-w) \leq \|P\|^{\frac{1}{n}} \|x-w\| < \|P\|^{\frac{1}{n}} \varepsilon. \ \square$$

Using this lemma we can obtain, in a simpler way, as we state in the next remark, the result in Proposition 2.1. However, we believe that the proof given above is more constructive and illustrates the ideas followed in the proof of Theorem 3.2, *(a)*. For a proof based on Lema 2.4 it is enough to consider linear combinations of three consecutive elements of the form $a_j^{-\frac{1}{n}} e_j$ when $a_j \neq 0$ to show that the canonical basis of ℓ_p, $(e_j)_{j \geq 1}$, belongs to $\operatorname{span}[P^{-1}(0)]$.

Remark 2.5. Let $1 \leq p < \infty$, n an odd integer, $n \geq p$ and $P(x) = \sum_{j=1}^{\infty} a_j x_j^n$ with $(a_j)_{j \geq 1} \in \ell_\infty$. If P is not w-continuous on bounded sets of ℓ_p, then $\operatorname{span}[P^{-1}(0)]$ is dense in ℓ_p and therefore $d_P \equiv 0$.

Now, we consider orthogonally additive polynomials of even degree. Note that we not need to impose conditions on the continuity of the polynomials.

Proposition 2.6. Let $1 \leq p < \infty$, n an even integer, $n \geq p$ and $P(x) = \sum_{j=1}^{\infty} a_j x_j^n$ with $(a_j)_{j\geq 1} \in \ell_\infty$.

(a) If the function $\mathrm{sign}(a_j)$, $a_j \neq 0$, is constant, then $d_p(x) = |P(x)|^{\frac{1}{n}}$ for all $x \in \ell_p$.

(b) If there exists a pair $i \neq j$ such that $a_i,\ a_j \neq 0$ and sign $(a_i) \neq \mathrm{sign}(a_j)$, then $d_p(x) = 0$ for all $x \in \ell_p$.

Proof. For the first case suppose that, for each $j \in I\!N$, $\mathrm{sign}(a_j) = 1$ or $a_j = 0$. Define $\|x\|_a = (\sum_{j=1}^{\infty} a_j x_j^n)^{\frac{1}{n}} = (\sum_{j=1}^{\infty} (\sqrt[n]{a_j} x_j)^n)^{\frac{1}{n}}$. By Minkowski's inequality we have, for all sum considered in the definition of d_P, that

$$\begin{aligned} |P(x - z_1)|^{\frac{1}{n}} + {}& |P(z_1 - z_2)|^{\frac{1}{n}} + \cdots + |P(z_k)|^{\frac{1}{n}} \\ &= \|x - z_1\|_a + \|z_1 - z_2\|_a + \ldots + \|z_k\|_a \geq \|x\|_a, \end{aligned}$$

So, $d_p(x) = |P(x)|^{\frac{1}{n}}$.
For the second case, let a_i and a_j be any two elements of the sequence with different signs. Without lost of generality suppose $a_i > 0, a_j < 0$. Let $x = \frac{1}{\sqrt[n]{a_i}} e_i$, $z = \frac{1}{\sqrt[n]{a_i}} e_i + \frac{1}{\sqrt[n]{-a_j}} e_j$ and $w = \frac{1}{\sqrt[n]{a_i}} e_i - \frac{1}{\sqrt[n]{-a_j}} e_j$. It is clear that both, z and w, are zeros of P. By Lemma 2.3, $\frac{1}{\sqrt[n]{a_i}} d_P(e_i) = d_P(x) = d_P(x + z + w) = 3\frac{1}{\sqrt[n]{a_i}} d_P(e_i)$, so that $d_P(e_i) = 0$. On the other hand, $\frac{1}{\sqrt[n]{a_i}} d_P(e_i) = d_P(x) = d_P(x - z) = \frac{1}{\sqrt[n]{-a_j}} d_P(e_j)$ and so $d_P(e_j) = 0$. Finally, for all those $j's$ such that $a_j = 0$ we have that $P(e_j) = 0$ and so is $d_P(e_j)$. Thus, $d_P(e_j) = 0$, for all $j \in I\!N$, which concludes the proof. □

We have described d_P for all diagonal n-homogeneous polynomials P that are not w-continuous on bounded sets of ℓ_p. Now we are able to characterize the τ-convergence of bounded nets in those spaces.

