THE MAXIMUM ANGLE CONDITION FOR MIXED AND NON CONFORMING ELEMENTS. APPLICATION TO THE STOKES EQUATIONS *

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Abstract. For the Lagrange interpolation it is known that optimal order error estimates hold for elements satisfying the maximum angle condition. The objective of this paper is to obtain similar results for the Raviart-Thomas interpolation arising in the analysis of mixed methods. We prove that optimal order error estimates hold under the maximum angle condition for this interpolation both in two and three dimensions and moreover, that this condition is indeed necessary to have these estimates.

Error estimates for the mixed approximation of second order elliptic problems and for the non conforming piecewise linear approximation of the Stokes equations are derived from our results.

Key words. Mixed, non conforming, maximum angle, Raviart-Thomas, Stokes.

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1. Introduction. This paper deals with the convergence of non conforming and mixed finite elements when narrow (or degenerate) elements are used in 2 and 3 dimensions. The use of this kind of elements is of interest in many situations, in particular, for non isotropic problems involving solutions which behaves different in each direction. We obtain optimal order error estimates for the Raviart-Thomas interpolation of the lowest degree for triangular and tetrahedral elements satisfying a maximum angle condition. Our arguments also apply to degenerate rectangular elements providing error estimates with constants independent of the relation between the mesh size in each direction.

We give two applications of our result. The first one follows immediately from the known error analysis of Raviart-Thomas mixed approximation of second order elliptic problems. Indeed, it is known [19, 10] that the error in that approximation can be bounded in terms of the error for the RT interpolation. Therefore, our results give optimal error estimates for degenerate elements.

Our second application concerns the approximation of the Stokes equations by the linear non conforming elements of Crouzeix and Raviart [9].

The standard finite element analysis requires the so called regularity assumption on the elements, i.e., bounded ratio between outer and inner diameter (see for example [7, 6]). This same hypothesis is needed for the analysis of mixed methods based on the Piola transform given in [19] and for the analysis of non conforming approximations given in [9].

However, for standard finite element approximations of coercive problems (such as scalar second order elliptic problems) it is now well known that the regularity assumption can be relaxed. Indeed, for 2d problems, optimal order error estimates can be proven under the weaker maximum angle condition (i.e. angles bounded away from π, see for example [3, 14]). Generalizations of these results to the 3d case were

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obtained in several papers [15, 21, 11] although the situation is not totally analogous to the 2d case. In [15] Krizek proved optimal order error estimates for the Lagrange interpolation on a tetrahedron K, for smooth functions ($u \in W^{2,\infty}$) provided the angles between faces and the angles in the faces are bounded away from π. This seems a natural extension of the 2d maximum angle condition. In [21] Al Shenk obtained general results which provide error estimates for Lagrange interpolation for functions in certain Sobolev spaces and valid for a wide class of narrow elements, more precisely this class is obtained by regular transformations of rectangular tetrahedra. However his results do not apply for functions in H^2 and elements of degree one (a very important case in finite element applications). Moreover he gave interesting counterexamples showing that the optimal order error estimate in H^1 does not hold uniformly in his class of narrow elements, in fact he showed that the constant in the estimate blows-up when a rectangular reference tetrahedron is compressed in one direction, in particular the optimal error estimate in H^1 does not hold under Krizek hypothesis.

For non coercive problems such as the Stokes equations the situation is quite different. It is very well known that to obtain good approximations for this problem in the standard velocity-pressure formulation, the finite element spaces for velocity and pressure have to be chosen appropriately (see [4, 12]), in order to satisfy the stability condition. In principle the constant appearing in the stability depends on the geometry of the mesh and so, the question of whether or not this constant can be taken independently of the minimum angle has to be analyzed for each method. As it is known the natural choice $\mathcal{P}_1 - \mathcal{P}_0$ is not stable and so, to obtain low order stable elements one have to enrich the velocity space in some way. This can be done by using non conforming linear elements for the velocity [9] or enriching the space $\mathcal{P}_1$ with some quadratic functions [4, 5]. Other possibility is to use the so called mini-element [2]. For this last case, A. Russo [20] showed that the method becomes unstable when narrow elements are used, more precisely, the stability constant blows up when the minimum angle goes to zero. For the method of [5], the usual proof [4, 12] gives a stability constant which depends on the minimum angle and so, it would be interesting to analyze whether this is essential or just a problem of the proof. For the non conforming method of Crouzeix and Raviart considered here, it is not difficult to show that the stability constant is independent of the geometry of the elements. However, since the method is non conforming, consistency terms arise in the error equations. The usual arguments to estimate these terms make use of trace theorems and scaling arguments and need the regularity assumption on the mesh. We give an alternative proof to bound the consistency terms estimating them by the error for the Raviart-Thomas interpolation operator [4, 19]. The relation between non conforming and RT method is well known for second order elliptic problems [1, 16] but, since we are interested in the Stokes equations, we do not make use of this relation explicitely. Consequently, we obtain optimal order error estimates for degenerate elements both for two and three dimensional problems. Our results hold under the standard maximum angle condition in 2d and under its generalization introduced by Krizek in 3d. Our estimates are valid for solutions in H^2 (so, in this sense, the behavior is better than that for Lagrange interpolation).

The arguments that we use for 2 and 3d are different. The reason for doing that is that our proof for the 2d case seems more direct and can be extended straightforward for rectangular elements but its natural extension to 3d would impose unnecessary restrictions on the class of narrow elements allowed. On the other hand, the argument that we use for the 3d case is optimal in the sense that it provides a necessary and sufficient condition on the elements in order to have the optimal error estimate.

2. Notation and some basic estimates. The object of this section is to recall some estimates which are basic for our subsequent argument and to introduce some notation.

We will denote with small letters the scalar functions, with small bold fonts the vectorial ones, and we keep the capital bold for the tensor fields. Given a general triangle (*resp.: tetrahedron*) K, $\mathbf{p}_0$ will be a vertex (arbitrary unless specified), $\mathbf{v}_i$ (with $\|\mathbf{v}_i\| = 1$) and l_i , $1 \le i \le n$ ($n = 2$ or 3) the directions and lengths respectively, of the edges sharing $\mathbf{p}_0$, i.e., $K = c.h.\{\mathbf{p}_0, \mathbf{p}_0 + l_i\mathbf{v}_i\}_{1\le i\le n}$, where $c.h.$ means the convex hull. We will denote with h_K the length of the longest edge of K and with ν_i, $1 \le i \le n$ the unit exterior normal vectors to the edges if $n = 2$ (*resp.: faces if* $n = 3$) sharing $\mathbf{p}_0$. The reference element $\hat{K}$ is defined as $\hat{K} = c.h.\{\mathbf{0}, \mathbf{e}_i\}_{1\le i\le n}$ with $\mathbf{e}_i$ being the cannonical vectors. $F : \hat{K} \to K$ will be the affine transformation $F(\hat{x}) = B\hat{x} + \mathbf{p}_0$ with $B\mathbf{e}_i = l_i\mathbf{v}_i$ and for a function $\hat{w}$ defined in $\hat{K}$ we associate w defined on K in the usual way: $\hat{w} = w \circ F$. Finally, we will denote with $\frac{\partial}{\partial v_i}$ the derivative in the direction of $\mathbf{v}_i$ both for scalar and vector functions.

