Error estimates for anisotropic finite elements and applications

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OUTLINE OF THE TALK

- Introduction to FEM
- Basic error analysis and examples
- The regularity hypothesis on the elements
- Necessity of relaxing the regularity hypothesis
- Error estimates for the Lagrange interpolation

- Differences between 2D and 3D cases

- Necessity of other interpolations

- An average interpolation

- Results for mixed finite element and non-conforming methods

- Application to the Stokes equations

- Application to problems with boundary layers

FINITE ELEMENT METHOD

GENERAL SETTING: V Hilbert space

$$B(u, v) = F(v) \qquad \forall v \in V$$

B continuous bilinear form, F continuous linear form.

APPROXIMATE SOLUTION:

V_h finite dimensional space , $\quad u_h \in V_h$

$$B(u_h, v) = F(v) \qquad \forall v \in V_h$$

ERROR ESTIMATES IN FINITE ELEMENT APPROXIMATIONS

They can be divided in two classes

- A PRIORI ESTIMATES

- A POSTERIORI ESTIMATES

GOALS OF A PRIORI ESTIMATES

- To prove convergence and to know the order of the error

- To know the dependence of the error on different things (geometry of the mesh, regularity of the solution, degree of the approximation)

A typical *a priori* error estimate is of the form

$$\|u - u_h\| \leq Ch^{\alpha}|||u|||$$

where h is a mesh size parameter.

A BASIC QUESTION IS:

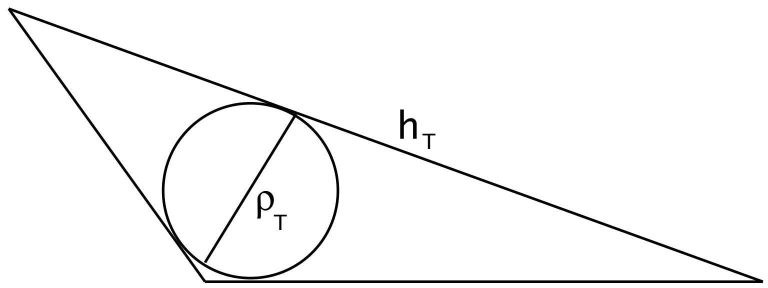
WHAT KIND OF ELEMENTS ARE ALLOWED?

or, in other words,

HOW DOES THE ERROR DEPEND ON THE GEOMETRY OF THE ELEMENTS?

The classic theory is based in the so-called

"REGULARITY ASSUMPTION"



$$\frac{h_T}{\rho_T} \le \sigma$$

h_T exterior diameter, ρ_T interior diameter

The constant in the error estimates depends on the regularity parameter σ

The advantages of the arguments based on this hypothesis are:

- It allows for very general results on error estimates for approximations of different kinds

- It implies the so called *inverse estimates* which simplify many arguments

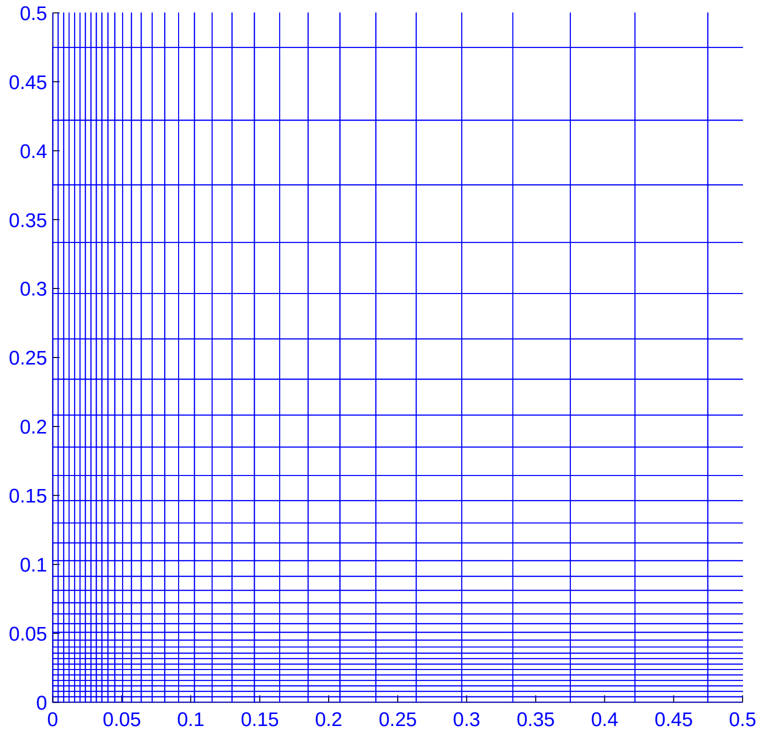
See for example the books by Ciarlet and Brenner-Scott

HOWEVER,

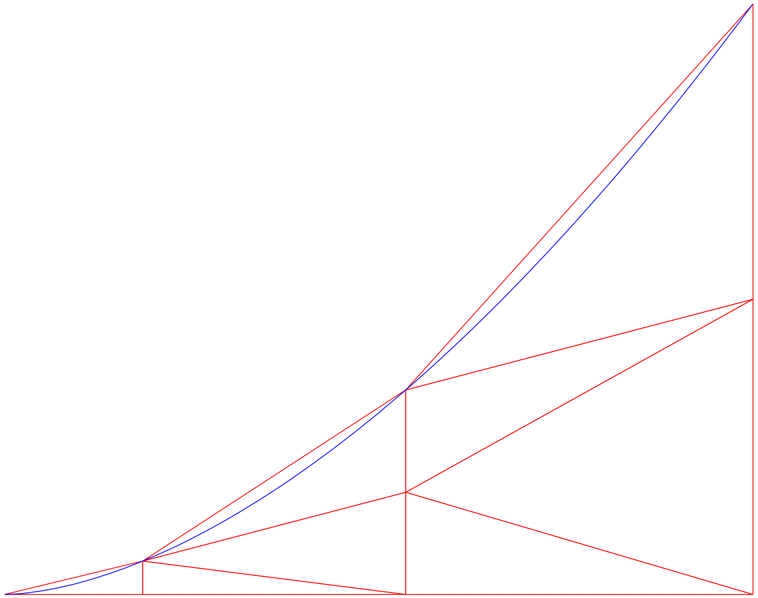
In many applications it is essential to remove the regularity hypothesis on the elements and to use

ANISOTROPIC OR FLAT ELEMENTS

EXAMPLE 1: PROBLEMS WITH BOUNDARY LAYERS



EXAMPLE 2: CUSPIDAL DOMAINS



The constants in error estimates depend on:

- CONSTANTS IN INTERPOLATION OR BEST APPROXIMATION ERROR

- STABILITY CONSTANTS

- BOUNDS OF CONSISTENCY TERMS IN NON-CONFORMING METHODS

In standard analysis the regularity hypothesis is used for all these steps

CASE 1:

COERCIVE FORMS AND CONFORMING METHODS

$$V_h \subset V$$

If

$$B(v,v) \geq \alpha \|v\|^2 \quad \forall v \in V$$

then

$$\|u - u_h\| \leq C \inf_{v \in V_h} \|u - v\|$$

The computed approximate solution is, up to a constant, like the best approximation.

CLASSIC EXAMPLES

Scalar second order elliptic equations:

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) = f & \text{in} \quad \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq M|\xi|^2 \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^n$$

$$V = H_0^1(\Omega)$$

The linear elasticity equations:

$$\begin{cases} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} = \mathbf{f} & \text{in} \quad \Omega \subset \mathbb{R}^n \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$B(\mathbf{u},\mathbf{v}) = \int_\Omega \{2\mu\varepsilon_{i,j}(\mathbf{u})\varepsilon_{i,j}(\mathbf{v}) + \lambda\mathrm{div}\mathbf{u}\,\mathrm{div}\mathbf{v}\}\,dx$$

where

$$\varepsilon_{i,j}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

$$V = H_0^1(\Omega)^n$$

CASE 2:

NON COERCIVE FORMS SATISFYING AN INF-SUP CONDITION AND CONFORMING METHODS

$$\inf_{u\in V_h} \sup_{v\in V_h} \frac{B(u,v)}{\|u\|\|v\|} \geq \alpha > 0$$

In this case we also have

$$\|u-u_h\| \leq C \inf_{v\in V_h} \|u-v\|$$

CLASSIC EXAMPLES

1-Mixed formulation of second order elliptic problems

$$\begin{cases} \mathrm{div}(a(x)\nabla p) = f & \text{in} \quad \Omega \subset \mathbb{R}^n \\ p = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$\begin{cases} \mathbf{u} = & -a(x)\nabla p \quad \text{in} \quad \Omega \\ \mathrm{div}\mathbf{u} = & f \\ p = & 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

$$B((\mathbf{u},p),(\mathbf{v},q)) := \int_\Omega a(x)^{-1}\mathbf{u}\cdot\mathbf{v} + \int_\Omega p\,\mathrm{div}\,\mathbf{v} + \int_\Omega q\,\mathrm{div}\,\mathbf{u}$$

$$V = H(\mathrm{div},\Omega)^n \times L^2(\Omega)$$

2-The Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \subset \mathbb{R}^n \\ \operatorname{div}\mathbf{u} = 0 & \text{in} \quad \Omega \subset \mathbb{R}^n \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = F(v)$$

$$B((\mathbf{u}, p), (\mathbf{v}, q)) := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p \operatorname{div}\mathbf{v} - \int_\Omega q \operatorname{div}\mathbf{u}$$

$$V = H_0^1(\Omega)^n \times L_0^2(\Omega)$$

CASE 3:

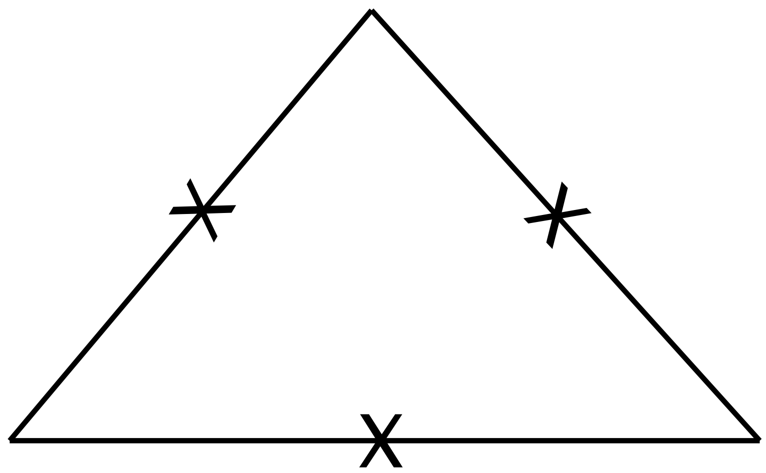
STABLE FORMS BUT NON-CONFORMING METHODS

$$V_h \not\subset V$$

STRANG'S LEMMA:

$$\|u - u_h\| \leq C \left\{ \inf_{v \in V_h} \|u - v\| + \sup_{w \in V_h} \frac{|B_h(u, w) - F(w)|}{\|w\|} \right\}$$

CLASSIC EXAMPLE Crouzeix-Raviart linear *non-conforming method*



For the Poisson equation:

$$B_h(u,v) = \sum_K \int_K \nabla u \cdot \nabla v$$

The arguments used in the original paper of CR use the regularity assumption on the elements.