Theorem 2.7. Let $X = \ell_p$, $1 \leq p < \infty$. Let k be the smallest integer verifying $p \leq 2k$. Then, for any bounded net (x_α) and $x \in X$

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad \begin{cases} (i) & x_\alpha \xrightarrow{w} x \quad \textit{and} \\ (ii) & \|x_\alpha - x\|_{2k} \to 0 \end{cases}$$

Proof. Let (x_α) be a bounded net such that $x_\alpha \xrightarrow{\tau} x$, for some x in X. Since every $\varphi \in X'$ is an orthogonally additive polynomial and $|\varphi(a)| = d_\varphi(a)$ then, (i) holds. To verify the second condition note that since $p \leq 2k$, $P(x) = \sum_{j=1}^{\infty} x_j^{2k}$ is a well defined $2k$-homogeneous orthogonally additive polynomial on ℓ_p, not w-continuous on bounded

sets. By Proposition 2.6, $\|x_\alpha - x\|_{2k} = |P(x_\alpha - x)|^{\frac{1}{2k}} = d_P(x_\alpha - x)$ that converges to zero by hypothesis.
Conversely, let P be an n-homogeneous orthogonally additive polynomial on ℓ_p. Assume that (x_α) is weakly convergent to x.
If $n < p$ then P is w-continuous on bounded sets and $d_P(x_\alpha - x) \to 0$. If $n \geq p$ then $P(x) = \sum_{j=1}^\infty a_j x_j^n$, with $(a_j) \in \ell_\infty$. If n is odd, by Proposition 2.1, P is w-continuous on bounded sets or $d_P \equiv 0$ so, $d_P(x_\alpha - x) \to 0$. On the other hand, if n is even, by Proposition 2.6, $d_P \equiv 0$ or $d_P(x) = |P(x)|^{\frac{1}{n}}$ for all $x \in \ell_p$. For the last situation, take $A > 0$ a constant such that $|a_j|^{\frac{1}{n}} \leq A$, for all $j \in I\!N$. Thus for any $y \in \ell_p$ since $p \leq 2k \leq n$ we have that

$$d_P(y) \leq |P(y)|^{\frac{1}{n}} = |\sum_{j=1}^{\infty} a_j y_j^n|^{\frac{1}{n}} \leq A\|y\|_n \leq A\|y\|_{2k}. \tag{1}$$

This inequality allows us to conclude the proof. $\square$

Corollary 2.8. Let $X = \ell_p$, $p = 1$ or $p \in (2k-1, 2k]$ for some $k \in I\!N$. Then, for any bounded net (x_α) and $x \in X$

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad x_\alpha \xrightarrow{wp_o} x.$$

Proof. Let x be in X and let (x_α) be a bounded net. Assume that $x_\alpha \xrightarrow{\tau} x$. To prove that $x_\alpha \xrightarrow{wp_o} x$ we have to show that $P(x_\alpha - x) \to 0$ for any n-homogeneous orthogonally additive polynomial P. As in Theorem 2.7, the case $n < p$ is due to w-continuity. Let $n \geq p$ and $P(x) = \sum_{j=1}^\infty a_j x_j^n$, with $(a_j) \in \ell_\infty$. Choose, as before, $A > 0$ such that $|a_j|^{\frac{1}{n}} \leq A$ for all $j \in I\!N$. If $p \neq 1$, then $n \geq p$, $n \geq 2k \geq p$. This k is the smallest integer such that $p \leq 2k$. The inequality $|P(y)|^{\frac{1}{n}} \leq A\|y\|_{2k}$ and Theorem 2.7 give $|P(x_\alpha - x)| \to 0$.
If $p = 1$ and $n = 1$, then P is a linear form and so $P(x_\alpha - x) \to 0$ since $x_\alpha \xrightarrow{w} x$. If $n \geq 2$, the result follows from the inequality $|P(y)|^{\frac{1}{n}} \leq A\|y\|_2$ and Theorem 2.7.
The converse is due to the property $d_p(x) \leq |P(x)|^{\frac{1}{n}}$. $\square$

Remark 2.9. Let $X = \ell_p$, with $p = 2k+1$ for some $k \in I\!N$. Then, the wp_o-topology is strictly stronger than the τ-topology.