We will use the usual notation $\|.\|_{m,K}, |.|_{m,K}$ for Sobolev norms and seminorms where the domain will be suppressed when it is Ω.

The following two results are the classic Poincaré inequality and a generalization of it written in a convenient way for our purposes. This generalization was given in [3] for the 2d case.

Lemma 2.1. *Let K be a triangle (resp.: tetrahedron) and let $w \in H^1(K)$ be a function with vanishing average on K. Then, there exists a constant C independent of K such that*

$$\|w\|_{0,K} \le C \sum_{j=1}^{n} l_j \|\frac{\partial w}{\partial v_j}\|_{0,K} \tag{2.1}$$

Proof. It follows from the Poincaré inequality for the function $\hat{w}$ on the reference element $\hat{K}$, the change of variable F and observing that $\frac{\partial \hat{w}}{\partial x_i} = l_i \frac{\partial w}{\partial v_i}$. ☐

Remark 1. *It is interesting to remark that the constant in the Poincaré inequality can be taken independent of the shape (i.e., depending only on the diameter) for a general convex domain. In fact, this has been proven in [13, 17].*

Lemma 2.2. *Let K be a triangle (resp.: tetrahedron) and S be any of its edges (resp.: faces). Let $w \in H^1(K)$ be a function with vanishing average on S. Then, there exists a constant C independent of K such that*

$$\|w\|_{0,K} \le C \sum_{j=1}^{n} l_j \|\frac{\partial w}{\partial v_j}\|_{0,K} \tag{2.2}$$

Proof. For the reference element $\hat{K}$ a usual argument gives $\|\hat{w}\|_{0,\hat{K}} \le \hat{C}\|\nabla\hat{w}\|_{0,\hat{K}}$. Indeed, if this inequality does not hold, by compactness there exists a sequence $\hat{w}_n$, with vanishing mean value on S, converging in L^2 to some $\hat{w}$ and such that $\|\hat{w}_n\|_{0,\hat{K}} = 1$, $\|\nabla\hat{w}_n\|_{0,\hat{K}} \le 1/n$. In particular, it follows from this last inequality that the convergence is in H^1 and that $\hat{w}$ is a constant. Now, by a trace theorem it follows that the restriction of $\hat{w}_n$ to S converges in $L^2(S)$ to that of $\hat{w}$. In particular $\hat{w}$ has vanishing mean value on S and therefore $\hat{w} \equiv 0$ which contradicts $\|\hat{w}\|_{0,\hat{K}} = 1$. Now, the proof ends by using the change of variable F. $\square$

REMARK 2. *The proof of the previous lemma on the reference element was given for the 2d case in [3]. The extension to 3d is straightforward if one replaces "edge" by "face" as we have done. We have included some details of the proof in order to emphasize this fact. Indeed, the result does not hold in 3d if S is an edge of the element K. Actually, the restriction to an edge in 3d does not make sense for a general function in H^1. This last extension is what would be needed in order to generalize the results of [3, 14] for the Lagrange interpolation to 3d (note that for regular functions, the derivative in the direction of an edge of the error has vanishing mean value on that edge). However, as was shown by the counterexamples in [21], that generalization is not possible without requiring some extra regularity on the function (see [11]).*

Let us now introduce some operators. We use the standard notation $\mathcal{P}_k(K)$ for polynomials of degree not greather than k restricted to K. Given an element K, the operator $P : L^2(K) \to \mathcal{P}_0(K)$ is the L^2 projection. For $w \in L^2(K)$ it is defined by:

$$\int_K Pw = \int_K w$$

We will also use the $\mathcal{P}_1$-edges (*resp.: faces*) mean average interpolant $\Pi : H^1(K) \to \mathcal{P}_1(K)$ which has been introduced in [9]. For $w \in H^1(K)$ it is defined by:

$$\int_S \Pi w = \int_S w$$

for any edge (*resp.: face*) S of K.

From Lemma 2.1 it follows immediately the following well known estimate for the L^2 projection:

$$\|w - Pw\|_{0,K} \leq C \sum_{j=1}^{n} l_j \|\frac{\partial w}{\partial v_j}\|_{0,K} \tag{2.3}$$

In general, Lemmas 2.1 and 2.2 are the basis of error estimates for several interpolation or projection operators. Indeed, in many situations, the error (or some derivative of it) satisfies the hypothesis of some of these lemmas. For example, for the operator Π introduced above we have the following results:

LEMMA 2.3. *The following error estimates hold with a constant C independent of K:*

$$\|w - \Pi w\|_{0,K} \leq C \sum_{j=1}^{n} l_j \|\frac{\partial w}{\partial v_j}\|_{0,K} \tag{2.4}$$

$$\|\frac{\partial \Pi w}{\partial \xi}\|_{0,K} \leq \|\frac{\partial w}{\partial \xi}\|_{0,K} \tag{2.5}$$

$$\|\frac{\partial (w - \Pi w)}{\partial \xi}\|_{0,K} \leq C \sum_{j=1}^{n} l_j \|\frac{\partial^2 w}{\partial v_j \partial \xi}\|_{0,K} \tag{2.6}$$

where $\frac{\partial}{\partial \xi}$ *is a derivative in any direction.*

Proof. Since $w - \Pi w$ has vanishing mean value on the edges (resp.: faces) of K it follows from Lemma 2.2 that

$$\|w - \Pi w\|_{0,K} \leq C \sum_{j=1}^{n} l_j \|\frac{\partial(w - \Pi w)}{\partial v_j}\|_{0,K} \tag{2.7}$$

Now, it follows from the definition of Π that

$$\int_K \frac{\partial \Pi w}{\partial \xi} = \int_K \frac{\partial w}{\partial \xi}$$

or in other words, $P\frac{\partial w}{\partial \xi} = \frac{\partial \Pi w}{\partial \xi}$ and so (2.5) holds and (2.6) follows from (2.3). Finally, (2.4) is a consequence of (2.7) and (2.5) with $v_j = \xi$. ☐

We end this section by introducing the Raviart-Thomas space of the lowest degree and the corresponding interpolation operator. The Raviart-Thomas and it generalization to 3d are defined on a triangular (*resp.: tetrahedral*) element K as [18, 19]

$$\mathcal{RT}_0(K) = \mathcal{P}_0^n(K) \oplus \mathbf{x}\mathcal{P}_0(K)$$

and, the interpolation operator $RT : H^1(K)^n \to \mathcal{RT}_0(K)$ is defined by

$$\int_S RT\mathbf{w} \cdot \nu = \int_S \mathbf{w} \cdot \nu$$

for all S edge if $n = 2$ (*resp.: face if* $n = 3$) of K, where ν denotes the unit normal vector to S.