MAIN TOOLS TO PROVE THE INF-SUP

1- Brezzi's theory for mixed methods

For example, for the Stokes problem

$$\mathbf{u}_h \in U_h \qquad p_h \in Q_h$$

it is enough to prove

$$\inf_{p \in Q_h} \sup_{\mathbf{v} \in U_h} \frac{\int_\Omega p \, \mathrm{div}\, \mathbf{v}}{\|p\| \|v\|} \geq \alpha > 0$$

or equivalently, the existence of the Fortin operator

$$\Pi_h : H_0^1(\Omega)^n \longrightarrow U_h$$

such that

$$\int_\Omega \mathrm{div}\,(\mathbf{u} - \Pi_h \mathbf{u})\, q = 0 \qquad \forall q \in Q_h$$

and

$$\|\Pi_h \mathbf{u}\|_{H_0^1} \leq C \|\mathbf{u}\|_{H_0^1}$$

Again, many of the arguments to obtain this result make use of the regularity of the elements.

LAGRANGE INTERPOLATION

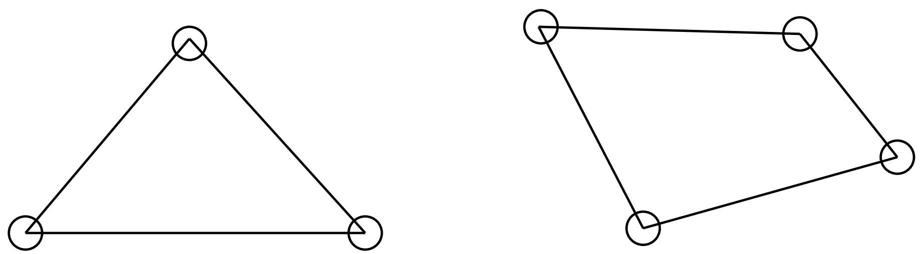
Consider the lowest order case:

K triangle , $\mathcal{P}_1$ interpolation

or

K quadrilateral , $\mathcal{Q}_1$ isoparametric interpolation

$$u_I(P_i) = u(P_i) \qquad P_i \text{ nodes}$$

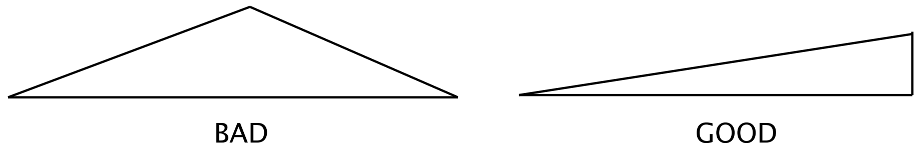


THE REGULARITY HYPOTHESIS CAN BE REPLACED BY WEAKER ASSUMPTIONS!

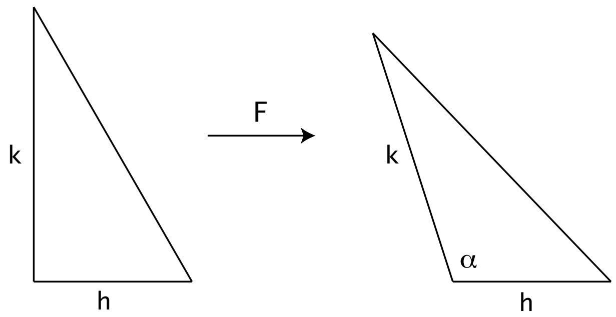
IN THE CASE OF TRIANGLES IT CAN BE REPLACED BY THE “MAXIMUM ANGLE CONDITION”

First results: Babuska-Aziz, Jamet (1976)

Other references: Krizek, Al Shenk, Dobrowolski, Apel, Nicaise, Formaggia, Perotto, Acosta, Lombardi, Durán, etc..

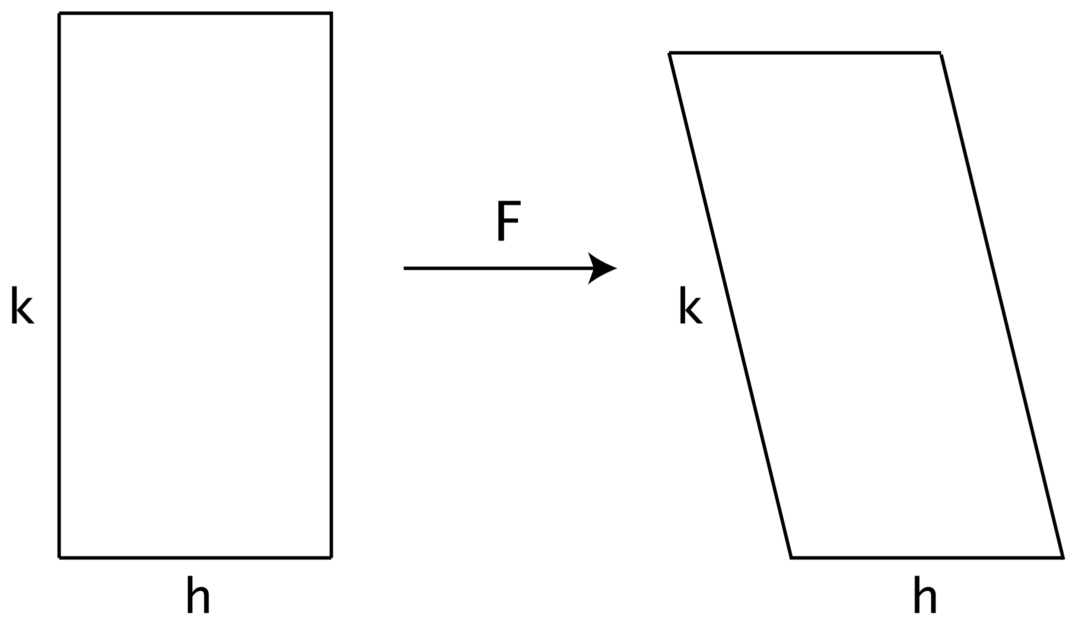


IDEA: WORK WITH AN APPROPRIATE REFERENCE FAMILY INSTEAD OF A FIXED REFERENCE ELEMENT

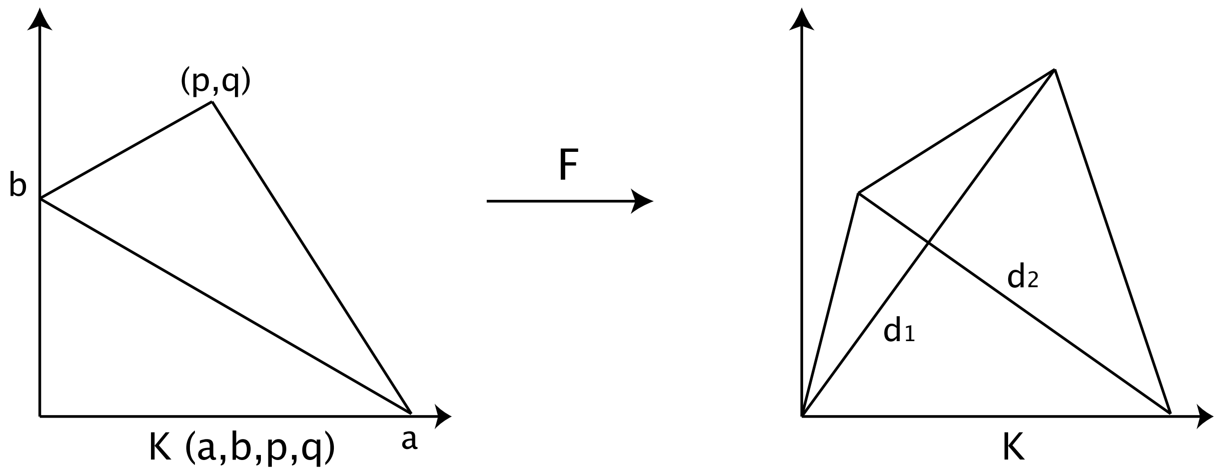


$$F : \tilde{T} \longrightarrow T$$

$$F(\tilde{x}) = B\tilde{x} + a \qquad B \in \mathbb{R}^{n\times n} \quad a \in \mathbb{R}^n$$



$$F(\tilde{x}) = B\tilde{x} + a \qquad B \in \mathbb{R}^{n \times n} \quad a \in \mathbb{R}^n$$



$$F(\tilde{x}) = B\tilde{x} + a \qquad B \in \mathbb{R}^{n \times n} \quad a \in \mathbb{R}^n$$

THE $\mathcal{P}_1$ CASE

Let $\widehat{T}$ be the triangle with vertices at $(0,0)$, $(0,1)$ and $(1,0)$

Poincaré type inequality: if $\widehat{\ell}$ is an edge of $\widehat{T}$ then

$$\int_{\widehat{\ell}} v = 0 \Longrightarrow \|v\|_{L^2(\widehat{T})} \leq C\|\nabla v\|_{L^2(\widehat{T})}$$

It follows from:

Standard Poincaré inequality:

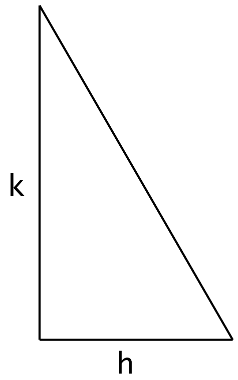
$$\int_{\widehat{T}} v = 0 \Longrightarrow \|v\|_{L^2(\widehat{T})} \leq C\|\nabla v\|_{L^2(\widehat{T})}$$

and

Trace theorem:

$$\|v\|_{L^2(\hat{\ell})} \leq C\|v\|_{H^1(\hat{T})}$$

Changing variables: $\tilde{x} = h\hat{x}$ and $\tilde{y} = k\hat{y}$ we have



$$\int_\ell v = 0 \Longrightarrow \|v\|_{L^2(\tilde{T})} \leq C\left\{h\left\|\frac{\partial v}{\partial x}\right\|_{L^2(\tilde{T})} + k\left\|\frac{\partial v}{\partial y}\right\|_{L^2(\tilde{T})}\right\}$$

but, if $\ell = \{0 \leq x \leq h, y = 0\}$, we have

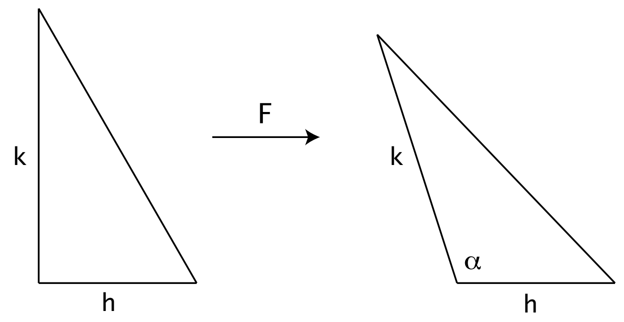
$$\int_\ell \frac{\partial}{\partial x}(u - u_I) = 0$$

and then

$$\left\| \frac{\partial}{\partial x}(u - u_I) \right\|_{L^2(\tilde{T})} \leq C \left\{ h \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(\tilde{T})} + k \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(\tilde{T})} \right\}$$

THE CONSTANT C IS INDEPENDENT OF h and k !