Proof. Consider $P(x) = \sum_{j=1}^\infty x_j^p$, which is a well defined p-homogeneous orthogonally additive polynomial on ℓ_p. Let $(x_m)_{m \in I\!N}$ be the sequence $x_m = \frac{1}{m}e_1 + \frac{1}{m}e_2 + \ldots + \frac{1}{m}e_{m^p}$. Thus, $\|x_m\|_{2k+2} = (\frac{1}{m})^{\frac{1}{2k+2}} \to 0$, if

$m \to \infty$, with $2k+2$ the smallest even integer greater than p. Since $(x_m) \subseteq \ell_p$ is a sequence with finite support and $\|x_m\|_p = |P(x_m)| = 1$, then it is weakly convergent to zero. So, by Theorem 2.7 (x_m) is convergent to zero in the τ-topology but not in the wp_o-topology. $\square$

3. τ on the spaces L_p.

In this section we investigate the topology τ on L_p spaces. We will use the representation of the space of orthogonally additive polynomials on L_p described in Example 1.6 to show the main result, Theorem 3.2. As a corollary a characterization of the τ convergence on L_p is given.

Before going on, note that next result is obtained as an immediate consequence of Example 1.6.

Remark 3.1. The τ-topology and the w-topology are the same on $X = L_p[0,1],\ 1 \leq p < 2$.

Theorem 3.2. Let $X = L_p[0,1],\ 1 \leq p < \infty$. For any n-homogeneous orthogonally additive polynomial on X (i.e. $P(x) = \int_0^1 \xi x^n d\mu$, $n \leq p$) we have:

(a) If n is an odd integer and $1 < n$ then, $d_P \equiv 0$ on $L_p[0,1]$.

(b) If n is an even integer then, when $\xi \geq 0$ a.e. or $\xi \leq 0$ a.e., $d_P(x) = |P(x)|^{\frac{1}{n}}$ for all $x \in L_p[0,1]$. Otherwise $d_P \equiv 0$ on $L_p[0,1]$.

Proof. By Sundaresan, Example 1.6, $\xi \in L_{\frac{p}{p-n}}$ if $n < p$, $\xi \in L_\infty$ if $n = p$. In order to prove (a) the proof is split in two parts.

Step 1: $\xi > 0$. We will prove that $d_P(\overline{1}) = 0$ and then show how the same argument can be applied to any function $\chi_{[a,b]}$ with $0 \leq a < b \leq 1$. Since the span of the set of characteristic functions is dense in $L_p[0,1]$, the result follows by density.
First note that $\xi \in L_r[0,1]$ and $\xi > 0$ with $r = \frac{p}{p-n}$ or $r = \infty$, if $p = n$. Then, $\xi \in L_1[0,1]$ and $\int_\alpha^\beta \xi d\mu > 0$ for any $0 \leq \alpha < \beta \leq 1$.
Let $(t_n)_{n \in \mathbb{N}}$ be a strictly decreasing sequence of numbers in $(0,1)$ such that $t_n \to 0$. Let us call $T_m(1) = \int_{t_m}^1 \xi d\mu,\ T_m(2) = \int_0^{t_m} \xi d\mu$. Thus, $T_m(1) \to P(\overline{1})$ and $T_m(2) \to 0$. For each $m \in \mathbb{N}$ define

$$h^m = \begin{cases} \frac{1}{2} & on \quad (t_m, 1] \\ \\ -\frac{1}{2}\sqrt[n]{\frac{T_m(1)}{T_m(2)}} & on \quad [0, t_m] \end{cases}$$

Table II.