We will also make use of the operator $\mathbf{RT}$ acting on tensor fields $\mathbf{M}$ obtained by applying RT row to row.

3. Applications of the Raviart-Thomas interpolation. In this section we present two applications of the Raviart- Thomas interpolation. The first is the standard one about error estimates for mixed approximations while the second one concerns the bound of the error in the non conforming approximation of the Stokes equations.

We assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a polygonal or polyhedral domain and that it is divided into an admissible partition $\mathcal{T}$ of triangular or tetrahedral elements. The finite element space $\mathcal{RT}_0$ is the subspace of $H(\text{div}\,, \Omega)$ of functions which are locally in $\mathcal{RT}_0(K)$ and $\mathcal{L}_0$ is the space of piecewise constant functions. The global versions of the operators introduced in the previous section will be denoted like the local ones. It is not difficult to see, and is in fact well known, that the RT interpolation satisfies the following commutative diagram property:

$$\begin{array}{ccccc} H^1(\Omega)^n & \xrightarrow{div} & L^2(\Omega) & & \\ RT\big\downarrow & & \big\downarrow P & & \\ \mathcal{RT}_0 & \xrightarrow{div} & \mathcal{L}_0 & \longrightarrow 0 & \end{array} \tag{3.1}$$

Given a second order elliptic problem,

$$\begin{cases} -div(a\nabla p) = f & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega \end{cases}$$

the $\mathcal{RT}_0$ space can be used in conjunction with $\mathcal{L}_0$ to approximate $\mathbf{u} = -a\nabla p$ and p by using the mixed formulation

$$\int_\Omega a^{-1}\mathbf{u}\cdot\mathbf{v} - \int_\Omega p\,\mathrm{div}\,\mathbf{v} = 0 \quad \forall\,\mathbf{v}\in H(div,\Omega)$$

$$\int_\Omega q\,\mathrm{div}\,\mathbf{u} = \int_\Omega f\,q \quad \forall\,q\in L^2(\Omega)$$

then, it is known [10] that, as a consequence of the commutative diagram property (3.1) the finite element approximation $\mathbf{u}_h \in \mathcal{RT}_0$ satisfies the error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\|\mathbf{u} - RT\mathbf{u}\|_0$$

where the constant C depends only on the coefficient $a(x)$. Therefore, error estimates on degenerate elements will hold whenever this is true for the RT interpolation.

Our second application is for the Stokes equations,

$$\begin{cases} -\mathbf{div}\,(\nabla\mathbf{u} - p\mathbf{Id}) = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

We consider the piecewise linear non conforming elements of Crouzeix and Raviart $(\mathcal{L}_1^{NC})^n$ for the velocity, where $\mathcal{L}_1^{NC}$ denotes the piecewise linear functions which are continuous at the midpoints of the edges (*resp.: barycenters of the faces*) between elements and vanishing at the midpoints of the edges (*resp.: barycenters of the faces*) on the boundary of Ω. The pressure is taken in $\mathcal{L}_0$. This combination provides a stable method (see [9, 12, 4]).

The Crouzeix-Raviart finite element approximation [9] $(\mathbf{u}_h, p_h) \in \mathcal{L}_1^{NC}\times\mathcal{L}_0$ is defined by,

$$\begin{cases} \sum_K \int_K (\nabla\mathbf{u}_h : \nabla\mathbf{v}_h - p_h\,\mathrm{div}\,\mathbf{v}_h) = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h \\ \sum_K \int_K q_h\,\mathrm{div}\,\mathbf{u}_h = 0 \end{cases} \tag{3.3}$$

for all $(\mathbf{v}_h, q_h) \in \mathcal{L}_1^{NC}\times\mathcal{L}_0$.

Our goal is to show that the error can be estimated in terms of an error for the RT interpolation plus other error terms which are of optimal order independently of the shape of the elements. Consequently, we will have that optimal error estimates hold if this is true for the RT interpolation.

We will make use of the vectorial versions of Π and P which are defined by applying them to each component and will be denoted with bold face. In particular

$\mathbf{\Pi} : H_0^1(\Omega)^n \to (\mathcal{L}_1^{NC})^n$ is the Fortin operator associated with this method, i.e., it satisfies

$$\int_\Omega \text{div } (\mathbf{u} - \mathbf{\Pi u})\, q_h = 0 \qquad \forall q_h \in \mathcal{L}_0 \tag{3.4}$$

Note that, in view of Lemma 2.3, estimates with constants independent of the shape of the element hold for this operator in L^2 and H^1 norms. In particular, the stability or inf-sup condition will be valid with a constant independent of the shape of the elements. Therefore, the key point is to estimate the consistency term arising in the error equations. Let us mention that the standard way used to estimate these terms in non conforming methods makes use of a trace theorem and a scaling argument which require the minimum angle condition. Our arguments will show that this condition is not necessary. To simplify notation we introduce

$$\mathbf{M} = \nabla \mathbf{u} - p\mathbf{Id}$$

In order to obtain the error equations, we multiply by $\mathbf{w}_h = \mathbf{\Pi u} - \mathbf{u}_h \in (\mathcal{L}_1^{NC})^n$ both sides of (3.2), and integrate by parts obtaining

$$\sum_K \{ \int_K (\nabla \mathbf{u} : \nabla \mathbf{w}_h - p \,\text{div } \mathbf{w}_h) - \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h \} = \int_\Omega \mathbf{f} \cdot \mathbf{w}_h \tag{3.5}$$

Subtracting the first equation of (3.3) from (3.5) and calling $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ we have

$$\sum_K \int_K (\nabla \mathbf{e}_h : \nabla \mathbf{w}_h - (p - p_h)\,\text{div } \mathbf{w}_h) = \sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h \tag{3.6}$$

But it follows from the second equations of (3.2) and (3.3) that

$$\sum_K \int_K q_h \,\text{div } (\mathbf{u} - \mathbf{u}_h) = 0 \qquad \forall q_h \in \mathcal{L}_0$$

and therefore, from (3.4)

$$\sum_K \int_K q_h \,\text{div } \mathbf{w}_h = 0 \qquad \forall q_h \in \mathcal{L}_0$$

Then, on each K, div $\mathbf{w}_h$ vanishes and so,

$$\sum_K \int_K (p - p_h)\,\text{div } \mathbf{w}_h = 0$$

Therefore, (3.6) takes the form

$$\sum_K \int_K \nabla \mathbf{e}_h : \nabla \mathbf{w}_h = \sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h \tag{3.7}$$

and writing

$$\mathbf{e}_h = \mathbf{u} - \mathbf{\Pi u} + \mathbf{w}_h$$

we finally obtain from (3.7) the error equation

$$\sum_K \int_K \nabla \mathbf{e}_h : \nabla \mathbf{e}_h = \sum_K \{ \int_K \nabla \mathbf{e}_h : \nabla(\mathbf{u} - \mathbf{\Pi u}) + \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h \} \tag{3.8}$$

In the next lemma we bound the consistency term $\sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h$ by using the Raviart-Thomas interpolation.