Now, for a general triangle T



$$F : \tilde{T} \longrightarrow T$$

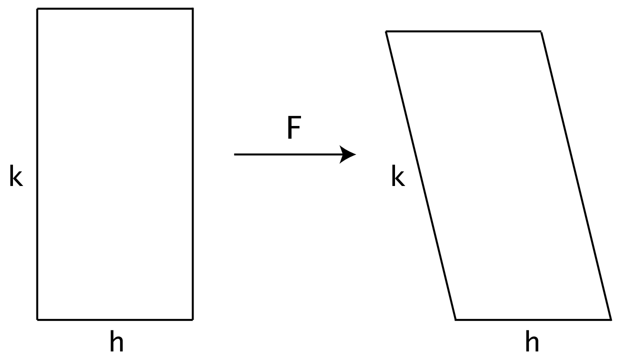
$$F(\tilde{x}) = B\tilde{x} + a \qquad B \in \mathbb{R}^{n\times n} \quad a \in \mathbb{R}^n$$

$$\|B\| \leq C \qquad \|B^{-1}\| \leq \frac{C}{\sin\,\alpha}$$

Then

$$\|\nabla(u - u_I)\|_{L^2(T)} \leq \frac{C}{\sin\,\alpha} h_T \|D^2 u\|_{L^2(T)}$$

THE CASE $\mathcal{Q}_1$ ON PARALLELOGRAMS



As in the case of triangles we obtain

$$\left\|\frac{\partial}{\partial x}(u-u_I)\right\|_{L^2(R)} \leq C\left\{h\left\|\frac{\partial^2(u-u_I)}{\partial x^2}\right\|_{L^2(R)} + k\left\|\frac{\partial^2(u-u_I)}{\partial x\partial y}\right\|_{L^2(R)}\right\}$$

$$\frac{\partial^2 u_I}{\partial x^2} = 0 \qquad \text{but} \qquad \frac{\partial^2 u_I}{\partial x\partial y} \neq 0$$

However,

$$\int_R \frac{\partial^2 u_I}{\partial x \partial y} = \int_R \frac{\partial^2 u}{\partial x \partial y}$$

and so

$$\left\| \frac{\partial^2 u_I}{\partial x \partial y} \right\|_{L^2(R)} \leq \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)}$$

REMARK: The fact that $D^2 u_I \neq 0$ introduces an extra difficulty. A similar difficulty arises in the analysis of mixed methods (and as we will see, that case is more complicated)

Then

$$\left\|\frac{\partial}{\partial x}(u-u_I)\right\|_{L^2(R)} \le C\left\{h\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^2(R)} + k\left\|\frac{\partial^2 u}{\partial x \partial y}\right\|_{L^2(R)}\right\}$$

and for a general parallelogram

$$\|\nabla(u-u_I)\|_{L^2(P)} \le \frac{C}{\sin\,\alpha} h_T \|D^2 u\|_{L^2(P)}$$

THE CASE OF QUADRILATERALS IS MORE COMPLICATED

SEVERAL CONDITIONS HAVE BEEN INTRODUCED

- Ciarlet-Raviart (1972): Regularity and non degeneracy of the angles.

- Jamet (1977): Regularity.

- Zenizek-Vanmaele (1995), Apel (1998): Allows anisotropic (flat) elements but far from triangles.

The most general condition seems to be

“THE REGULAR DECOMPOSITION PROPERTY” (G. Acosta, R.Durán, SIAM J. Numer. Anal. 2000)

RDP: K convex quadrilateral. Divide it in two triangles by the diagonal d_1. Then, the constant in the error estimate depends on the ratio $|d_2|/|d_1|$ and on the maximum angle of the two triangles

In particular the maximum angle condition is a sufficient condition

REMARK: The situation is different for L^p based Sobolev norms. Recently Acosta and Monzon showed that the RDP is not sufficient to have the error estimate for $p > 3$

THE 3D CASE

ANALOGOUS ESTIMATES IN 3D ARE NOT TRUE!!

WHAT FAILS IN THE ARGUMENT?

$$\|u\|_{L^2(s)} \leq C\|u\|_{H^1(R)},$$

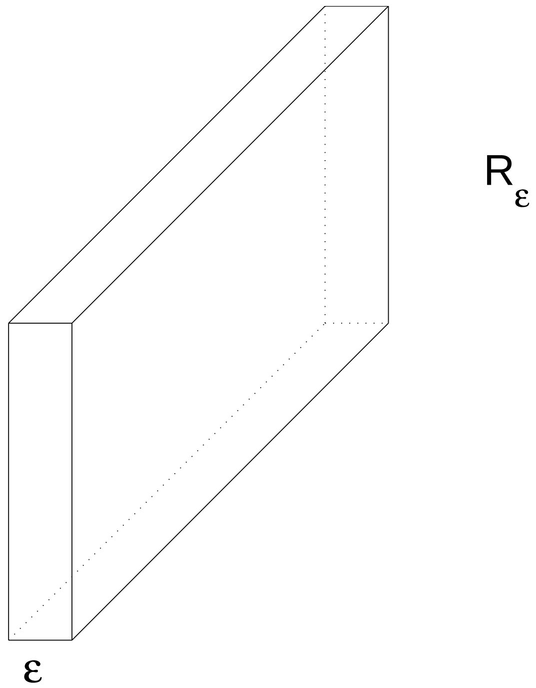
WHERE s IS AN EDGE OF R IS NOT TRUE

COUNTEREXAMPLES FOR THE INTERPOLATION ERROR ESTIMATE WERE GIVEN BY

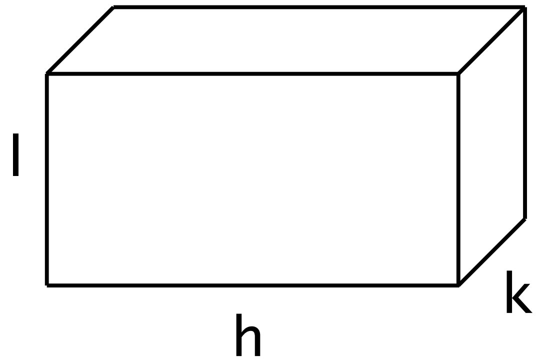
Apel-Dobrowolski (Computing 1992), Al Shenk (Math. Comp. 1994).

$$\int_{R_\varepsilon} |\nabla(u - u_I)|^2 \sim C_\varepsilon h_{R_\varepsilon}^2 \int_{R_\varepsilon} |D^2 u|^2$$

C_ε goes to ∞ when $\varepsilon \to 0$



REMARK: If the interpolated function u is slightly more regular, for example $u \in W^{2,p}$, for some $p > 2$ then an estimate analogous to those valid in the 2D case holds. For example:



$$\left\| \frac{\partial}{\partial x}(u - u_I) \right\|_{L^2(R)} \leq C_p \left\{ h \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(R)} + k \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2(R)} l \left\| \frac{\partial^2 u}{\partial x \partial z} \right\|_{L^2(R)} \right\}$$

NATURAL QUESTION: IS THERE A BETTER APPROXIMATION?

YES !!

AVERAGE INTERPOLANTS

Originally they were introduced to approximate non smooth functions for which Lagrange interpolation is not even defined (P. Clement, 1976)

Many works have been written constructing different types of average interpolants (see for example the book by Apel and its references)

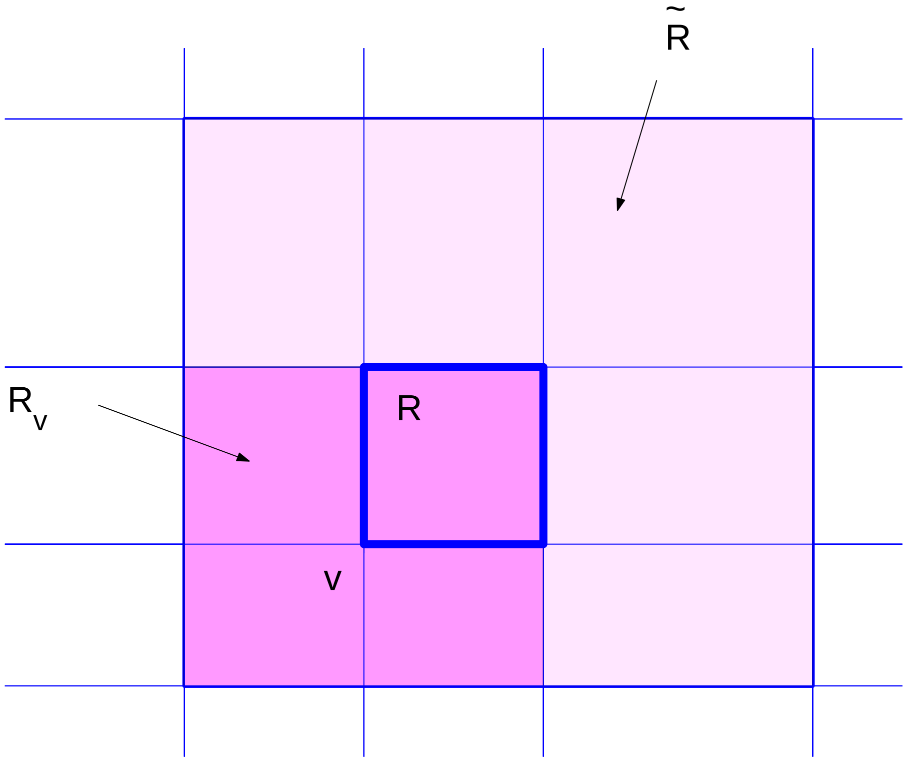
AN AVERAGE INTERPOLANT FOR RECTANGLULAR ELEMENTS (A. Lombardi- R.Durán, Math. Comp. 2005)

HYPOTHESIS

R, S neighbor elements.

$$\boxed{\frac{h_{R,i}}{h_{S,i}} \leq \sigma \qquad 1 \leq i \leq n}$$

THE CONSTANT IN THE ERROR ESTIMATE DEPENDS ONLY ON σ.