[0,1]:	I_{2k}	I_{2k-1}	$\cdots$	I_{k+1}	I_k	$\cdots$	I_2	I_1
h_1^m :	$-\frac{2k-1}{2k}w_{2k}$	$-\frac{2k-3}{2k}w_{2k-1}$	$\cdots$	$-\frac{1}{2k}w_{k+1}$	$\frac{1}{2k}w_k$	$\cdots$	$\frac{2k-3}{2k}w_2$	$\frac{2k-1}{2k}$
h_2^m :	$\frac{2k-2}{2k}w_{2k}$	$\frac{2k-4}{2k}w_{2k-1}$	$\cdots$	0	$-\frac{2}{2k}w_k$	$\cdots$	$-\frac{2k-2}{2k}w_2$	0

so that $P(h^m) = 0$. Since $P(\overline{1} - h^m) = (\frac{1}{2})^n T_m(1) + (\sqrt[n]{T_m(2)} + \frac{1}{2}\sqrt[n]{T_m(1)}\ )^n$ and $d_P(\overline{1}) \leq |P(\overline{1} - h^m)|^{\frac{1}{n}}$ for all $m \in I\!N$ we obtain, taking the limit on m, that $d_P(\overline{1}) \leq \frac{\sqrt[n]{2}}{2}|P(\overline{1})|^{\frac{1}{n}}$.
Now, for each fixed $k \in I\!N$, consider the partition of $[0,1]$ given by $\Pi_{2k} = \{0, t_{m+2k-2}, \ldots, t_{m+1}, t_m, 1\}$. We proceed as we did before.
Call $T_m(1) = \int_{t_m}^1 \xi d\mu,\ T_m(2) = \int_{t_{m+1}}^{t_m} \xi d\mu,\ \ldots, T_m(2k) = \int_0^{t_{m+2k-2}} \xi d\mu$, so $T_m(1) \to P(\overline{1})$ and $T_m(j) \to 0$, for all $2 \leq j \leq 2k$, for every fixed k, when m goes to infinity.
Now, let $(I_j)_j$ be the ordered sequence of intervals defined by the partition Π_{2k} related to $T_m(j)$, with $I_1 = [t_m, 1]$ and let $w_j = \sqrt[n]{\frac{T_m(1)}{T_m(j)}}$ for every $j \geq 2$. Define the step functions $(h_1^m)_{m\in I\!N}$ and $(h_2^m)_{m\in I\!N}$ following Table II. As in Proposition 2.1, the weights $(w_j)_j$ were chosen so that $P(h_1^m) = P(h_2^m) = 0$. If $h^m = h_1^m + h_1^m$ we have that $h^m = \frac{2k-1}{2k}\chi_{I_1} + \sum_{j=2}^{2k}(-\frac{1}{2k})w_j\chi_{I_j}$ and

$$
\begin{aligned}
P(\overline{1} - h^m) &= (\tfrac{1}{2k})^n T_m(1) + \textstyle\sum_{j=2}^{2k}\ (1 + \tfrac{1}{2k}w_j)^n\ T_m(j) \\
&= (\tfrac{1}{2k})^n T_m(1) + \textstyle\sum_{j=2}^{2k}\ (\sqrt[n]{T_m(j)} + \tfrac{1}{2k}\sqrt[n]{T_m(1)})^n
\end{aligned}
$$

for all $m \in I\!N$ and k fixed. Since $d_P(\overline{1}) \leq |P(\overline{1} - h^m)|^{\frac{1}{n}}$ for all $m \in I\!N$, letting m go to infinity, we have that $d_P(\overline{1}) \leq \dfrac{\sqrt[n]{2k}}{2k}|P(\overline{1})|^{\frac{1}{n}}$ for all $k \in I\!N$ whence, $d_P(\overline{1}) = 0$.
To show that $d_P(\chi_{[a,b]}) = 0$ it is enough to consider a suitable sequence $(t_m) \subset (a,b)$ and repeat the construction for the new partition. In this situation the estimate obtained is $d_P(\chi_{[a,b]}) \leq \dfrac{\sqrt[n]{2k}}{2k}|P(\chi_{[a,b]})|^{\frac{1}{n}}$, for all $k \in I\!N$.

Step 2: $\xi \not\equiv 0$ arbitrary. We want to show that $d_P(\chi_{[a,b]}) = 0$ for all subintervals $[a,b]$ of $[0,1]$.
In Step 1 we only use that $\xi > 0$ *a.e.* to assure that $T_m(j) \neq 0$ for all $2 \leq j \leq 2k$, which makes possible the construction of the functions

$h_1^m,\ h_2^m$. To proceed in the same way we have to prove that there exists a sequence (t_m) satisfying that condition. Observe that we also need (t_m) converging to the left endpoint of the interval. In order to do this recall the following lemma.