LEMMA 3.1. *Let* $\mathbf{w}_h = \mathbf{\Pi u} - \mathbf{u}_h$. *Then,*

$$|\sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h| \le C \sum_K \{ h_K \|\mathbf{f}\|_{0,K} + \|\mathbf{M} - \mathbf{RT}(\mathbf{M})\|_{0,K} \} \|\nabla \mathbf{w}_h\|_{0,K} \tag{3.9}$$

Proof. Since $\mathbf{w}_h$ is continuous at the midpoints (*resp.: barycenters*) of the interior edges (*resp.: faces*), vanishes at the midpoints (*resp.: barycenters*) on the boundary and $\mathbf{RT}(\mathbf{M})\nu$ is constant on each edge (*resp.: face*) and continuous across the interelement boundaries, it is easy to see that

$$\sum_K \int_{\partial K} \mathbf{RT}(\mathbf{M})\nu \cdot \mathbf{w}_h = 0$$

then,

$$\sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h = \sum_K \int_{\partial K} (\mathbf{M} - \mathbf{RT}(\mathbf{M}))\nu \cdot \mathbf{w}_h \tag{3.10}$$

Now,

$$\int_{\partial K} (\mathbf{M} - \mathbf{RT}(\mathbf{M}))\nu \cdot \mathbf{w}_h = \int_K \mathbf{div}\ (\mathbf{M} - \mathbf{RT}(\mathbf{M})) \cdot \mathbf{w}_h + \int_K (\mathbf{M} - \mathbf{RT}(\mathbf{M})) : \nabla \mathbf{w}_h \tag{3.11}$$

To bound the first term we observe that from the commutative diagram property (3.1) we have

$$\mathbf{div}\ \mathbf{RT}(\mathbf{M}) = \mathbf{P}(\mathbf{div}\ \mathbf{M})$$

and so, from (3.2) we obtain

$$\int_K \mathbf{div}\ (\mathbf{M} - \mathbf{RT}(\mathbf{M})) \cdot \mathbf{w}_h = -\int_K (\mathbf{f} - \mathbf{Pf}) \cdot \mathbf{w}_h = -\int_K \mathbf{f} \cdot (\mathbf{w}_h - \mathbf{P}\mathbf{w}_h)$$

and so, it follows from (2.3) that

$$|\int_K \mathbf{div}\ (\mathbf{M} - \mathbf{RT}(\mathbf{M})) \cdot \mathbf{w}_h| \le C h_K \|\mathbf{f}\|_{0,K} \|\nabla \mathbf{w}_h\|_{0,K}$$

and therefore, (3.9) follows from this last inequality, and (3.10) and (3.11). □

As a consequence of this lemma we have the following error estimate for the Stokes equations:

THEOREM 3.2. *The following estimate holds with a constant C independent of the shape of the elements:*

$$(\sum_K \|\nabla \mathbf{e}_h\|_{0,K}^2)^{1/2} \le C(\sum_K \|\nabla(\mathbf{u} - \mathbf{\Pi u})\|_{0,K}^2 + \|\mathbf{M} - \mathbf{RT(M)}\|_{0,K}^2 + h_K^2 \|\mathbf{f}\|_{0,K}^2)^{1/2} \tag{3.12}$$

and in particular,

$$(\sum_K \|\nabla \mathbf{e}_h\|_{0,K}^2)^{1/2} \le C(\sum_K h_K^2(\|\mathbf{u}\|_{2,K}^2 + \|\mathbf{f}\|_{0,K}^2) + \|\mathbf{M} - \mathbf{RT(M)}\|_{0,K}^2)^{1/2} \tag{3.13}$$

Proof. It follows from the error equation (3.8) that

$$\sum_K \|\nabla \mathbf{e}_h\|_{0,K}^2 \le \sum_K \|\nabla \mathbf{e}_h\|_{0,K} \|\nabla(\mathbf{u} - \mathbf{\Pi u})\|_{0,K} + |\sum_K \int_{\partial K} \mathbf{M}\nu \cdot \mathbf{w}_h|$$

and therefore, (3.12) is an immediate consequence of (3.9). Now, (3.13) follows from (3.12) and (2.6). □

4. Error estimates for the RT interpolation: the 2d case. In this section we prove optimal order error estimates for the Raviart-Thomas interpolation in the 2d case valid for triangles satisfying the maximum angle condition. First, as a consequence of Lemma 2.2 we obtain error estimates for the normal direction components of the RT interpolation.

LEMMA 4.1. *Let K be a triangle and $\mathbf{w} \in H^1(K)^2$ then, there exists a constant C independent of K such that*

$$\|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K} \le C \sum_{k=1}^{2} l_k (\|\frac{\partial \mathbf{w}}{\partial v_k}\|_{0,K} + \|div\ \mathbf{w}\|_{0,K} |\nu_i \cdot \mathbf{v}_k|)$$

Proof. Since $(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i$ has zero mean value on the edge $l_i \mathbf{v}_i$ we can apply Lemma 2.2 to obtain

$$\|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K} \le C \sum_{k=1}^{2} l_k \|\frac{\partial (\mathbf{w} - RT\mathbf{w}) \cdot \nu_i}{\partial v_k}\|_{0,K} \tag{4.1}$$

with C independent of K.