$\tilde{R}$
R_v
R
v

Consider the Taylor polynomial of degree 1 around $(\overline{x}, \overline{y})$

$$p_{\overline{x},\overline{y}}(x,y) = u(\overline{x},\overline{y}) + \frac{\partial u}{\partial x}(\overline{x},\overline{y})(x-\overline{x}) + \frac{\partial u}{\partial y}(\overline{x},\overline{y})(y-\overline{y})$$

For each node V we take an average of $p_{\overline{x},\overline{y}}(x,y)$ around V obtaining the polynomial $q(x,y)$:

$$q(x,y) = \frac{1}{|R_V|}\int_{R_V} p_{\overline{x},\overline{y}}(x,y) d\overline{x}\, d\overline{y}$$

And define the approximation Πu of u by

$$\Pi u(V) = q(V)$$

ERROR ESTIMATES

Analogous to those for the Lagrange interpolation but:

- The error on one element depends also on the values of u in neighbor elements

- Valid also in 3D

$$\left\|\frac{\partial}{\partial x_j}(u - \Pi u)\right\|_{L^2(R)} \leq C \sum_{i=1}^{n} h_{R,i} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^2(\tilde{R})}$$

The proof is very technical!

MIXED METHODS

APPROXIMATION OF SECOND ORDER ELLIPTIC PROBLEMS

The 2D case

Raviart-Thomas spaces: for $k = 0, 1, 2, \cdots$

$$\mathcal{R}T_k(T) = \mathcal{P}_k^2(T) \oplus (x, y)\mathcal{P}_k(T)$$

$$H(\mathrm{div}, \Omega) = \{\mathbf{u} \in L^2(\Omega) : \mathrm{div}\mathbf{u} \in L^2\}$$

$$\mathcal{R}T_k = \{\mathbf{u} \in H(\mathrm{div}, \Omega) : \mathbf{u}|_T \in \mathcal{R}T_k(T)\}$$

COMMUTATIVE DIAGRAM PROPERTY:

$$P_k : L^2(T) \to \mathcal{P}_k(T) \qquad RT_k : H^1(T)^2 \to \mathcal{R}\mathcal{T}_k(T)$$

$$\begin{array}{ccccc} H^1(T)^2 & \xrightarrow{\mathrm{div}} & L^2(T) & & \\ RT_k \big\downarrow & & \big\downarrow P_k & & \\ \mathcal{R}\mathcal{T}_k & \xrightarrow{\mathrm{div}} & \mathcal{P}_k(T) & \longrightarrow & 0 \end{array}$$

$$\int_T \mathrm{div}\,(\mathbf{u} - RT_k\mathbf{u})\, q = 0 \qquad \forall q \in \mathcal{P}_k(T)$$

CONSIDER THE CASE $k = 0$

From the definition of RT_0

$$\int_{\ell_i} (\mathbf{u} - RT_0\mathbf{u}) \cdot \nu_i = 0 \qquad \forall \ell_i \quad \text{edge of} \quad T$$

Then, if ℓ_1 and ℓ_2 are the edges contained in $\{x = 0\}$ and $\{y = 0\}$

for $i = 1, 2$

$$\frac{\partial (RT_0\mathbf{u})_i}{\partial x} = \frac{\partial (RT_0\mathbf{u})_i}{\partial y} = \frac{\mathrm{div} RT_0\mathbf{u}}{2}$$

But, from the commutative diagram property we have

$$\mathrm{div} RT_0\mathbf{u} = P_0 \mathrm{div}\mathbf{u}$$

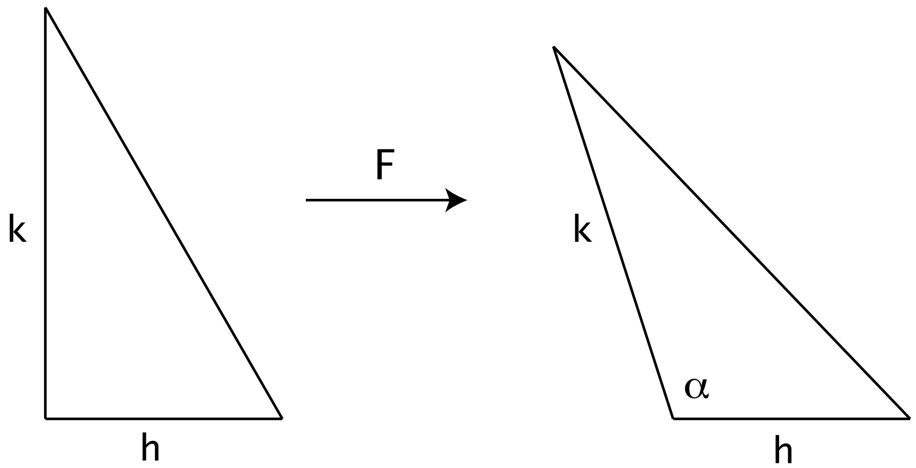
and so

$$\|\mathsf{div} RT_0 \mathbf{u}\|_{L^2(T)} \leq \|\mathsf{div}\mathbf{u}\|_{L^2(T)}$$

Then,

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq C\left\{h\left\|\frac{\partial \mathbf{u}}{\partial x}\right\|_{L^2(T)} + k\left\|\frac{\partial \mathbf{u}}{\partial y}\right\|_{L^2(T)} + (h+k)\|\mathsf{div}\mathbf{u}\|_{L^2(T)}\right\}$$

Therefore, making the change of variables



and using the Piola transform $\mathbf{u}(x,y) = \frac{1}{|\text{det } DF|} DF\, \tilde{\mathbf{u}}(\tilde{x},\tilde{y})$, $(x,y) \in T$ we obtain, for a general triangle T with maximum angle α,

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq \frac{C}{\text{sin } \alpha} h_T \|D\mathbf{u}\|_{L^2(T)}$$

THE 3D CASE

The same argument does not give the optimal result! Two generalizations of the MAXIMUM ANGLE CONDITION:

- REGULAR VERTEX PROPERTY

  A family of tetrahedra satisfies the RVP if for some vertex, the three edges containing that vertex remain "Uniformly linearly independent".

- MAXIMUM ANGLE CONDITION

  A family of tetrahedra satisfies the MAC if the angles between edges and between faces remain uniformly bounded away from π.

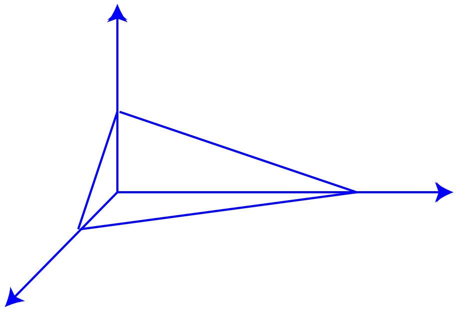
REMARK:

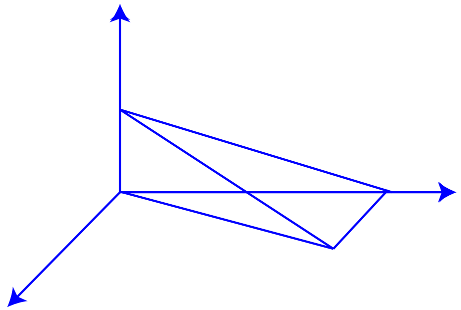
In 2D RVP $\Longleftrightarrow$ MAC

But,

In 3D RVP $\Longrightarrow$ MAC

BUT NOT CONVERSELY





A straightforward generalization of the argument given in 2D proves the error estimate under the RVP property!

NATURAL QUESTION: Does the estimate hold under the MAC hypothesis?

YES!

A DIFFERENT ARGUMENT: Reduction to a finite dimensional problem!

Introduce the FACE MEAN AVERAGE INTRPOLANT

$$\Pi : H^1(T)^3 \to \mathcal{P}_1(T)^3$$

$$\int_S \Pi \mathbf{u} = \int_S \mathbf{u}$$

It is easy to see:

- $\|\nabla \Pi \mathbf{u}\|_{\leq} \|\nabla \mathbf{u}\|_{L^2(T)}$

- $\|\mathbf{u} - \Pi \mathbf{u}\|_{L^2(T)} \leq C h_T \|\nabla \mathbf{u}\|_{L^2(T)}$ C independent of the shape

- $RT_0 \mathbf{u} = RT_0 \Pi \mathbf{u}$

Then

$$\|\mathbf{q} - RT_0\mathbf{q}\|_{L^2(T)} \leq C_1 h_T \|\nabla \mathbf{q}\|_{L^2(T)} \qquad \forall \mathbf{q} \in \mathcal{P}_1(T)^3$$

$$\Longrightarrow \|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq (C + C_1) h_T \|\nabla \mathbf{u}\|_{L^2(T)}$$

with a constant C independent of T!

Indeed

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{2(T)} \leq \|\mathbf{u} - \Pi\mathbf{u}\|_{L^2(T)} + \|\Pi\mathbf{u} - RT_0\Pi\mathbf{u}\|_{L^2(T)}$$

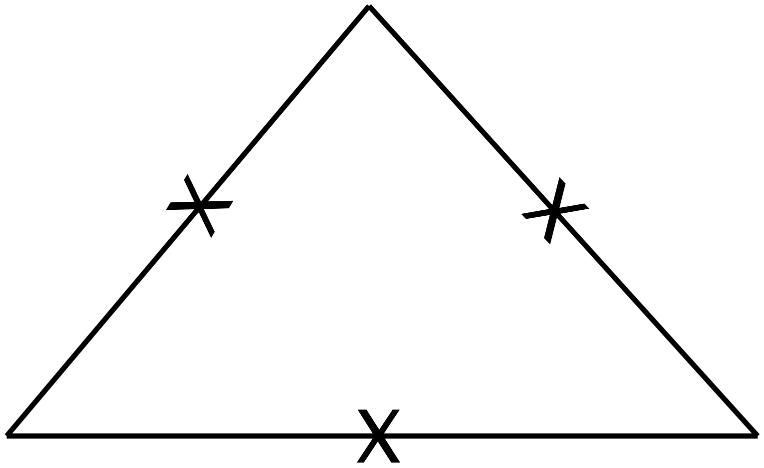
In this way we obtain:

$$\|\mathbf{u} - RT_0\mathbf{u}\|_{L^2(T)} \leq C(\alpha)\, h_T \|\nabla\mathbf{u}\|_{L^2(T)}$$

where α is the maximum angle of T.

