Error estimates for an average interpolation on anisotropic $\mathcal{Q}_1$ elements

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Joint work with Ariel Lombardi

- The classic Hardy inequality and its dual inequality.

- Their application in error estimates for functions in weighted Sobolev spaces.

- Advantages over compactness arguments.

- Example of application in 1D. Graded meshes for singularly perturbed problems.

- Error estimates for narrow elements in 2D and 3D. Necessity of Average Interpolants.

- The generalized Hardy inequality in 2D and 3D.

- Estimates for average interpolants in anisotropic rectangular elements (in 2D and 3D).

- Applications.

- Numerical examples.

We are interested in estimates with weighted norms on the right hand side.

REASONS: to approximate singular functions or functions with large derivatives.

THE USE OF HARDY INEQUALITY

THE 1D CASE:

CLASSIC HARDY INEQUALITY:

$$\left\|\frac{v}{d}\right\|_{L^2(a,b)} \leq 2\|v'\|_{L^2(a,b)}$$

$v \in H_0^1(a,b)$ $\quad$ $d(x)$ $\quad$ distance to the boundary of (a,b)

DUAL INEQUALITY:

$$u \in H^1(a,b) \quad , \quad \int_a^b u = 0$$

then,

$$\|u\|_{L^2(a,b)} \leq 2\|du'\|_{L^2(a,b)}$$

Proof : Define $v \in H_0^1(a,b)$

$$v(x) = -\int_a^x u(y)dy$$

Using the Hardy inequality for v, we have

$$\|u\|_{L^2}^2 = \int_a^b u'(x)v(x)dx$$

$$\leq \left\|\frac{v}{d}\right\|_{L^2} \|du'\|_{L^2} \leq 2\|u\|_{L^2}\|du'\|_{L^2}$$

ERROR ESTIMATE FOR THE DERIVATIVE:

$$I = (a,b) \quad u_I \in \mathcal{P}_1 \text{ Lagrange interpolation}$$

$$u \in H^2(I) \qquad 0 \le \alpha \le 1$$

$$\|(u-u_I)'\|_{L^2(I)} \le 2|I|^{1-\alpha}\|d^\alpha u''\|_{L^2(I)}$$

Proof : Use that

$$\int_I (u-u_I)' = 0$$

and the DUAL HARDY INEQUALITY.

REMARKS: Estimates of this kind can be proved by compactness arguments. However, our method has the following advantages:

1- In the n-dimensional case we obtain explicit information on the dependence of the constants on the geometry of the elements. This is important in our analysis for anisotropic elements.

2- Our argument gives better results: One can not obtain the case $\alpha = 1$ by compactness. This case is of interest in some applications.

$$H^{1,d} = \{v \in L^2(I) : dv' \in L^2(I)\}$$

with norm

$$\|u\|_{H^{1,d}} = \|u\|_{L^2}^2 + \|du'\|_{L^2}^2$$

The inclusion $H^{1,d} \subset L^2$ IS NOT COMPACT!

EXAMPLE (Ariel Lombardi): $I = (0, 1)$

Consider the sequence

$$u_n(x) = \begin{cases} nx & 0 < x < \frac{1}{n} \\ 2 - nx & \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \frac{2}{n} \leq x < 1 \end{cases}$$

and $w_n = \sqrt{n} u_n$. Then, $\|w_n\|^2_{H^{1,d}} = 10/3$. If $H^{1,d} \subset L^2$ is compact, there exists a subsequence w_n such that

$$w_n \rightarrow w \qquad \text{in} \quad L^2$$

but $w_n(x) \rightarrow 0 \quad \forall x \in I$ and so $w = 0$. But,

$$\|w_n\|^2_{L^2} = 2/3$$

CONTRADICTION!

APPLICATIONS

GRADED MESHES: AN EXAMPLE IN 1D

CONVECTION-DIFFUSION EQUATION:

$$\begin{aligned} -\varepsilon u'' - b(x)u' + c(x)u = f \quad & \text{in } (0,1) \\ u(0) = u(1) = 0 & \end{aligned}$$

$$b(x) \geq b_0 > 0 \ \forall\, x \in (0,1)$$

There is a boundary layer at $x = 0$.

GRADED MESH:

$$x_0 = 0 < x_1 < \cdots < x_N$$

u_I piecewise $\mathcal{P}_1$ Lagrange interpolation

Error estimate for the first interval $(0, x_1)$:

$$\varepsilon\|(u-u_I)'\|^2_{L^2(0,x_1)} \leq 4\varepsilon\|xu''\|^2_{L^2(0,x_1)}$$

$$\leq 4\varepsilon^{-2\beta}x_1^{2(1-\alpha)}\varepsilon^{1+2\beta}\|x^\alpha u''\|^2_{L^2(0,x_1)}$$

REMARK: We will use this estimate for $\alpha < 1$, but it is important to have a constant independent of α.