An easy computation shows that

$$\frac{\partial (RT\mathbf{w} \cdot \nu_i)}{\partial v_k} = \frac{1}{2}(\mathrm{div}\ RT\mathbf{w})\, \mathbf{v}_k \cdot \nu_i$$

and therefore, the result follows from (4.1) and the fact that div $RT\mathbf{w} = P\mathrm{div}\ \mathbf{w}$ which implies,

$$\|\text{div } RT\mathbf{w}\|_{0,K} \le \|\text{div } \mathbf{w}\|_{0,K}$$

□

Up to now, no restrictions were required on the shape of K. In order to have a bound for $\|\mathbf{w} - RT\mathbf{w}\|_{0,K}$, it is reasonable to expect some kind of restrictions on the elements. In view of the previous lemma, it will be enough to have a control of $\mathbf{w} - RT\mathbf{w}$ in terms of its components in the directions of the normals to the edges. Of course, for a fixed triangle, an estimate of $\|\mathbf{w} - RT\mathbf{w}\|_{0,K}$ in terms of the norms of two normal components, say $\|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_1\|_{0,K}$ and $\|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_2\|_{0,K}$, follows by a change of variables. However, if we want to have a uniform constant for a class of elements, we have to see the dependence of the constant involved in that estimate on the geometry of K. Clearly, this constant will depend on the angle between ν_1 and ν_2 and will not degenerate if this angle does not go to 0 or π or, equivalently, if the angle between the corresponding edges does not go to 0 or π. As it is easy to see, this condition will be satisfied if the "maximum angle condition" (i.e.: maximum angle bounded away from π) holds. Indeed, we have the following:

LEMMA 4.2. *Let K be a triangle with maximum angle θ_0. Then, there exists a constant C independent of K such that, for any $\mathbf{w} \in H^1(K)^2$,*

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \le \frac{C}{sin\ \theta_0} \sum_{i=1}^{2} \|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K}$$

where ν_1 and ν_2 are the normals to the edges sharing the vertex corresponding to the maximum angle.

Proof. Let N be the matrix made up with ν_1 and ν_2 as it rows. Then,

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \le \|N^{-1}\| \sum_{i=1}^{2} \|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K}$$

where $\| \ \|$ denotes the matrix norm associated with the euclidean norm. But, since ν_i are unit vectors, it follows that $\|N^{-1}\| \le \frac{C}{|det\,N|}$ and $|det\,N| = \sin\,\theta_1$ where θ_1 is the angle between ν_1 and ν_2. The proof concludes by observing that $\theta_1 = \pi - \theta_0$. □

Consequently, we have the following error estimate for the RT interpolation:

THEOREM 4.3. *Let K be a triangle with maximum angle θ_0. Let l_1, l_2, $\mathbf{v}_1$ and $\mathbf{v}_2$ be the lenghts and directions of the two edges sharing the vertex corresponding to the maximum angle. Then, there exists a constant C independent of K such that, for any $\mathbf{w} \in H^1(K)^2$,*

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \le C \sum_{k=1}^{2} l_k (\|\frac{\partial \mathbf{w}}{\partial v_k}\|_{0,K} + \|div\ \mathbf{w}\|_{0,K})$$

Proof. The result follows immediately from Lemmas 4.1 and 4.2. □

REMARK 3. *Similar arguments can be applied for rectangular elements (see [4] for the definition of the $\mathcal{RT}_0$ space on rectangles and its corresponding projection).*

Indeed, let K be a rectangle with sides of lengths l_1 and l_2. Proceeding as in Lemma 4.1 and using that in this case,

$$\|\frac{\partial (RT\mathbf{w})_i}{\partial x_i}\|_{0,K} \le \|\frac{\partial \mathbf{w}}{\partial x_i}\|_{0,K}$$

we can prove that

$$\|(\mathbf{w} - RT\mathbf{w})_i\|_{0,K} \le C \sum_{k=1}^{2} l_k \|\frac{\partial \mathbf{w}_i}{\partial x_k}\|_{0,K}$$

with a constant C independent of the relation between l_1 and l_2.

5. The 3d case. The objective of this section is to obtain error estimates for the RT interpolation in 3d. First, we show that the arguments given in the previous section can be generalized to obtain a result analogous to Theorem 4.3 for a class of degenerate elements. However, the condition required on the elements for that argument is too restrictive. In fact, using a different argument we show that a sligthly weaker estimate, but still providing optimal order convergence, holds under the maximum angle condition introduced by Krizek [15] which is the natural extension of the 2d one. Moreover, we prove that this condition is necessary and sufficient to have error estimates for the RT interpolation.

As in the 2d case we can obtain a bound for the components of $\mathbf{w} - RT\mathbf{w}$ in the normal directions.

LEMMA 5.1. *Let K be a tetrahedron and $\mathbf{w} \in H^1(K)^3$ then, there exists a constant C independent of K such that*

$$\|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K} \le C \sum_{k=1}^{3} l_k (\|\frac{\partial \mathbf{w}}{\partial v_k}\|_{0,K} + \|div\ \mathbf{w}\|_{0,K} |\nu_i \cdot \mathbf{v}_k|)$$

Proof. It follows from Lemma 2.2 as in the 2d case. $\square$

Now, in order to bound $\|\mathbf{w} - RT\mathbf{w}\|_{0,K}$ we can proceed as in the 2d case. Again, for a fixed element, we can estimate the norm of $\mathbf{w} - RT\mathbf{w}$ in terms of the norms of its components in three normal directions and we have to see the dependence of the constant in this estimate in terms of the geometry of the element. Clearly, the constant will not degenerate for a family of elements having three "uniformly" linearly independent normal directions in the sense that the volume determined by them is uniformly bounded away from zero. The following definition gives a geometric property on the elements which implies that (and is in fact equivalent).

Definition: We say that a tetrahedron K satisfies the *"regular vertex property"* with a constant $\bar{c} > 0$ if K has a vertex $\mathbf{p}_0$ such that $|\det M| \ge \bar{c} > 0$, where M is the matrix made up with $\mathbf{v}_i$, $i = 1, 2, 3$ as rows (where we have used the notation introduced in Section 2).

Now, the error estimate for the RT interpolation will be a consequence of the following lemma and Lemma 5.1.

LEMMA 5.2. *Let K be a tetrahedron satisfying the "regular vertex property" with a constant $\bar{c} > 0$. Then, there exists a constant C depending only on $\bar{c}$ such that for any $\mathbf{w} \in H^1(K)^3$*

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \leq C \sum_{i=1}^{3} \|(\mathbf{w} - RT\mathbf{w}) \cdot \nu_i\|_{0,K}$$

where ν_i are the normal vectors corresponding to the faces sharing the regular vertex $\mathbf{p}_0$.