APPLICATION TO THE STOKES EQUATIONS

CROUZEIX-RAVIART NON-CONFORMING ELEMENTS



Velocity $\mathbf{u}_h \in \mathcal{P}_1^{nc}$, Pressure $p_h \in \mathcal{P}_0^{d}$

STABILITY:

THE FORTIN OPERATOR

$$\Pi_h : H_0^1(\Omega)^n \longrightarrow \mathcal{P}_1^{nc}$$

$$\int_\Omega \mathrm{div}\,(\mathbf{u} - \Pi_h \mathbf{u})\, q = 0 \qquad \forall q \in \mathcal{P}_0^d$$

IS THE FACE (OR EDGE IN 2D) MEAN AVERAGE INTERPOLANT which satisfies

$$\|\Pi_h \mathbf{u}\|_{H_0^1} \leq C \|\mathbf{u}\|_{H_0^1}$$

with C independent of the shape of the elements!

THEREFORE: The inf-sup holds with a constant independent of the shape of the elements.

PROBLEM: Consistency terms!

THEY CAN BE BOUNDED BY USING THE RT_0 OPERATOR

(the relation between non-conforming and mixed methods is well known: Arnold-Brezzi)

CONSEQUENTLY: we obtain error estimates of optimal order with a constant which depends only on the maximum angle.

HIGHER ORDER RAVIART-THOMAS ELEMENTS

Applying similar arguments than for RT_0 (a generalized Poincaré inequality)

we can prove

$$\|\mathbf{u} - RT_k\mathbf{u}\|_{L^2(T)} \leq C h_T^{k+1} \|D^{k+1}(\mathbf{u} - RT_k\mathbf{u})\|_{L^2(T)}$$

PROBLEM:

HOW DO WE BOUND $\|D^{k+1} RT_k\mathbf{u}\|_{L^2(T)}$?

TRICK: $D^{k+1}RT_k\mathbf{u} = D^k \text{div}\mathbf{u}$

But, from the commutative diagram property we know that

$$\text{div}\,\mathcal{R}T_k\mathbf{u} = P_k \text{div}\,\mathbf{u}$$

$$\|D^{k+1}\Pi_k\mathbf{u}\|_{L^2(T)} \leq C\|D^k P_k \text{div}\mathbf{u}\|_{L^2(T)}$$

BUT, WE CAN PROVE

$$\|D^k P_k f\|_{L^2(T)} \leq C(\alpha)\|D^k f\|_{L^2(T)}$$

SUMMING UP:

$$\|\mathbf{u} - RT_k\mathbf{u}\|_{L^2(T)} \leq C(\alpha) h_T^{k+1}\|D^{k+1}\mathbf{u}\|_{L^2(T)}$$

where α is the maximum angle

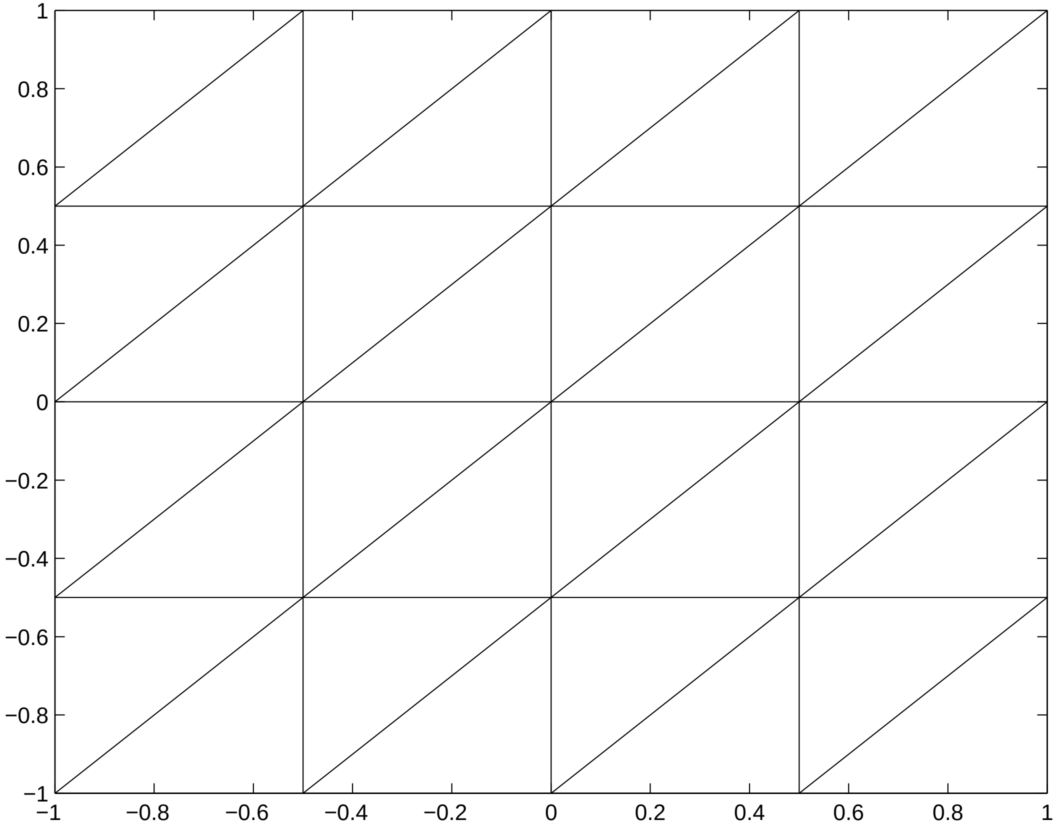
WE ARE NOT ABLE TO PROVE:

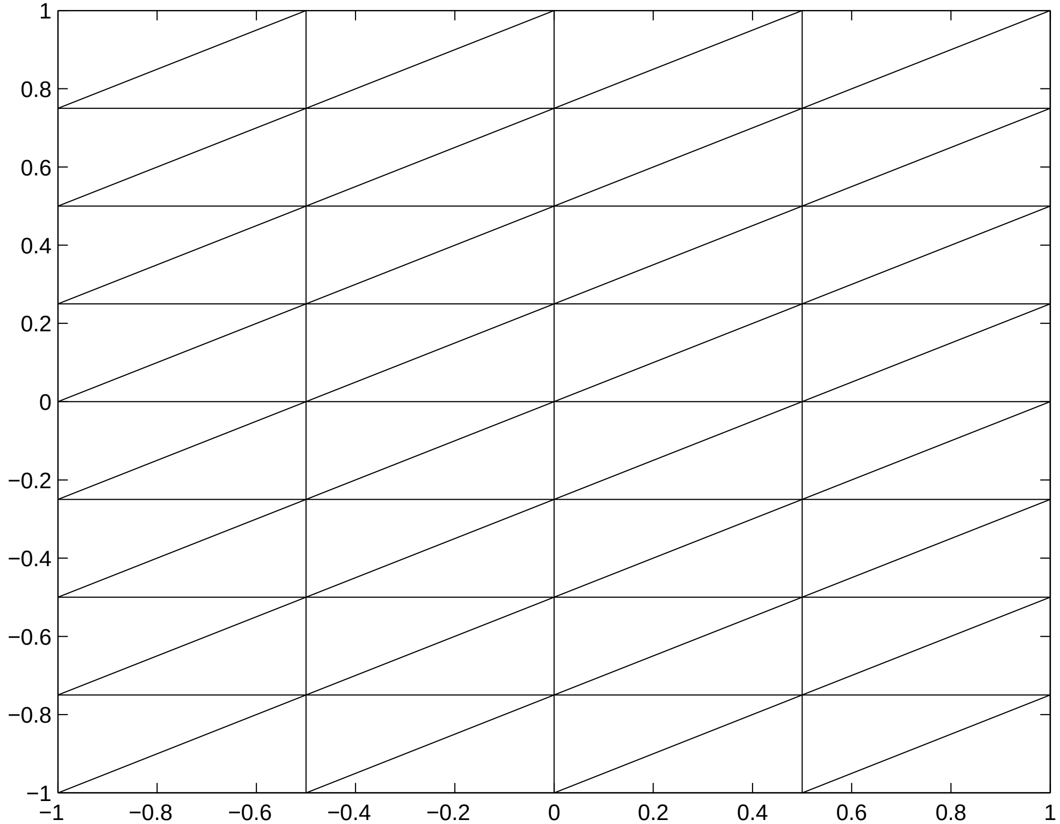
- THE INF-SUP FOR $k \geq 1$

- $\|\mathbf{u} - RT_k\mathbf{u}\|_{L^2(T)} \leq C(\alpha)h_T^m\|D^m\mathbf{u}\|_{L^2(T)}$ for $m < k + 1$

However, numerical experiments suggest that the *inf-sup* holds!

NUMERICAL RESULTS FOR RT_1 (by Ariel Lombardi)