Choose:

$$\beta = 1 - \alpha = \frac{1}{\log(\frac{1}{\varepsilon})}$$

So, $\varepsilon^{-\beta} = e$ and then,

$$\varepsilon\|(u-u_I)'\|^2_{L^2(0,x_1)} \leq Cx_1^{2(1-\alpha)}\varepsilon^{1+2\beta}\|x^\alpha u''\|^2_{L^2(0,x_1)}$$

Take $h > 0$ and $x_1 \le h^{\frac{1}{1-\alpha}}$. Then,

$$\varepsilon \|(u - u_I)'\|^2_{L^2(0,x_1)} \le C h^2 \varepsilon^{1+2\beta} \|x^\alpha u''\|^2_{L^2(0,x_1)}$$

Error estimate for the other intervals (x_j, x_{j+1}):

$$\begin{aligned} \varepsilon \|(u - u_I)'\|^2_{L^2(x_j,x_{j+1})} \\ \le 4\varepsilon^{-2\beta}(x_{j+1} - x_j)^2 \varepsilon^{1+2\beta} \|u''\|^2_{L^2(x_j,x_{j+1})} \end{aligned}$$

Now choose x_j such that:

$$\boxed{x_{j+1} \le x_j + h x_j^\alpha}$$

Then,

$$\varepsilon\|(u-u_I)'\|^2_{L^2(x_j,x_{j+1})} \le Ch^2 x_j^{2\alpha}\varepsilon^{1+2\beta}\|u''\|^2_{L^2(x_j,x_{j+1})}$$

$$\le Ch^2\varepsilon^{1+2\beta}\|x^\alpha u''\|^2_{L^2(x_j,x_{j+1})}$$

WEIGHTED A PRIORI ESTIMATE:

$$\varepsilon^{1+2\beta}\|x^\alpha u''\|^2_{L^2} \le C$$

if $\alpha \ge 0,\ \beta \ge 0,\ \alpha+\beta = 1$

Consequently,

$$\boxed{\varepsilon\|(u-u_I)'\|^2_{L^2(0,1)} \le Ch^2}$$

with C independent of ε.

N: Number of nodes in graded mesh $\Longrightarrow \quad h \sim \frac{\log N}{N}$

Therefore,

$$\varepsilon \|(u - u_I)'\|^2_{L^2(0,1)} \leq C \frac{\log N}{N}$$

Similar weighted estimates, but with different powers of $d(x)$, can be proved for the L^2 interpolation error.

L^2- ERROR ESTIMATE:

$$||u - u_I||_{L^2(I)} \leq \frac{C}{1 - 2\alpha} |I|^{1-\alpha} ||d^\alpha u'||_{L^2(I)}$$

for $0 \leq \alpha < \frac{1}{2}$.

The following example shows that the estimate is not true for $\alpha > \frac{1}{2}$:

$$u_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

Then,

$$\|u_n - u_{n,I}\|_{L^2(0,1)} \to \left(\int_0^1 (1-x)^2 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

while

$$\|x^\alpha u_n'\|^2_{L^2(0,1)} = \int_0^{\frac{1}{n}} n^2 x^{2\alpha} dx = \frac{1}{2\alpha+1} n^{1-2\alpha} \to 0$$

for $\alpha > \frac{1}{2}$

Using these estimates and the weighted a priori estimate

$$\varepsilon^{2\beta}\|x^\alpha u'\|_{L^2}^2 \leq C$$

$$\text{if} \quad \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta = \frac{1}{2}$$

Choosing,

$$\beta = \frac{1}{2} - \alpha = \frac{1}{\log \frac{1}{\varepsilon}}$$

ERROR ESTIMATE IN ENERGY NORM

$$\|v\|_\varepsilon^2 = \|v\|_{L^2}^2 + \varepsilon^2\|v'\|_{L^2}^2$$

$$\|u - u_I\|_\varepsilon \leq C \log \frac{1}{\varepsilon} \frac{\log N}{N}$$

THE 2D AND 3D CASES

Classic theory uses "regularity assumption":

$$\frac{h_T}{\rho_T} \leq C$$

h_T exterior diameter, ρ_T interior diameter. For both Lagrange and Average Interpolants.

BUT: IT'S KNOWN THAT IT IS NOT NEEDED! First works: Babuska-Aziz, Jamet (1976).

Other references: Krizek, Al Shenk, Dobrowolski, Apel, Nicaise, Formaggia, Perotto, Acosta, D., etc..

FOR EXAMPLE: RECTANGULAR ELEMENTS

K reference element

Given $u \in H^2(K)$, let $p \in \mathcal{P}_1$ be such that

$$\left\| \frac{\partial}{\partial x}(u - p) \right\|_{L^2(K)} \leq C \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)}$$

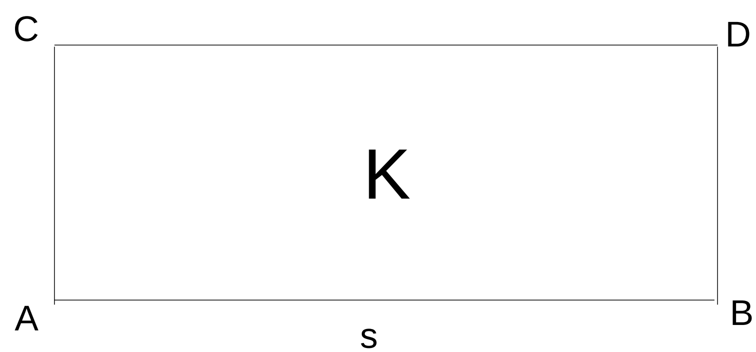
For example: p_1 the averaged Taylor polynomial of degree 1.