Lemma 3.3. (Royden) (Lemma 8, page 105.) For any $\xi \in L_1[a,b]$ consider $\gamma(t) = \int_a^t \xi(s)d\mu(s)$, $t \in [a,b]$. If $\gamma(t) = 0$ for all t, then $\xi \equiv 0$ almost everywhere on $[a,b]$.

Let $h = \chi_{[a,b]}$ and $\alpha = \sup\{\delta \geq a \ / \ \xi|_{[a,\delta]} \equiv 0 \ a.e.\}$. If $\alpha > a$, $d_P(h) = d_P(\chi_{[\alpha,b]})$ so, we would have to estimate this last value where there exists a $t_0 > \alpha$ such that $\xi|_{[\alpha,t_0]} \neq 0$. To simplify notation assume that $\alpha = a = 0$. Then, $\xi|_{[0,t_0]} \neq 0$.

Put $\gamma(t) = \int_0^t \xi(s)d\mu(s),\ t \in [0,t_0]$. By Lemma 3.3 there exists $t_1 \leq t_0$ such that $\gamma(t_1) \neq 0$ and $\xi|_{[0,t]} \neq 0$ for all $t \in (0,t_1)$.

To choose the next element of the sequence put $I_1 = [0, \frac{t_1}{2}]$. There exists $w \in I_1$ with $\gamma(w) \neq 0$. Note that γ is a continuous function defined on I_1, connected set, $\gamma(0) = 0$ and $\gamma(w) \neq 0$. Since γ is not a step function there exists $t_2 \in I_1$ such that $\gamma(t_2) \neq 0$ and $\gamma(t_2) \neq \int_0^{t_1} \xi(s)d\mu(s)$. Then, $\gamma(t_2) \neq 0$ and $\int_{t_2}^{t_1} \xi(s)d\mu(s) \neq 0$.

Put $I_2 = [0, \frac{t_2}{2}]$. Then, t_2 and I_2 satisfy the same conditions as t_1 and I_1. Now, it is possible to find $t_3 \in I_2$ satisfying $\gamma(t_3) \neq 0$ and $\int_{t_3}^{t_2} \xi(s)d\mu(s) \neq 0$.

An inductive procedure provides a sequence (t_m) such that $0 < t_{m+1} \leq \frac{t_m}{2}$, with $\gamma(t_{m+1}) \neq 0$ and $\int_{t_{m+1}}^{t_m} \xi(s)d\mu(s) \neq 0$. Clearly $t_m \to 0$, and has the desired properties.

To prove the statement (b) we consider, for a function $\xi \not\equiv 0$, the natural decomposition $\xi = \xi^+ - \xi^-$ and call $A^+ = supp\ \xi^+,\ A^- = supp\ \xi^-,\ A_0 = [0,1] - (A^+ \cup A^-)$.

1^{st} *Case:* $\mu(A^+) = 0$ or $\mu(A^-) = 0$. In this case we show that $d_P(x) = |P(x)|^{\frac{1}{n}}$. Assume, without lost of generality, that $\mu(A^-) = 0$. Then, $\mu(A^+) \neq 0$ provided that $\xi \not\equiv 0$.

Since $P(x) = \int_{A^+} \xi^+ x^n d\mu$, considering the function $\sqrt[n]{\xi^+}$ we define, as in Proposition 2.6, $\|x\|_\xi = (\int_{A^+} (\sqrt[n]{\xi^+}x)^n d\mu)^{\frac{1}{n}} = |P(x)|^{\frac{1}{n}}$. By Minkowski's inequality we have that

$$\begin{aligned} |P(x - z_1)|^{\frac{1}{n}} + \ & |P(z_1 - z_2)|^{\frac{1}{n}} + \cdots + |P(z_k)|^{\frac{1}{n}} \\ & = \|x - z_1\|_\xi + \|z_1 - z_2\|_\xi + \ldots + \|z_k\|_\xi \geq \|x\|_\xi. \end{aligned}$$

Thus, $|P(x)|^{\frac{1}{n}} = \|x\|_\xi \leq d_P(x) \leq |P(x)|^{\frac{1}{n}}$.