Proof. Proceeding as in the proof of Lemma 4.2 we see that we have to bound $\|N^{-1}\|$ where N is the matrix with ν_i as its rows. Now, since the coefficients of N are bounded by one we have $\|N^{-1}\| \leq C/|\text{det } N|$ and so, it is enough to see that $|\text{det } N| \geq \overline{\overline{c}} > 0$ with $\overline{\overline{c}}$ depending only on $\overline{c}$. But, an elementary computation shows that

$$|\text{det } N| = \frac{(\text{det } M)^2}{\sin\,\theta_{12}\sin\,\theta_{23}\sin\,\theta_{31}}$$

where θ_{ij} is the angle between $\mathbf{v}_i$ and $\mathbf{v}_j$. Therefore, since the denominator is less than or equal to one, we get

$$|\text{det } N| \geq (\text{det } M)^2 \geq \overline{c}^2$$

which concludes the proof. ☐

Remark 4. *Note that the uniform boundedness below of $|det\ N|$ and $|det\ M|$ are actually equivalent although we have used only one implication. Indeed, with the same argument one can prove that $|det\ M| \geq (det\ N)^2$. So, the "regular vertex property" is a reasonable assumption in order to have the result of the previous lemma.*

Collecting the previous lemmas we have the following error estimate for the RT interpolation:

Theorem 5.3. *Let K be a tetrahedron satisfying the "regular vertex property" with a constant $\overline{c} > 0$. Let l_i and $\mathbf{v}_i$, $i = 1, 2, 3$ be the lenghts and directions of the three edges sharing the regular vertex $\mathbf{p}_0$. Then, there exists a constant C depending only on $\overline{c}$ such that for any $\mathbf{w} \in H^1(K)^3$*

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \leq C \sum_{k=1}^{3} l_k (\|\frac{\partial \mathbf{w}}{\partial v_k}\|_{0,K} + \|div\ \mathbf{w}\|_{0,K}) \tag{5.1}$$

Proof. The result follows immediately from Lemmas 5.1 and 5.2. ☐

Remark 5. *As in the 2d case, the same arguments can be applied to hexahedral elements with faces parallel to the coordinate directions to obtain an estimate analogous to that in Remark 3.*

In the 2d case the *"regular vertex property"* is equivalent to the *"maximum angle condition"*. Indeed, the vertex corresponding to the maximum angle will be a "regular" one.

However, the situation is different in the 3d case. A natural extension of the *"maximum angle condition"* to 3d is the following one introduced by Krizek in [15] for the analysis of error estimates for the Lagrange interpolation.

Definition 5.4. *We say that a tetrahedron satisfies the "maximum angle condition" with a constant $\overline{\psi} < \pi$ if the angles inside the faces and the angles between faces are bounded above by $\overline{\psi}$.*

Remark 6. *In [15], Krizek showed that a regular family of tetrahedra (with the usual definition: ratio between outer and inner diameter bounded) satisfies the "maximum angle condition" with a uniform constant $\overline{\psi}$. It is easy to see that the family shown in Figure 1* a)*, with arbitrary lengths r_1, r_2, r_3, is not regular (for example we can choose $r_1 = r_3 = h^2$, and $r_2 = h$), but it satisfies uniformly both the "maximum angle condition" and the "regular vertex property". Instead, the family in Figure 1* b)*, with arbitrary lengths r_1, r_2, r_3, satisfies uniformly the "maximum angle condition" but not the "regular vertex property" (take for example $r_1 = r_3 = h^2$, and $r_2 = h$). Therefore, the conditions are not equivalent. On the other hand, we will show below that the "regular vertex property" implies the "maximum angle condition".*

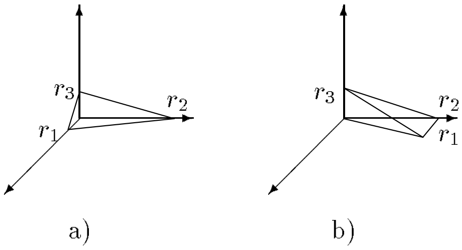

Figure 1

At this point, a natural question is whether or not error estimates for the RT interpolation hold uniformly for a family satisfying the *"maximum angle condition"*. In what follows we will give a positive answer to this question. We will obtain a slightly weaker estimate than that in Theorem 5.3 but still providing optimal order convergence. Moreover, we will show that the *"maximum angle condition"* is necessary and sufficient for the error estimates to hold.

To obtain this result we will use a different approach. Given a function $\mathbf{w} \in H^1(K)^3$ the idea is to find a linear vector function having the same RT interpolation as that of $\mathbf{w}$. This can be done by using the operator $\mathbf{\Pi}$ introduced in Section 2. Indeed, we have the following:

Lemma 5.5. *Let K be a tetrahedron and $\mathbf{w} \in H^1(K)^3$ then,*

$$RT\mathbf{\Pi}\mathbf{w} = RT\mathbf{w}$$

Proof. From the definition of $\mathbf{\Pi}$ it follows that

$$\int_S \mathbf{\Pi}\mathbf{w} \cdot \nu = \int_S \mathbf{w} \cdot \nu$$

for all face S of K. But $\int_S \mathbf{w} \cdot \nu$ are the degrees of freedom defining RT and so the lemma is proven. ∎

The following lemma shows that it is possible to reduce the problem to a finite dimensional one.

LEMMA 5.6. *Let K be a tetrahedron and assume that*

$$\|\mathbf{q} - RT\mathbf{q}\|_{0,K} \le C_1 h_K \|\nabla \mathbf{q}\|_{0,K} \tag{5.2}$$

for all $\mathbf{q} \in \mathcal{P}_1(K)^3$. Then,

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \le (C + C_1) h_K \|\nabla \mathbf{w}\|_{0,K}$$

for all $\mathbf{w} \in H^1(K)^3$, with a constant C independent of K.

Proof. From Lemma 5.5 we have

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \le \|\mathbf{w} - \mathbf{\Pi}\mathbf{w}\|_{0,K} + \|\mathbf{\Pi}\mathbf{w} - RT\mathbf{\Pi}\mathbf{w}\|_{0,K} \tag{5.3}$$

Now, from (5.2) and (2.5) we obtain

$$\|\mathbf{\Pi}\mathbf{w} - RT\mathbf{\Pi}\mathbf{w}\|_{0,K} \le C_1 h_K \|\nabla \mathbf{w}\|_{0,K}$$

and the proof concludes by using (2.4) to bound the first term in (5.3). ∎

In particular, it follows from this lemma that the error estimate for functions in $H^1(K)^3$ will be valid uniformly for a class of elements if this is true for linear polynomials. Therefore, the problem is reduced to a finite dimensional one and we will analize this problem for a family of elements satisfying the "*maximum angle condition*".

In order to do that, we will first find an analytic characterization of the "*maximum angle condition*". In our subsequent arguments we will use the following lemmas.