Let $u_I \in \mathcal{Q}_1$ be the Lagrange interpolation.

$$\left\|\frac{\partial}{\partial x}(u - u_I)\right\|_{L^2(K)} \le \left\|\frac{\partial}{\partial x}(u - p)\right\|_{L^2(K)} + \left\|\frac{\partial}{\partial x}(p - u_I)\right\|_{L^2(K)}$$

So, it is enough to estimate $\left\|\frac{\partial}{\partial x}(p - u_I)\right\|_{L^2(K)}$

We use: for $\quad v = p - u_I \in \mathcal{Q}_1(K)$



$$\left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(K)} \sim \quad |v(B) - v(A)|^2 + |v(D) - v(C)|^2$$

$$|v(B) - v(A)| = |(p(B) - u(B)) - (p(A) - u(A))|$$

$$= \left| \int_s \frac{\partial}{\partial x}(p - u) \right| \leq C \left\{ \left\| \frac{\partial}{\partial x}(p - u) \right\|_{L^2(K)} + \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)} \right\}$$

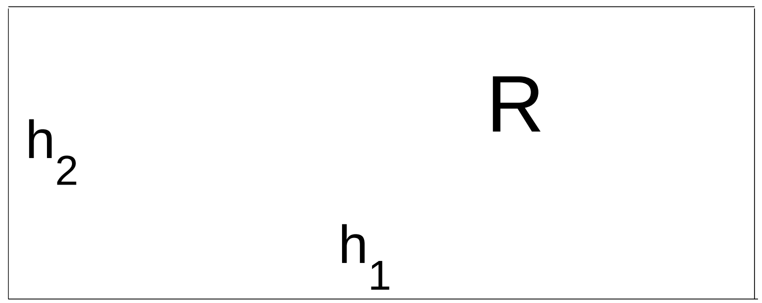
where we have used a trace theorem.

Analogously we bound $|v(D) - v(C)|$ and so we obtain:

$$\left\| \frac{\partial}{\partial x}(u - u_I) \right\|_{L^2(K)} \leq C \left\| \nabla \frac{\partial u}{\partial x} \right\|_{L^2(K)}$$

$\dfrac{\partial^2 u}{\partial y^2}$ DOES NOT APPEAR!

Therefore, changing variables we obtain for a rectangle R:



$$\left\| \frac{\partial}{\partial x}(u - u_I) \right\|_{L^2(R)} \leq C \left\{ h_1 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + h_2 \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} \right\}$$

THE CONSTANT C IS INDEPENDENT OF THE RELATION BETWEEN h_1 and h_2 !

A SIMILAR ESTIMATE IN 3D IS NOT TRUE !!

WHAT FAILS IN 3D IN THE ARGUMENT GIVEN ABOVE?

THE TRACE THEOREM:

$$\|u\|_{L^2(s)} \leq C\|u\|_{H^1(R)},$$

WHERE s IS AN EDGE OF R, IS NOT TRUE!

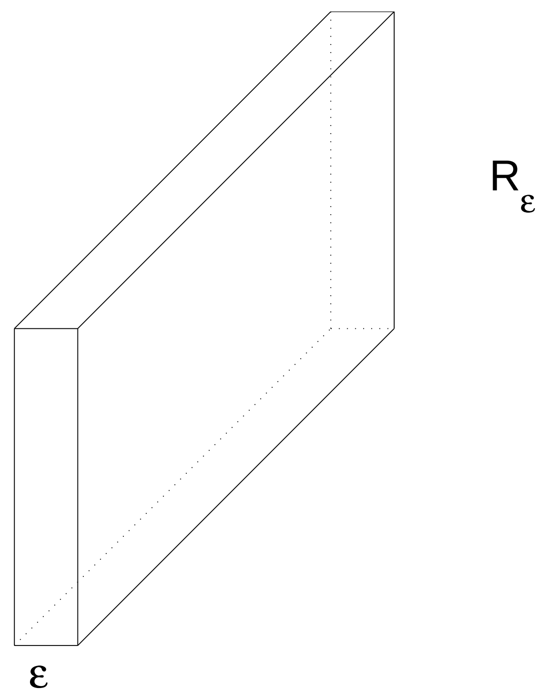
Counterexamples for the interpolation error estimate were given by:

Apel-Dobrowolski (Computing 1992), Al Shenk (Math. Comp. 1994).

They showed that the constant in the estimate

$$\|u - u_I\|_{H^1(R_\varepsilon)} \leq C_\varepsilon h\|u\|_{H^2(R_\epsilon)}$$

goes to ∞ when $\varepsilon \to 0$



THIS IS ONE REASON TO WORK WITH AVERAGE INTERPOLANTS.

The other reason is the classic one: to approximate non-smooth functions.