2^{nd} *Case:* $\mu(A^+) > 0$ and $\mu(A^-) > 0$. We will show that $d_P \equiv 0$. Again, it is enough to show that $d_P(h) = 0$ for any $h = \chi_{[a,b]}$.

Put $p_1 = \mu(A^+ \cap [a,b])$ and $p_2 = \mu(A^- \cap [a,b])$.
If $p_1 = p_2 = 0$, then *supp* $h \subseteq A_0$, $P(h) = 0$ and so is $d_P(h)$.
If $p_1 > 0$ and $p_2 = 0$, then $\int_a^b \xi^+ d\mu > 0$. Since $\int_{A^-} \xi^- d\mu > 0$ there exists a constant $\delta > 0$ such that $\int_a^b \xi^+ d\mu = \delta \int_{A^-} \xi^- d\mu > 0$.

Let us consider $g_1 = h + \sqrt[n]{\delta}\chi_{A^-}$. P is orthogonally additive and g_1 is a sum of two support disjoint functions. Then,

$$P(g_1) = P(h) + \delta P(\chi_{A^-}) = \int_{A^+} \xi^+ h \; d\mu - \delta \int_{A^-} \xi^- d\mu = 0.$$

Analogously, $g_2 = h - \sqrt[n]{\delta}\chi_{A^-}$ is a zero of P. By Lemma 2.3 $d_P(h) = d_P(h + g_1 + g_2) = 3d_P(h)$ so $d_P(h) = 0$.
If $p_2 > 0$ and $p_1 = 0$ we proceed as above. Now it is left to prove the result for $p_1 > 0$ and $p_2 > 0$. There exists a constant $\delta > 0$ such $\int_a^b \xi^+ d\mu = \delta \int_a^b \xi^- d\mu$. Proceeding as before but considering $g_1 = h\chi_{A^+} + \sqrt[n]{\delta}h\chi_{A^-}$ and $g_2 = h\chi_{A^+} - \sqrt[n]{\delta}h\chi_{A^-}$ we obtain that $d_P(h\;\chi_{A^+}) = d_P(h\;\chi_{A^-}) = 0$. Write $h = h\;\chi_{A^+} + h\;\chi_{A^-} + h\;\chi_{A_0}$. Since $d_P(h\;\chi_{A_0}) = 0$ and d_P is a seminorm we have that $d_P(h) \leq d_P(h\;\chi_{A^+}) + d_P(h\;\chi_{A^-}) + d_P(h\;\chi_{A_0}) = 0$. This ends the proof of the theorem. $\square$

The next corollary, whose proof follows easily from the above, is a first attempt to give a characterization for the topology τ on bounded sets of spaces $L_p[0,1]$.

Corollary 3.4. Let $X = L_p[0,1],\ 1 \leq p < \infty$. Then, for any bounded net (x_α) and $x \in X$

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad \begin{cases} (i)\ x_\alpha \xrightarrow{w} x \quad \textit{and} \\ (ii)\ \int_0^1 \xi(x_\alpha - x)^n d\mu \to 0, & \text{for all } \xi \geq 0, \xi \in L_{\frac{p}{p-n}} \\ & (\xi \in L_\infty \text{ if } n = p.) \text{ and} \\ & \text{for all even integers } n \leq p. \end{cases}$$

Theorem 3.5. Let $X = L_p[0,1],\ 1 \leq p < \infty$. Let k be the largest integer verifying $2k \leq p$. Then, for any bounded net (x_α) and $x \in X$

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad \begin{cases} (i) & x_\alpha \xrightarrow{w} x \quad \textit{and} \\ (ii) & \|x_\alpha - x\|_{2k} \to 0 \end{cases}$$

Proof. Since every $\varphi \in X'$ is an orthogonally additive polynomial, then (i) holds. For the second condition, since $2k \leq p$ then, $P(x) = \int_0^1 x^{2k} d\mu$ is a well defined $2k$-homogeneous orthogonally additive polynomial on L_p. By Theorem 3.2, $\|x_\alpha - x\|_{2k} = |P(x_\alpha - x)|^{\frac{1}{2k}} = d_P(x_\alpha - x)$ which converges to zero by hypothesis.