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0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
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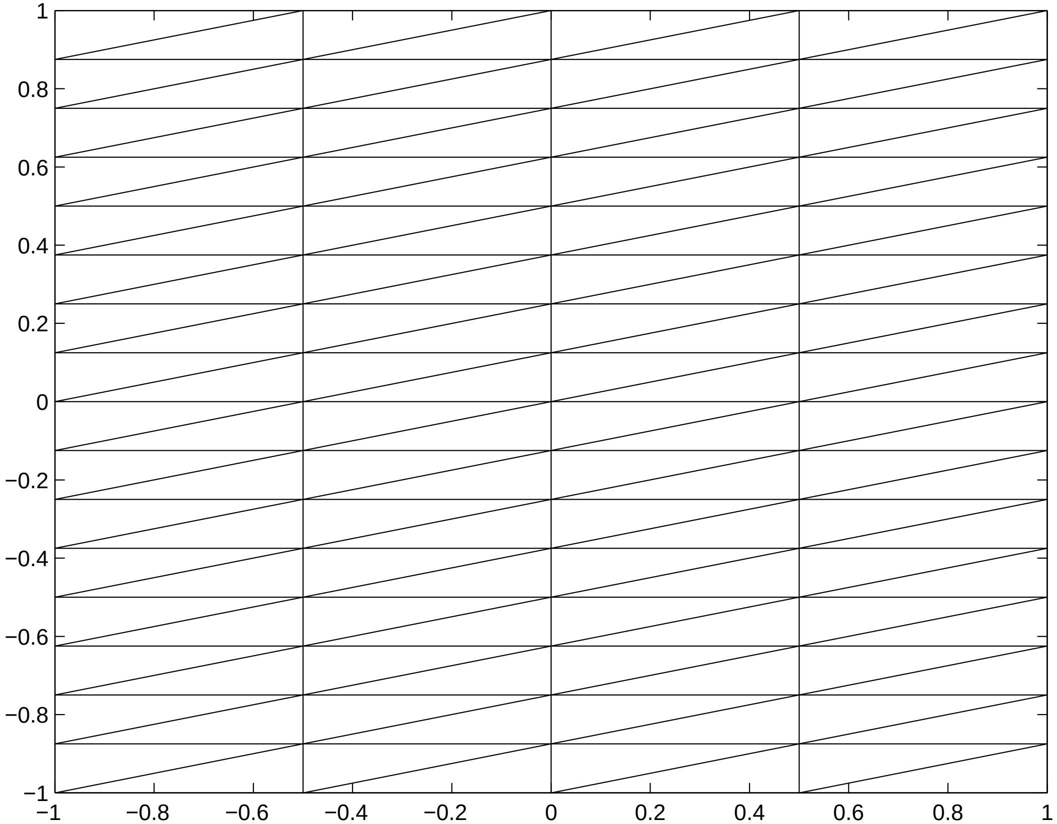
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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−0.8
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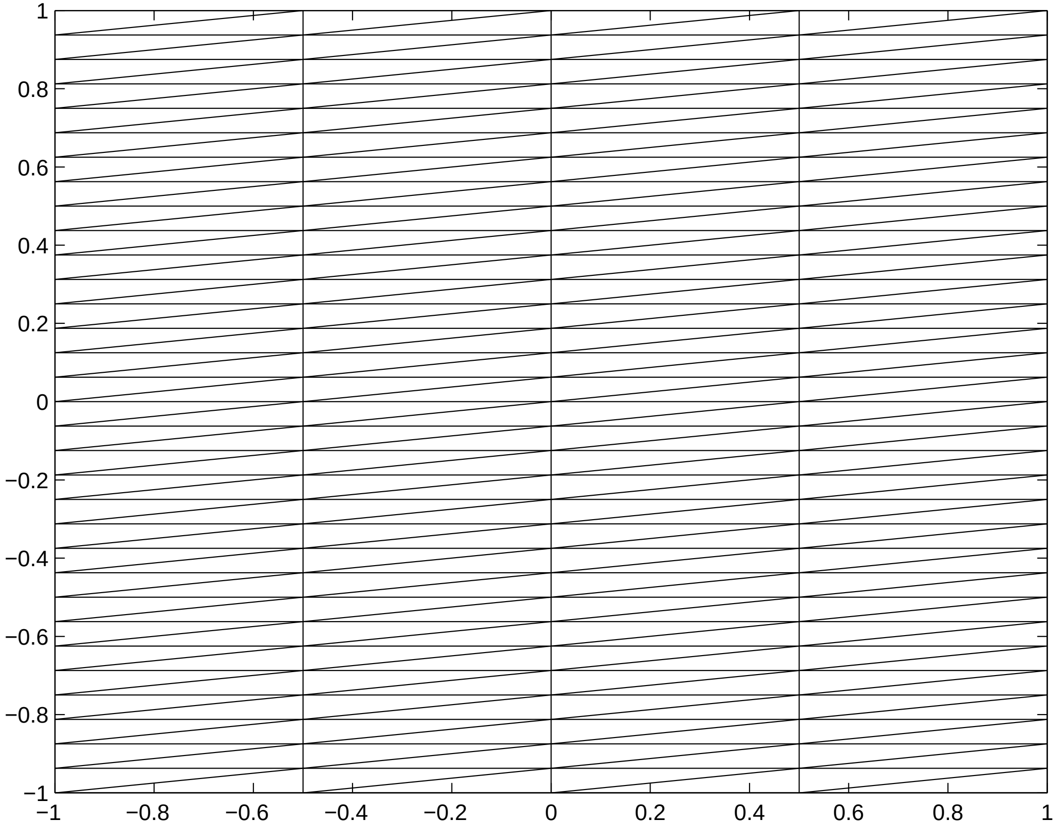
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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−0.6
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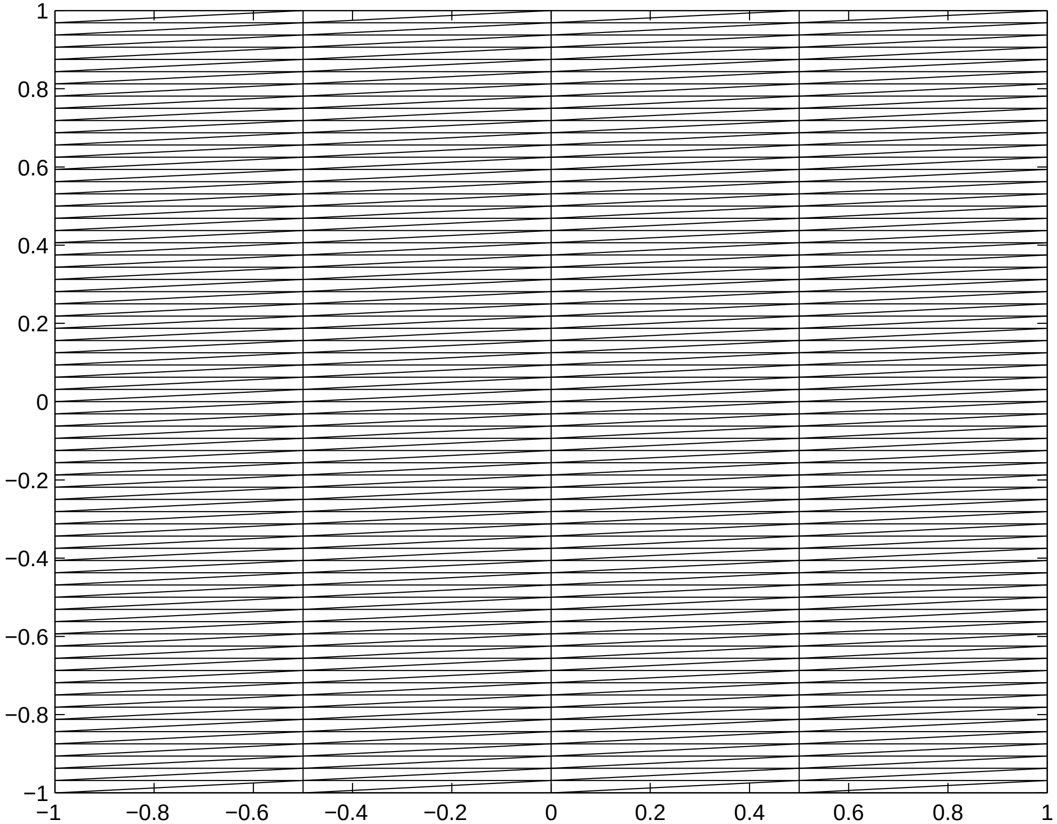
−0.2
0
0.2
0.4
0.6
0.8
1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0.4
0.2
0
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−0.4
−0.6
−0.8
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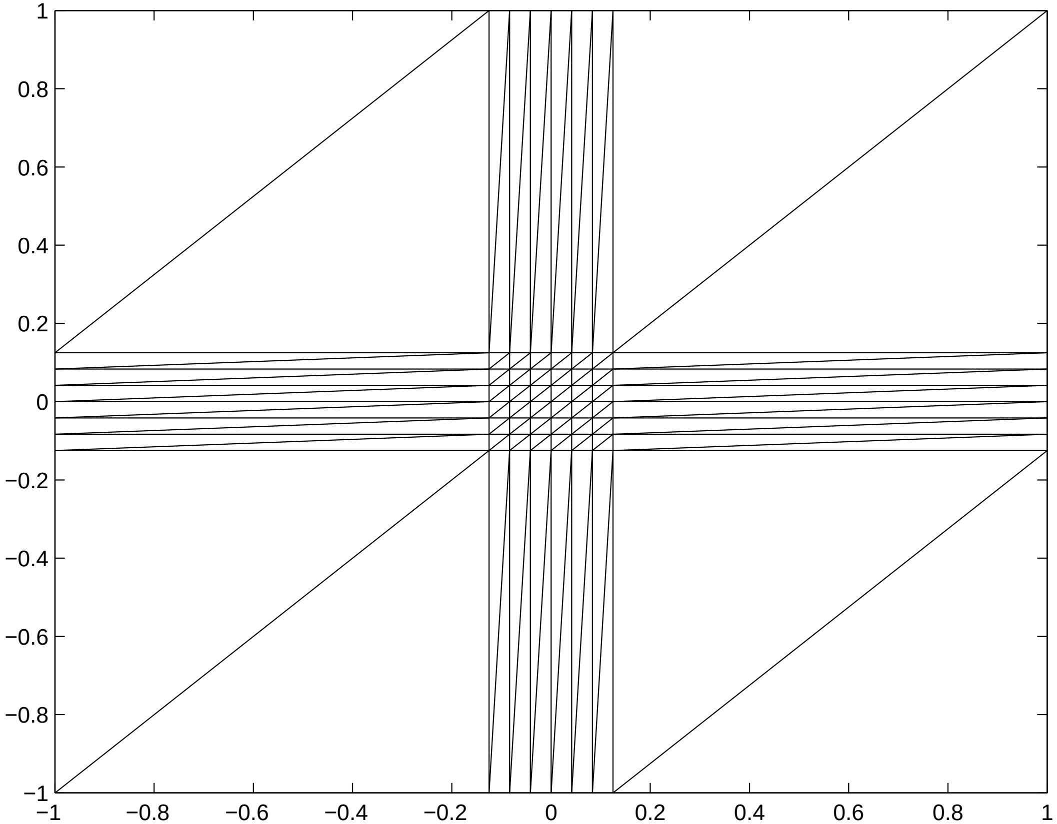