LEMMA 5.7. *Let K be a tetrahedron satisfying the "maximum angle condition" with a constant $\overline{\psi}$. Given an arbitrary vertex $\mathbf{p}$ of K, let $\theta_1 \ge \theta_2 \ge \theta_3$ the angles between faces sharing $\mathbf{p}$. Then,*

$$\theta_1 > \frac{\pi}{3} \qquad \textit{and,} \qquad \theta_1, \theta_2 \in \left(\frac{\pi - \overline{\psi}}{2}, \overline{\psi}\right] \tag{5.4}$$

Proof. See [15]. ∎

LEMMA 5.8. *Let K be a tetrahedron and $\tilde{K}$ obtained from K by a linear transformation given by a matrix A. If K satisfies the "maximum angle condition" with a constant $\overline{\psi}$ then, $\tilde{K}$ satisfies it with the constant $(1 - \frac{2}{\pi\kappa(A)})\pi + \frac{2}{\pi\kappa(A)}\overline{\psi}$ where $\kappa(A)$ is the condition number of A in the euclidean norm.*

Proof. Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ let us call ϕ the angle between them and $\tilde{\phi}$ the angle between $A\mathbf{v}_1$ and $A\mathbf{v}_2$. Calling $t = \langle A\mathbf{v}_1, A\mathbf{v}_2\rangle / \|A\mathbf{v}_2\|^2$ we have

$$\sin\tilde{\phi} = \frac{\|A(\mathbf{v}_1 - t\mathbf{v}_2)\|}{\|A\mathbf{v}_1\|} \ge \frac{1}{\kappa(A)} \frac{\|\mathbf{v}_1 - t\mathbf{v}_2\|}{\|\mathbf{v}_1\|} \ge \frac{1}{\kappa(A)} \sin\phi$$

where in the last inequality we have used that

$$\sin\,\phi = \min_{t\in\mathbb{R}} \frac{\|\mathbf{v}_1 - t\mathbf{v}_2\|}{\|\mathbf{v}_1\|}$$

therefore,

$$\tilde{\phi} \geq \frac{2}{\pi\kappa(A)}\phi \tag{5.5}$$

Indeed, for $0 \leq \phi \leq \pi/2$, this inequality follows from the previous one and the facts that $2/\pi \leq \sin\,\phi/\phi$ and $\sin\,\tilde{\phi} \leq \tilde{\phi}$. For $\phi > \pi/2$ we divide the angle in two halves and apply the argument to each of them.

On the other hand, applying (5.5) to the complementary angle $\pi - \phi$ we obtain

$$\tilde{\phi} \leq (1 - \frac{2}{\pi\kappa(A)})\pi + \frac{2}{\pi\kappa(A)}\phi \tag{5.6}$$

Now, the bound for the maximum angle of each face of $\tilde{K}$ follows immediately from (5.6). On the other hand, the bound for the angle between two faces of $\tilde{K}$ follows from (5.5) applied to the angle between the corresponding normal vectors and taking into account that these normal vectors are obtained from the corresponding normal vectors of K by the transformation given by the matrix of cofactors of A (see [8, page 53]), which has the same condition number than A. $\square$

It is easy to see that, after a rigid movement, any tetrahedron can be transformed into one like that in Figure 2.

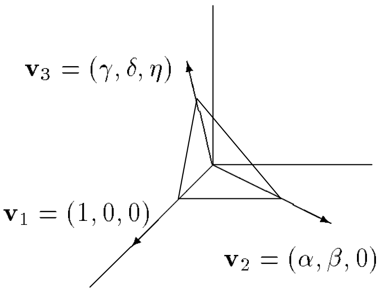

Figure 2

Moreover, we have the following:

LEMMA 5.9. *A tetrahedron K satisfies the "maximum angle condition" with a constant $\overline{\psi}$ if and only if there exists a rigid movement which transforms K in $\tilde{K}$ as in Figure 2, i.e., $\tilde{K} = c.h.\{\mathbf{0}, l_1\mathbf{v}_1, l_2\mathbf{v}_2, l_3\mathbf{v}_3\}$ where $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (\alpha, \beta, 0)$, $\mathbf{v}_3 = (\gamma, \delta, \eta)$ are unitary vectors, $\beta > 0$, $\eta > 0$, $l_i > 0$ and there exists a constant $m_0 = m_0(\overline{\psi})$ such that,*
1) $|\delta/\eta| \leq m_0$
2) $1/\beta \leq m_0$ *or,* $|\frac{\alpha l_2 - l_1}{l_2\beta}| \leq m_0$
3) $1/\eta \leq m_0$ *or,* $|\frac{\gamma l_3 - l_1}{l_3\eta}| \leq m_0$

Proof. First, we prove that the "maximum angle condition" implies the existence of the rigid movement such that *1), 2), 3)* holds.

Let $\mathbf{p}$ be any vertex of K. From Lemma 5.7 we know that the largest angles θ_1, θ_2 between faces sharing $\mathbf{p}$ satisfies (5.4). Let us call F the face such that θ_1 and θ_2 are the angles between it and its neighbor faces . Clearly, there is a rigid movement which transforms $\mathbf{p}$ into the origin and takes the face F to the plane $z = 0$ as in Figure 2 and such that $\beta, \eta > 0$.

We will denote with S_{ij} the faces determined by $\mathbf{v}_i$ and $\mathbf{v}_j$ and with ν_{ij} the corresponding normal vectors.

To prove *1)*, observe first that

$$\nu_{13} = \frac{1}{(\eta^2 + \delta^2)^{1/2}}(0, -\eta, \delta)$$

and $\nu_{12} = (0, 0, -1)$. Now, calling θ the angle between S_{12} and S_{13} (recall that $\theta = \theta_1$ or $\theta = \theta_2$, with θ_i as before) we have

$$\sin^2\theta = \frac{1}{1 + (\frac{\delta}{\eta})^2}$$

and, since $\overline{\psi} \geq \theta > \frac{\pi - \overline{\psi}}{2}$, *1)* holds.

To prove *2)* let us call μ_1 the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ and μ_2 the other angle adyacent to $\mathbf{v}_1$ on the face S_{12}. We have

$$\sin^2\mu_1 = \beta^2$$

and,

$$\sin^2\mu_2 = \frac{1}{(1 + (\frac{l_2\alpha - l_1}{l_2\beta})^2)}$$

Now, since the maximum angle of the face S_{12} is bounded by $\overline{\psi}$, it follows that

$$\sin^2\mu_1 + \sin^2\mu_2 = \beta^2 + \frac{1}{(1 + (\frac{l_2\alpha - l_1}{l_2\beta})^2)}$$

is bounded by below by a positive constant depending on $\overline{\psi}$ and so *2)* holds.