To prove the converse we apply Corollary 3.4. So, it is enough to verify condition (ii) for every ξ continuous function, $\xi \geq 0$. Since $n \leq p$, n even integer, then $n \leq 2k$ and there exists $A > 0$ such that $\|\xi\|_\infty \leq A$. Now, an inequality similar to (1) is obtained and the result follows. □

Corollary 3.6. Let $L_p[0,1],\ 1 \leq p < 2$ or $p \in [2k, 2k+1)$ for some $k \in I\!N$. Then, for any bounded net (x_α) and $x \in X$

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad x_\alpha \xrightarrow{wp_o} x.$$

Proof. Note that, by Theorem 3.2, for any orthogonally additive polynomial of odd degree $d_p \equiv 0$ and the same happens to any one of even degree when the polynomial is given by a function ξ with non constant sign. Now, the argument follows as in the above corollary. □

Remark 3.7. Let $X = L_p[0,1]$, with $p = 2k+1$ for some $k \in I\!N$. Then, the wp_o-topology is strictly stronger than the τ-topology.

Proof. Let $(x_m)_{m \in I\!N}$ be the sequence $x_m = \sqrt[2k+1]{m}\, \chi_{[0,\frac{1}{m}]}$, that verifies $\|x_m\|_{2k+1} = 1$ and $\|x_m\|_{2k} = (\frac{1}{m^{2k+1}})^{\frac{1}{2k}} \to 0$, if $m \to \infty$, with $2k$ the largest even integer smaller than p. Also, for any continuous function ξ on $[0,1]$, $\int_0^1 \xi\, x_m d\mu \to 0$, with m, so, $x_m \xrightarrow{w} 0$. Then, by Theorem 3.5, $x_m \xrightarrow{\tau} 0$.
Now, consider the orthogonally additive polynomial on L_p, $P(x) = \int_0^1 x^{2k+1} d\mu$. Then, $x_m \overset{wp_o}{\not\to} = 0$ since $|P(x_m)| = 1$.□

Corollary 3.8. Let $X = L_p[0,1],\ 1 \leq p < \infty$. Then, for any net (x_α) and $x \in X$

(a) If $1 \leq p < 3$, then $\quad x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad x_\alpha \xrightarrow{wp_o} x.$

(b) If $p \geq 3$ and $(x_\alpha),\ x$ are uniformly bounded, then

$$x_\alpha \xrightarrow{\tau} x \quad \textit{if and only if} \quad x_\alpha \xrightarrow{wp_o} x.$$

Proof. It is clear that we only have to show the *only if* implication since $d_P(x) \leq |P(x)|^{\frac{1}{n}}$ gives the other one.
To prove statement (a) note that if $1 \leq p < 3$, an orthogonally additive polynomial P is a linear form or $P(x) = \int_0^1 \xi x^2 d\mu$, with $\xi \in L_{\frac{p}{p-2}}$. For the latest, consider the application given by $Q(x) = \int_0^1 |\xi| x^2 d\mu$. Then, Q is a well defined orthogonally additive polynomial of second degree satisfying, by (b) of Theorem 3.2, that $|P(x_\alpha - x)| \leq \int_0^1 |\xi|(x_\alpha - x)^2 d\mu = d_Q^2(x_\alpha - x)$.

For statement (b), let $P(x) = \int_0^1 \xi x^n d\mu$, with $\xi \in L_{\frac{p}{p-n}}$, or $\xi \in L_\infty$ if $p = n$. Consider $2 \leq n \leq p$ since $n = 1$ gives linear forms. Whence, the result follows.
When n is an even integer, the result follows by Corollary 3.4 since $|\xi| \geq 0$ and $|P(x_\alpha - x)| \leq \int_0^1 |\xi|(x_\alpha - x)^n d\mu$. When n is odd $(n-1)$ is even and $n - 1 \geq 2$. Suppose $\|x_\alpha\|_\infty, \|x\|_\infty \leq A$ for some $A > 0$. Then, $|P(x_\alpha - x)| \leq \int_0^1 |\xi|(x_\alpha - x)^{n-1}|x_\alpha - x| d\mu \leq 2A \int_0^1 |\xi|(x_\alpha - x)^{n-1} d\mu$ and Corollary 3.4 implies $|P(x_\alpha - x)| \to 0$, which completes the proof. $\square$

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