GENERALIZED HARDY INEQUALITY:

$D \subset \mathbb{R}^n$ convex domain, $u \in H_0^1(D)$

$d(x)$ distance to the boundary

$$\left\|\frac{u}{d}\right\|_{L^2(D)} \leq 2\|\nabla u\|_{L^2(D)}$$

ANISOTROPIC VERSION

$R = \prod_{i=1}^n (a_i, b_i) \qquad h_i = b_i - a_i$

$u \in H_0^1(R)$, δ_R is a "normalized distance":

$$\delta_R(x) = \min\left\{\frac{x_i - a_i}{h_{R,i}}, \frac{b_i - x_i}{h_{R,i}} : 1 \leq i \leq n\right\}$$

$$\left\|\frac{u}{\delta_R}\right\|_{L^2(R)} \leq 2\sum_{i=1}^{n} h_i \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(R)} .$$

DUAL INEQUALITY

$\frac{1}{\delta} \leq h_i \leq \delta$, $\quad \psi \in C_0(R)$, $\quad \int_R \psi = 1$.

$u \in H^1(R)$ such that $\int_R u\psi = 0$

$$\|u\|_{L^2(R)} \leq C\|d\nabla u\|_{L^2(R)}$$

with C depending only on δ and ψ.

Proof : REPEAT THE ARGUMENT GIVEN IN 1D:

$v := u - (\int_R u)\psi$ has vanishing mean value.

So, there exists $F \in H_0^1(R)^2$ such that

$$-\text{div}\, F = v$$

and

$$\|F\|_{H_0^1(R)^2} \leq C\|v\|_{L^2(R)}$$

C DEPENDS ONLY ON δ : It follows from the explicit bound given in DM.

Since $\int_R u\psi = 0$, then

$$\|u\|_{L^2(R)}^2 = \int_R uv = -\int_R u\, \text{div}\, F$$

and the proof finish as in the 1D case.

AN AVERAGE INTERPOLANT

Our construction is a modification of that in D. (Math. Comp. 1999).

DIFFERENCE: We do not use reference elements for the definition!

In this way we can relax the regularity assumptions on the mesh.

ASSUMPTION: local regularity in each direction

R, S neighboring elements

$$\frac{h_{R,i}}{h_{S,i}} \leq \sigma \qquad 1 \leq i \leq n.$$

OUR ERROR ESTIMATES DEPEND ONLY ON σ.

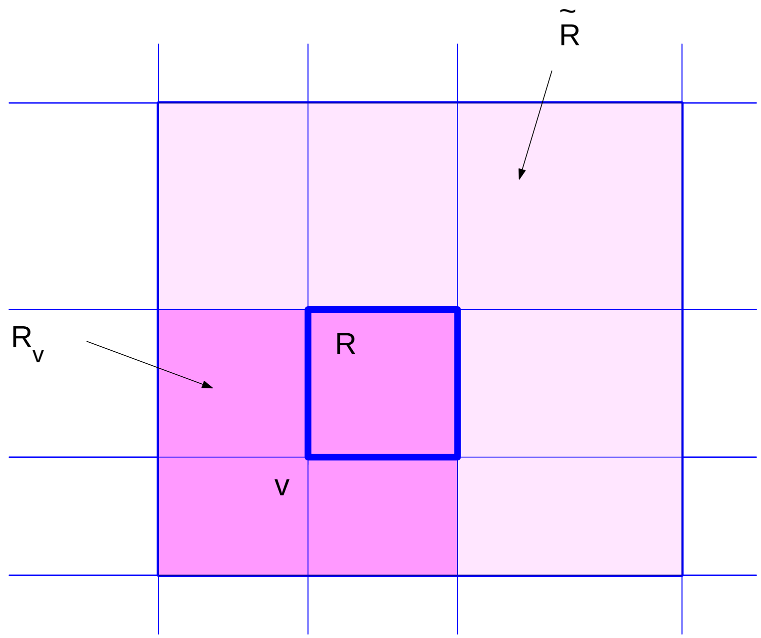
$\mathcal{N}_{in}$ set of interior nodes. For $z \in \mathcal{N}_{in}$:

$$h_{z,i} = \min\{h_{R,i} : z \text{ is a vertex of } R\}, \qquad 1 \le i \le n$$

$$\tilde{R} = \bigcup\{S \in \mathcal{T} : S \text{ is a neighboring element of } R\}.$$

and

$$R_v = \bigcup\{S \in \mathcal{T} : v \text{ is a vertex of } S\}.$$



$Tu(x,y)$ Taylor polynomial of u of degree 1 at the point x:

$$Tu(x,y) = u(x) + \nabla u(x) \cdot (y - x)$$

<u>AVERAGED TAYLOR POLYNOMIAL</u>

$$\psi \in C^\infty(\mathbb{R}^n) \qquad \int \psi = 1$$

$$\operatorname{supp} \psi \subset B(0,r) \qquad r \leq 1/\sigma$$

For $z \in \mathcal{N}_{in}$ define:

$$\psi_z(x) = \frac{1}{h_{z,1}h_{z,2}h_{z,3}} \psi \left(\frac{z_1 - x_1}{h_{z,1}}, \frac{z_2 - x_2}{h_{z,2}}, \frac{z_3 - x_3}{h_{z,3}} \right)$$