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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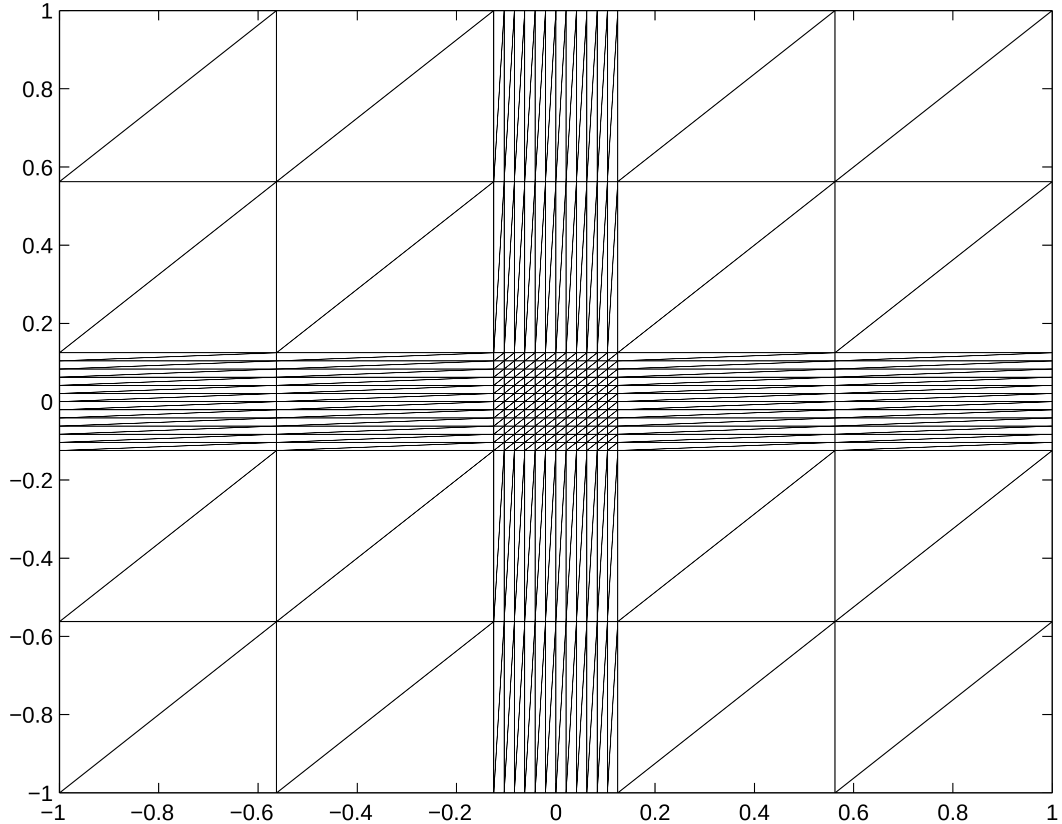
−0.2
0
0.2
0.4
0.6
0.8
1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−0.8
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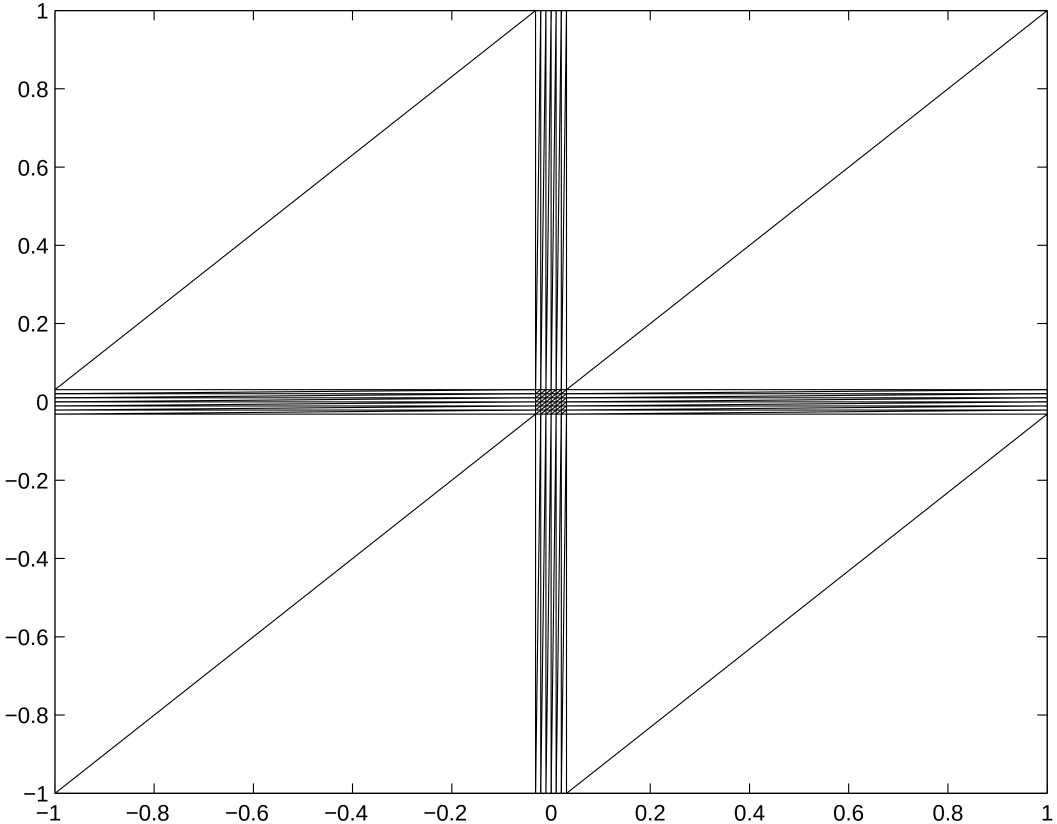
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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−0.6
−0.8
−1
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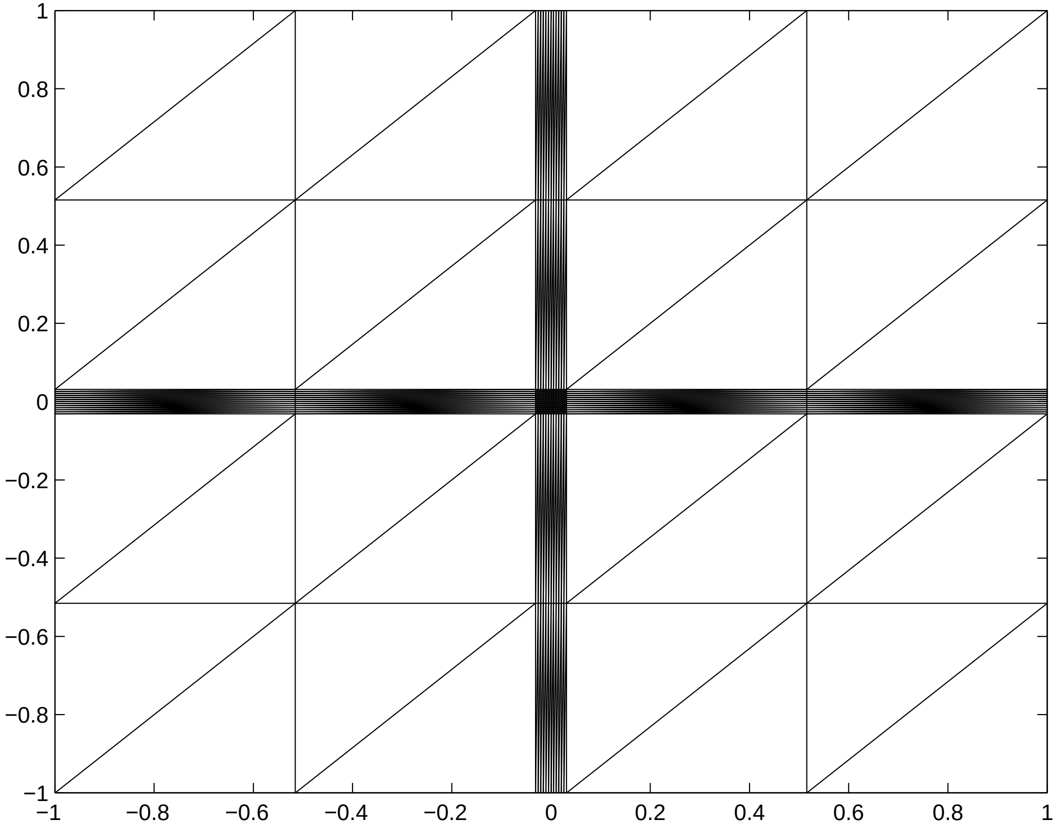
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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0.4
0.2
0
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−0.4
−0.6
−0.8
−1
−1
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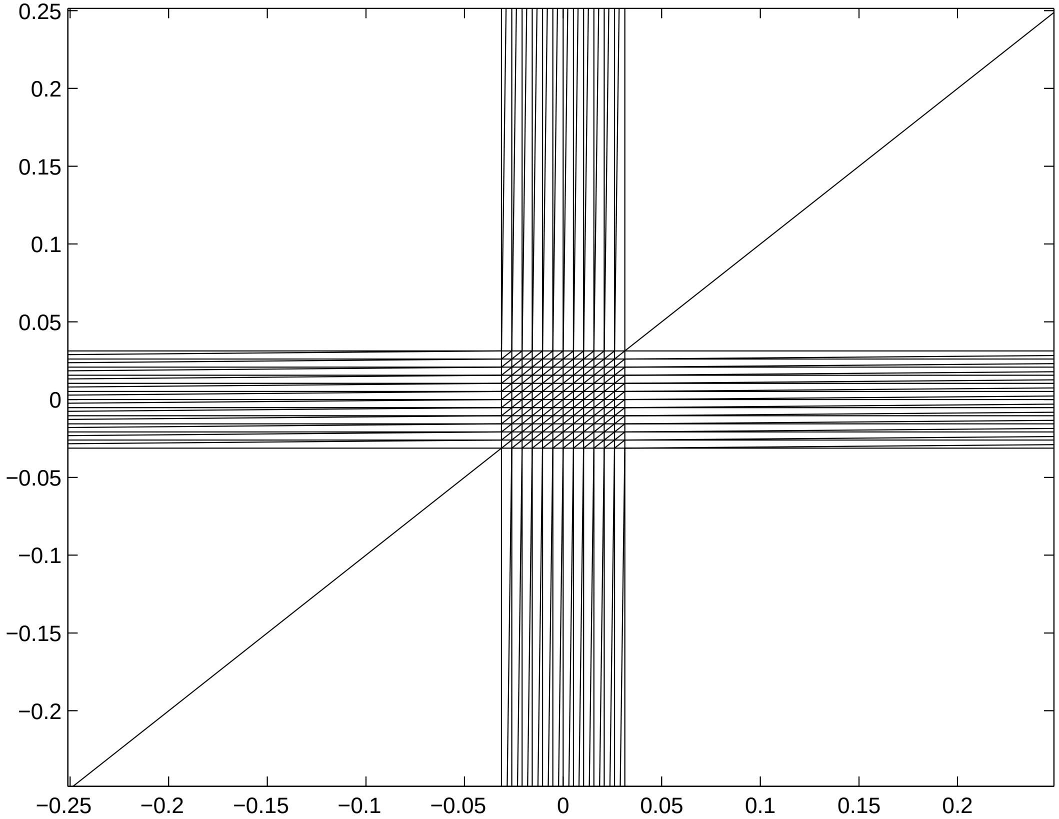
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1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0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2