We introduce the averaged Taylor polynomial of order 1 of u at $z \in \mathcal{N}_{in}$:

$$T_{1,z}(u)(y) = \int Tu(x,y)\psi_z(x)dx$$

Analogously, we introduce the average of u at $z \in \mathcal{N}_{in}$:

$$T_{0,z}(u) = \int u(x)\psi_z(x)dx$$

INTERPOLANT:

For $u \in H_0^1(\Omega)$ define Πu as the unique piecewise $\mathcal{Q}_1$ function such that, for $z \in \mathcal{N}_{in}$,

$$\Pi u(z) = T_{1,z}(u)(z)$$

$\Pi u(z) = 0$ for boundary nodes z.

$$\Pi u(x) = \sum_{z \in \mathcal{N}_{in}} T_{1,z}(u)(z)\lambda_z(x)$$

λ_z: standard basis functions

ERROR ESTIMATES IN WEIGHTED NORMS

L^2- ERROR ESTIMATES:

THEOREM: If R is an interior element,

$$\|u - \Pi u\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\| \delta_{\tilde{R}} \frac{\partial u}{\partial x_i} \right\|_{L^2(\tilde{R})}$$

$$C = C(\sigma, \psi)$$

Proof : First we prove stability:

$$\|\Pi u\|_{L^2(R)} \leq C \|u\|_{L^2(\tilde{R})}$$

then, for z_1 a vertex of R,

$$\begin{aligned}\|u - \Pi u\|_{L^2(R)} &\leq \|u - T_{0,z_1}(u)\|_{L^2(R)} \\ &+ \|\Pi(T_{0,z_1}(u) - u)\|_{L^2(R)} \\ &\leq C\|u - T_{0,z_1}(u)\|_{L^2(R)}\end{aligned}$$

and use the dual Hardy inequality. □

H^1- ERROR ESTIMATES:

THEOREM: If R is an interior element,

$$\left\|\frac{\partial}{\partial x_j}(u - \Pi u)\right\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\|\delta_{\tilde{R}} \frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^2(\tilde{R})}$$

Proof : VERY TECHNICAL!

IDEA: Decompose the error in two parts (as in the proof of the estimate for Lagrange interpolation)

$$u - \Pi u = \Big(u - T_{1,z_1}(u)\Big) + \Big(T_{1,z_1}(u) - \Pi u\Big)$$

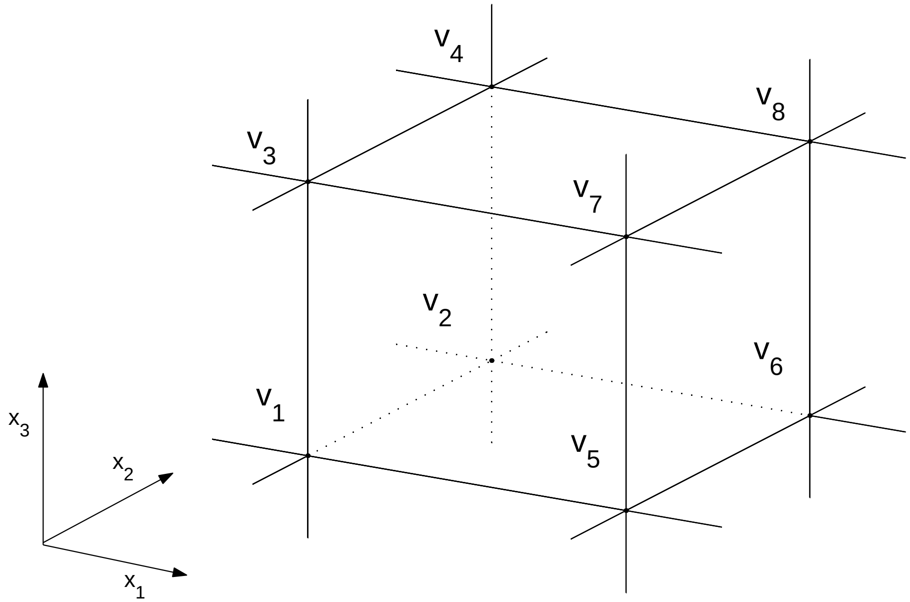
First term:

$$\left\| \frac{\partial(u - T_{1,z_1}(u))}{\partial x_1} \right\|_{L^2(R)}$$

and use the dual Hardy inequality.

Second term: $w := T_{1,z_1}(u) - \Pi u \in \mathcal{Q}_1$ then

$$\frac{\partial w}{\partial x_1} = \sum_{i=1}^{4} \Big(w(z_i) - w(z_{i+4})\Big) \frac{\partial \lambda_{z_i}}{\partial x_1}$$



So,

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{L^2(R)} \leq \sum_{i=1}^{4} |w(z_i) - w(z_{i+4})| \left\| \frac{\partial \lambda_{z_i}}{\partial x_1} \right\|_{L^2(R)}$$

But,

$$\left\| \frac{\partial \lambda_{z_i}}{\partial x_1} \right\|_{L^2(R)} \leq C \left(\frac{h_{z_i,2} h_{z_i,3}}{h_{z_i,1}} \right)^{\frac{1}{2}}$$

So, we have to estimate $|w(z_i) - w(z_{i+4})|$

For example,

$$w(z_1) - w(z_5) = T_{1,z_5}(u)(z_5) - T_{1,z_1}(u)(z_5)$$

$$= \int Tu(x, z_5)\psi_{z_5}(x)dx - \int Tu(x, z_5)\psi_{z_1}(x)dx$$

Which, after long technical details! can be bounded by

$$C\frac{1}{h_{z_1,2}h_{z_1,3}}\sum_{i=1}^{3} h_{z_1,i}\int \left|\frac{\partial^2 u}{\partial x_1 \partial x_i}(\overline{x})\right| \psi(\overline{x})d\overline{x}$$

where the function $\psi(\overline{x})$ has support in $\tilde{R}$. Then, use the Cauchy-Schwarz inequality and the Hardy inequality for $\psi(\overline{x})$. $\square$

APPLICATIONS

REACTION-DIFFUSION EQUATION:

$$-\varepsilon^2 \Delta u + u = f \qquad \text{in} \quad \Omega = (0,1)^2$$
$$u = 0 \qquad \text{on} \quad \partial\Omega$$

$$\|v\|_\varepsilon^2 = \|v\|_{L^2}^2 + \varepsilon^2 \|\nabla v\|_{L^2}^2$$

$$\|u - u_h\|_\varepsilon \le \|u - \Pi u\|_\varepsilon$$

THEOREM: With graded mesh:

$$\|u - u_h\|_\varepsilon \le C \, \frac{\log N}{\sqrt{N}}$$

CONVECTION-DIFFUSION EQUATION:

$$\begin{aligned} -\varepsilon\Delta u - u_x - u_y &= 1 \qquad \text{in } \Omega = (0,1)^2 \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

For graded meshes it follows from our weigthed error estimates:

$$\|u - \Pi u\|_\varepsilon \le C \log \frac{1}{\varepsilon} \frac{\log N}{\sqrt{N}}$$

However, in this case we don't have:

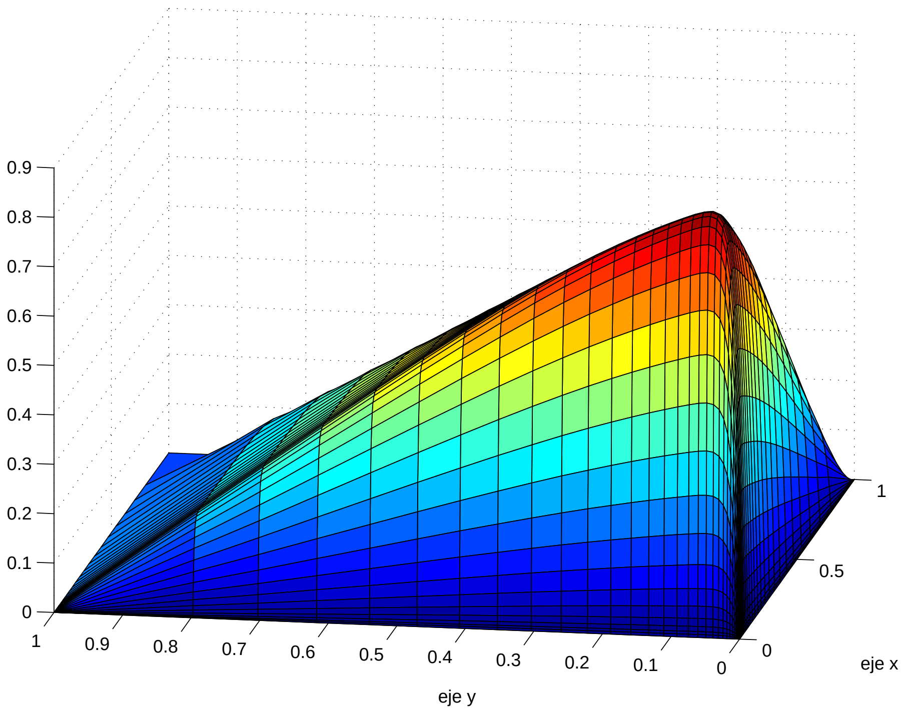
$$\|u - u_h\|_\varepsilon \le C\|u - \Pi u\|_\varepsilon$$

So

$$\|u - u_h\|_\varepsilon \quad ? \qquad \text{WE DON'T KNOW!}$$

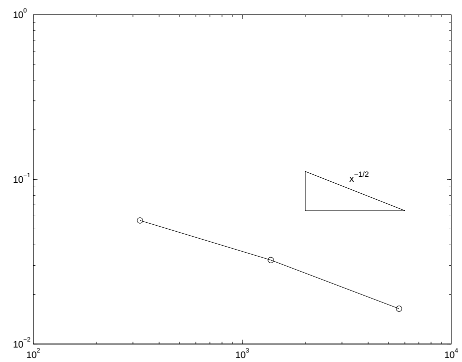
PRELIMINARY NUMERICAL EXPERIMENTS SHOW GOOD RESULTS!