Example	inf-sup
(1)	0.49905797195785
(2)	0.49929292121011
(3)	0.49932521957619
(4)	0.49933289504315
(5)	0.49933479989259
(6)	0.49734012930349
(7)	0.49917541929084
(8)	0.49719590019379
(9)	0.49911691360397

APPLICATIONS

PROBLEMS WITH BOUNDARY LAYERS

Consider the convection-diffusion problem

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

$$b_i < -\gamma \qquad \text{with} \qquad \gamma > 0 \qquad \text{for} \qquad i = 1, 2 \tag{2}$$

It is known that the solution obtained by standard FE with uniform meshes present oscillations unless the mesh is too fine.

SOLUTIONS?

Several special techniques have been introduced: up-wind, stream-line diffusion, Petrov-Galerkin, etc.

But, is it possible to obtain good results with the standard method by using appropriate meshes?

We prove error estimates valid uniformly in ε if graded meshes are used.

What is the difficulty in this problem? Recall the FE theory:

The bilinear form is:

$$B(v,w) = \int_\Omega (\varepsilon \nabla v \cdot \nabla w + b \cdot \nabla v \, w + c \, vw) \; dx.$$

Consider the norm:

$$\|v\|_\varepsilon^2 = \|v\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v\|_{L^2(\Omega)}^2.$$

Assuming

$$c - \frac{\operatorname{div} b}{2} \geq \mu > 0$$

the bilinear form is coercive with a constant α independent of ε.

But:

1- The constant M in the continuity of the form depends on ε.

2- The second derivatives arising in the standard error estimates depends on ε.

Using a graded mesh we have proved that

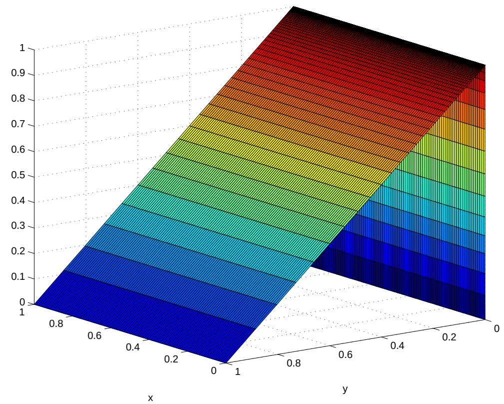
$$\boxed{\|u - u_N\|_\varepsilon \leq C\frac{(\log(1/\varepsilon)^2}{\sqrt{N}}}$$

where N is the number of nodes in the mesh. The order with respect to the number of nodes is optimal in the sense that it is the same than the order obtained for a problem with a smooth solution with uniform meshes.

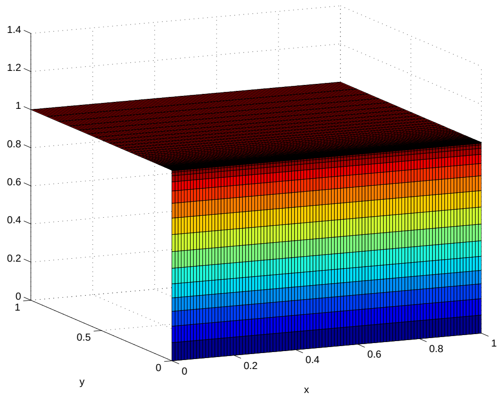
NUMERICAL EXAMPLES

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u + cu &= f && \text{in } \Omega \\ u &= u_D && \text{in } \Gamma_D \\ \frac{\partial u}{\partial n} &= g && \text{in } \Gamma_N, \end{aligned}$$

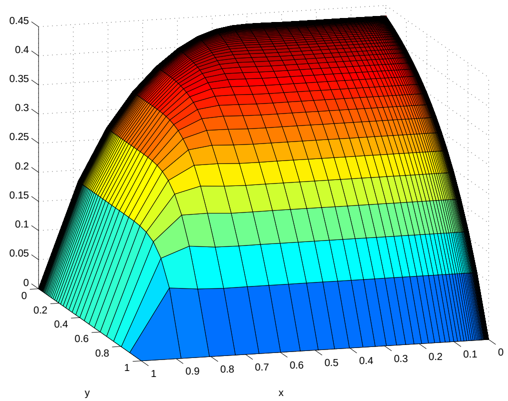
With different coefficients and data.


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x
y

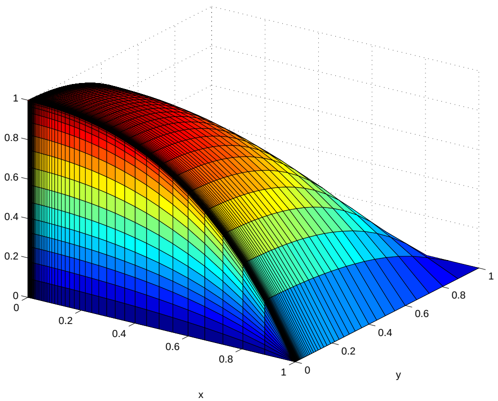
(1)


1.4
1.2
1
0.8
0.6
0.4
0.2
0
y
x

(2)


0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
y
x

(3)


1
0.8
0.6
0.4
0.2
0
x
y

(4)

No oscillations are observed.

For one of the examples we know the exact solution

$$u(x,y) = \left[\left(x - \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \left(y - \frac{1 - e^{-\frac{y}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \right) \right] e^{x+y},$$

and so we can compute the order of convergence.

N	Error
324	0.16855
961	0.097606
3249	0.052696
12100	0.025912
45796	0.013419

$\varepsilon = 10^{-4}$

N	Error
676	0.16494
2025	0.094645
6889	0.050256
25281	0.026023
96100	0.013427

$\varepsilon = 10^{-6}$

The orders computed from these tables are 0.513738 for the first case and 0.507040 for the second one as predicted by the theoretical results.

ADVANTAGE OVER SHISHKIN MESHES

The graded meshes designed for a given ε work well also for larger values of ε. This is not the case for the Shishkin meshes!

Errors for different values of ε with the mesh corresponding to $\varepsilon = 10^{-6}$:

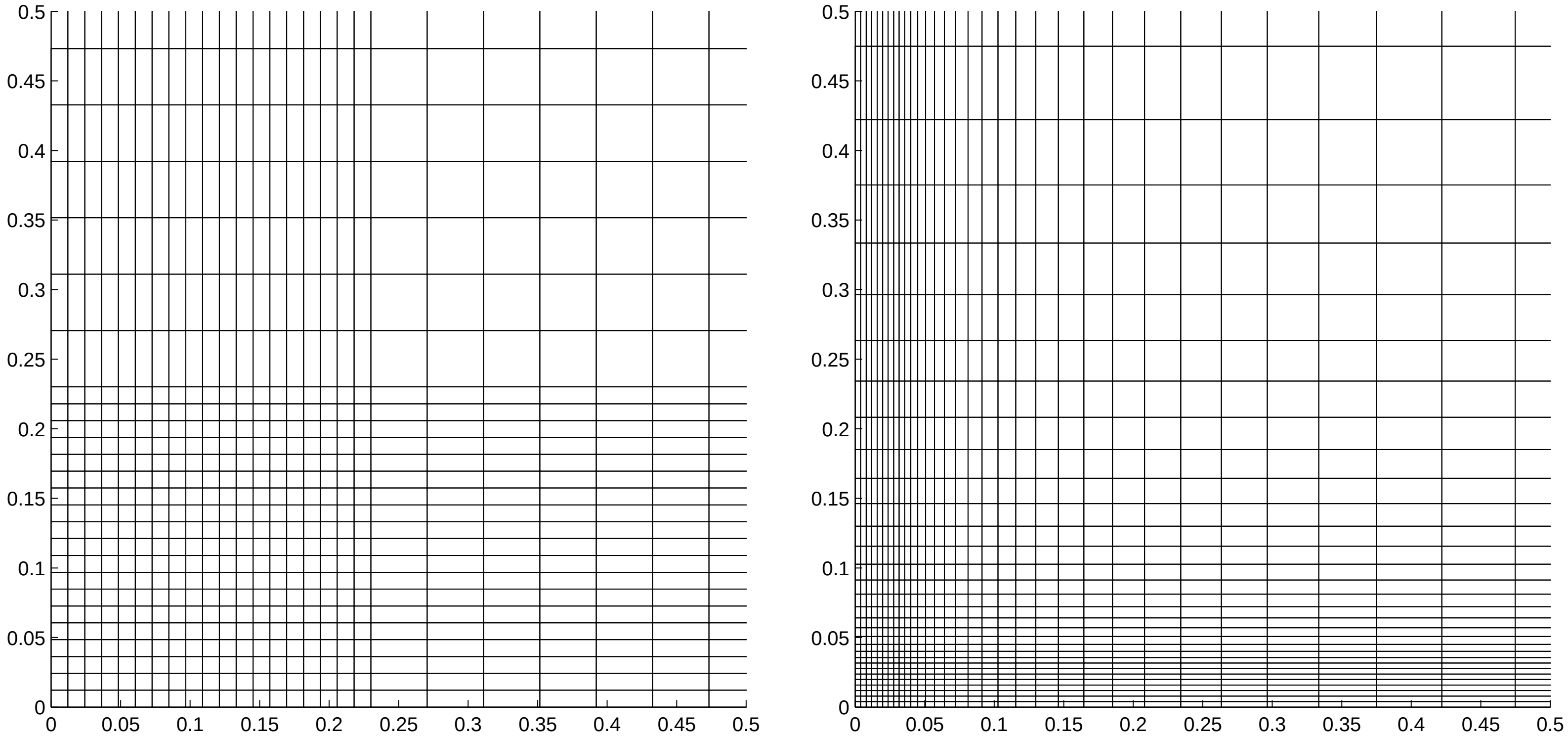
ε	Error
10^{-6}	0.040687
10^{-5}	0.033103
10^{-4}	0.028635
10^{-3}	0.024859
10^{-2}	0.02247
10^{-1}	0.027278

Graded meshes, $N = 10404$

ε	Error
10^{-6}	0.0404236
10^{-5}	0.249139
10^{-4}	0.623650
10^{-3}	0.718135
10^{-2}	0.384051
10^{-1}	0.0331733

Shishkin meshes, $N = 10609$

Different structure of the well known Shishkin meshes and our meshes:



FURTHER RESEARCH

- Average interpolants for more general domains (there are difficulties with boundary conditions).

- Results for other mixed methods (for example for BDM spaces our arguments do not apply!).

- Conforming methods for Stokes (there are some results for $\mathcal{Q}_{k+2} - \mathcal{Q}_k$ methods but not for Taylor-Hood elements although there is numerical evidence that they work on anisotropic meshes).

Our results are contained in the following references:

http://mate.dm.uba.ar/~rduran/

- R. G. Durán, *Error estimates for narrow 3-d finite elements*, Math. Comp. **68**(225), 187-199, 1999.

- G. Acosta, R. G. Durán, *The maximum angle condition for mixed and non conforming elements: Application to the Stokes equations*, SIAM J. Numer. Anal. **37**(1), 18-36, 2000.

- G. Acosta, R. G. Durán, *Error estimates for $\mathcal{Q}_1$ isoparametric elements satisfying a weak angle condition*, SIAM J. Numer. Anal. **38**(4), 1073-1088, 2000.

- R. G. Durán , A. L. Lombardi, *Error estimates on anisotropic $\mathcal{Q}_1$ elements for functions in weighted Sobolev spaces*, Math. Comp. **74**(252), 1679-1706, 2005.

- R. G. Durán , A. L. Lombardi, *Finite element approximation of convection diffusion problems using graded meshes*, Appl. Numer. Math.. **56**(10-11), 1314-1325, 2006.

- R. G. Durán , A. L. Lombardi, *Error estimates for the Raviart-Thomas interpolation under the maximum angle condition*, submitted.