Numerical Solution with graded mesh



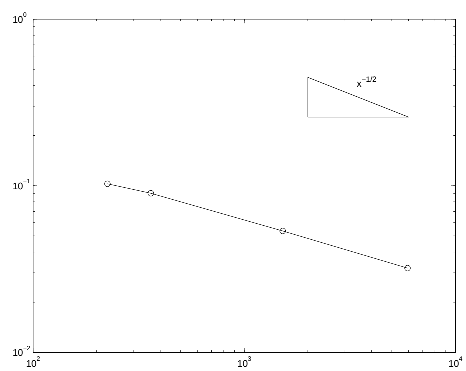
$$\varepsilon = 0.01$$

Order of convergence with graded mesh and $\mathcal{Q}_1$ elements:



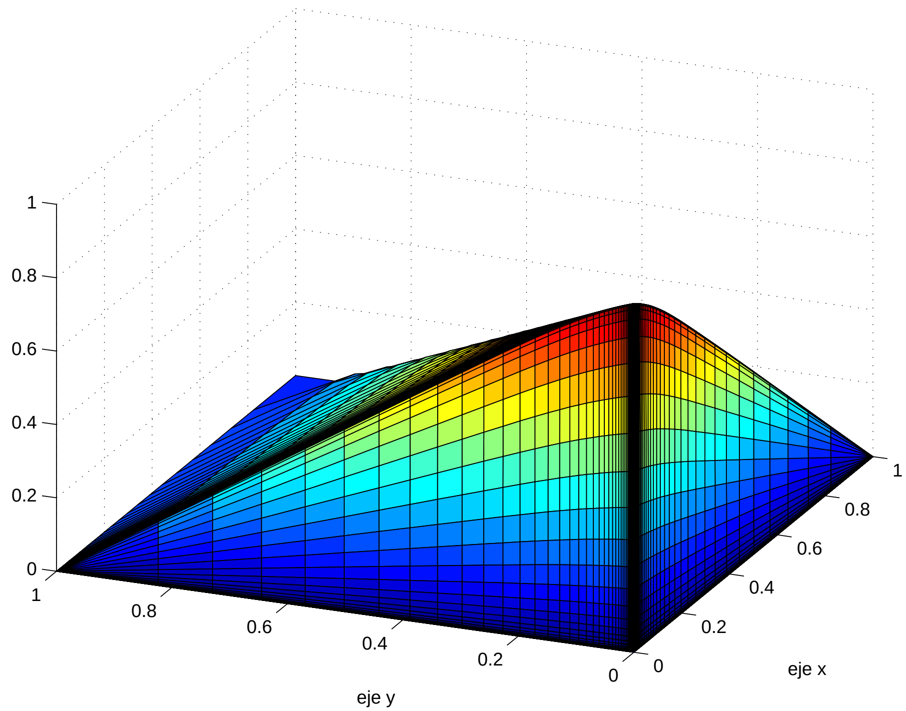
$$\varepsilon = 0.01$$

Order of convergence with Shishkin mesh and $\mathcal{Q}_1$ elements:



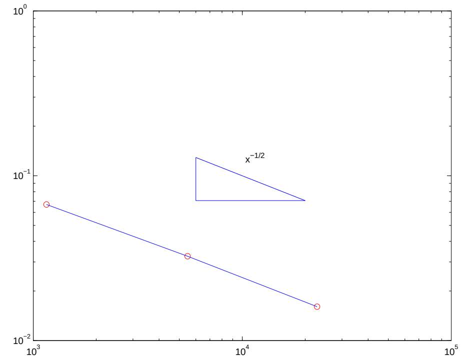
$$\varepsilon = 0.01$$

Numerical Solution with graded mesh



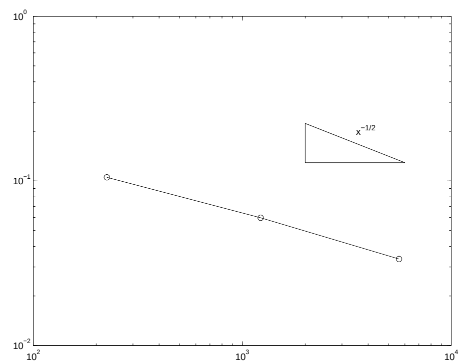
$$\varepsilon = 0.0001$$

Order of convergence with graded mesh and $\mathcal{Q}_1$ elements:



$$\varepsilon = 0.0001$$

Order of convergence with Shishkin mesh and $\mathcal{Q}_1$ elements:



$$\varepsilon = 0.0001$$

ORDER OF CONVERGENCE

Graded Meshes, $\varepsilon = 0.01$

# Nodes	Est. Error	Est. Order
324	0.056315	
1369	0.032338	0.38
5625	0.016383	0.48

Shishkin Meshes, $\varepsilon = 0.01$

# Nodes	Est. Error	Est. Order
225	0.102680	
361	0.090082	0.27
1521	0.053507	0.36
5929	0.032026	0.37

ORDER OF CONVERGENCE

Graded Meshes, $\varepsilon = 0.0001$

# Nodes	Est. Error	Est. Order
1156	0.066923	
5476	0.032484	0.46
22801	0.016070	0.49

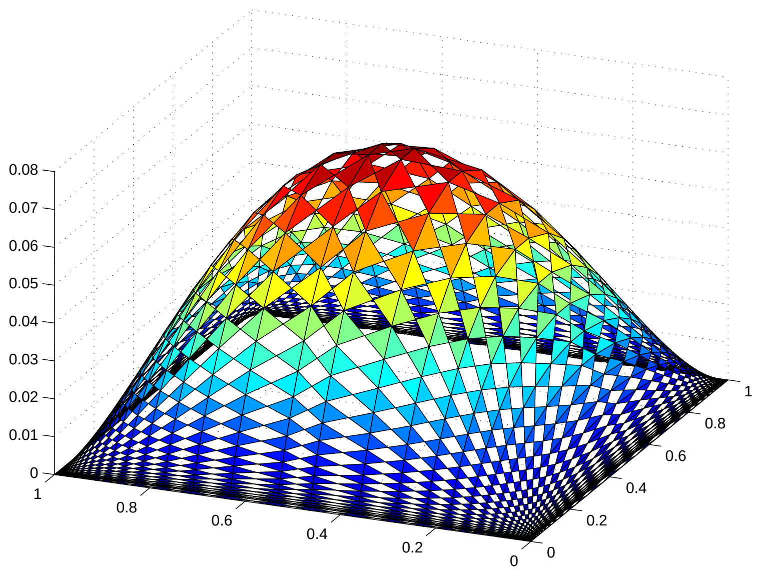
Shishkin Meshes, $\varepsilon = 0.0001$

# Nodes	Est. Error	Est. Order
225	0.105182	
1225	0.059729	0.33
5625	0.033547	0.37

FOURTH ORDER MODEL EQUATION

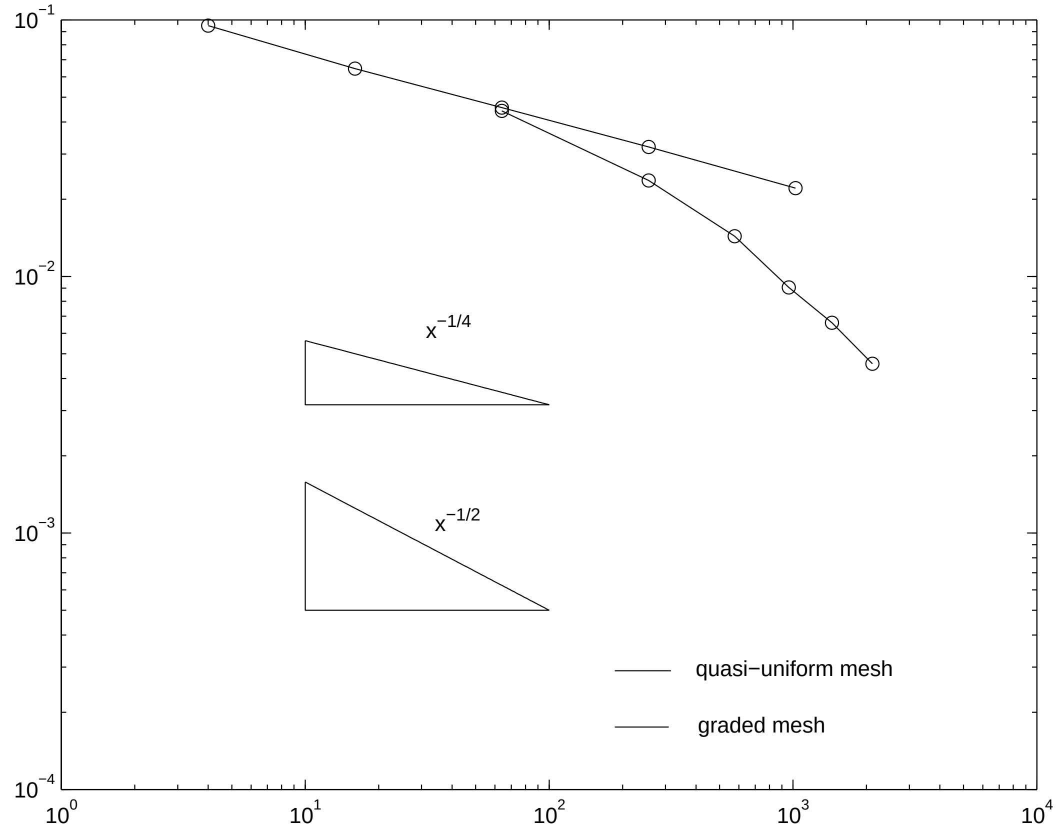
$$\begin{aligned} -\varepsilon^2\Delta^2 u + \Delta u &= 1 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Numerical solution with Adini's element:



$\varepsilon = 0.001$

Order of convergence with uniform and graded meshes:



$\varepsilon = 0.001$

For uniform meshes:

$$\|u - u_h\|_\varepsilon \leq \frac{C}{\sqrt[4]{N}}$$

this can be proved (but not yet written!).

Expected order for graded meshes:

$$\|u - u_h\|_\varepsilon \leq \frac{C}{\sqrt{N}}$$

BUT: NO THEORY FOR ANISOTROPIC ELEMENTS!

WE ARE WORKING ON THAT!