Calling now μ_1 the angle between $\mathbf{v}_1$ and $\mathbf{v}_3$ and μ_2 the other angle adyacent to $\mathbf{v}_1$ on the face S_{13} we have

$$\sin^2\mu_1 = \delta^2 + \eta^2$$

and,

$$\sin^2\mu_2 = \frac{1}{1 + (\frac{l_3\gamma - l_1}{l_3\eta})^2(\frac{1}{1 + (\frac{\delta}{\eta})^2})}$$

and, since the maximum angle of S_{13} is bounded by $\overline{\psi}$,

$$\sin^2\mu_1 + \sin^2\mu_2 = \eta^2(1+\frac{\delta^2}{\eta^2}) + \frac{1}{1+(\frac{l_3\gamma-l_1}{l_3\eta})^2(\frac{1}{1+(\frac{\delta}{\eta})^2})}$$

is bounded by below by a positive constant depending on $\overline{\psi}$ and *3)* follows from this fact together with *1)*.

We now prove the converse. We have to show that, if *1), 2)* and *3)* hold then, the tetrahedron in Figure 2 satisfies the maximum angle condition with a constant depending only on m_0.

Consider the transformations defined by the matrices

$$A_1 = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & \beta & \delta \\ 0 & 0 & \eta \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & (l_2\alpha - l_1)/l_2\beta & \gamma/\eta \\ 0 & 1 & \delta/\eta \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & \alpha & (l_3\gamma - l_1)/l_3\eta \\ 0 & \beta & \delta/\eta \\ 0 & 0 & 1 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & (l_2\alpha - l_1)/l_2\beta & (l_3\gamma - l_1)/l_3\eta \\ 0 & 1 & \delta/\eta \\ 0 & 0 & 1 \end{pmatrix}$$

A_1, A_2, A_3 and A_4 transform into the tetrahedron $\tilde{K}$ of Figure 2 the tetrahedra given in Figures 3 a), b), c) and d) respectively. The tetrahedra of Figure 3 a) and d) are rigid movements of the tetrahedron of Figure 1 a) while those of Figure 3 b) and c) are rigid movements of the tetrahedron of Figure 1 b) which satisfy the maximum angle condition with a constant independent of r_1, r_2 and r_3. On the other hand, under the hypothesis *1), 2)* and *3)*, at least one of the matrices A_j has a condition number bounded by a constant depending only on m_0. Indeed, this follows easily from the fact that $\kappa(A) \le C max|a_{ij}|^3/|\det A|$. So, the proof concludes by using Lemma 5.8. $\square$

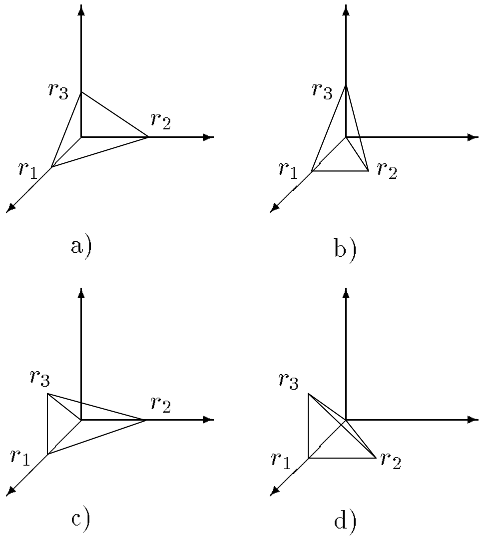

Figure 3

Remark 7. *An immediate consequence of Lemma 5.9 is that the "regular vertex property" implies the "maximum angle condition". Indeed, by applying a rigid movement, it is enough to consider the case in which the tetrahedron is as in Figure 2* *and the origin is a regular vertex (recall that the vertex* $\mathbf{p}$ *transformed into the origin in the proof of Lemma 5.9 was arbitrary). Now, since the origin is a regular vertex it follows that the determinant of the matrix with rows* $\mathbf{v}_i$*, which is equal to* $\beta\eta$*, is bounded by below by a positive constant and then,* β *and* η *are bounded by below by the same constant (recall that they are less than or equal to one). Therefore, the conditions equivalent to the "*maximum angle condition*" given in Lemma 5.9 follows immediately.*

REMARK 8. *In [21] Al Shenk proved that error estimates for the Lagrange interpolation, for several finite element spaces, hold uniformly for a class of narrow elements. This class is obtained from the family of elements in Figure 1* a) *(with arbitrary* r_1*,* r_2 *and* r_3*) by linear transformations with a condition number bounded by a fixed constant. It is not difficult to see that this class is exactly that of elements satisfying the "*regular vertex property*" with a uniform constant* $\overline{c}$*. Our results show that the family of all elements satisfying the "*maximum angle condition*" with a constant* $\overline{\psi}$ *can be obtained in a similar way, but now, transforming the two families given in Figure 1.*

We can now prove the Theorem about convergence for the RT interpolation.

THEOREM 5.10. *Let* K *be a tetrahedron satisfying the "*maximum angle condition*" with a constant* $\overline{\psi}$*. Then, the following estimate holds with a constant* C *depending only on* $\overline{\psi}$*:*

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K} \leq Ch_K\|\nabla\mathbf{w}\|_{0,K} \tag{5.7}$$

for all $\mathbf{w} \in H^1(K)^3$*.*

Proof. In view of Lemma 5.6 we only have to verify the estimate for functions in $\mathcal{P}_1(K)^3$. On the other hand, taking into account Lemma 5.9 we can assume that K is like in Figure 2 and we have to prove that the error estimate holds with a constant depending only on the constant $m_0(\overline{\psi})$ given by that lemma.

First we will check the estimate for the basis

$$B = \{(1,0,0),(0,1,0),(0,0,1),(x,0,0),(y,0,0),(z,0,0),$$

$$(0,x,0),(0,y,0),(0,z,0),(0,0,x),(0,0,y),(0,0,z)\}$$

Since RT interpolates exactly the first three vectors of B, we will deal only with the others.

A straightforward computation gives

$$RT(x,0,0) = \frac{1}{3}(\frac{2(l_2\alpha + l_3\gamma)}{3} + x, y, z)$$

$$RT(y,0,0) = \frac{1}{3}(l_2\beta + l_3\delta, 0, 0)$$

$$RT(z,0,0) = \frac{1}{3}(l_3\eta, 0, 0)$$

$$RT(0,x,0) = \frac{1}{3}(\frac{l_2\alpha - l_1}{\beta}, l_1 + l_3\gamma, 0)$$

$$RT(0,y,0) = \frac{1}{3}(-l_2\alpha + x, l_3\delta + y, z)$$

$$RT(0,z,0) = \frac{1}{3}(0, l_3\eta, 0)$$

$$RT(0,0,x) = \frac{1}{3}(\frac{(l_1 - l_3\gamma)\gamma}{\eta} + \frac{(l_2\alpha - l_1)\delta\alpha}{\beta\eta}, \frac{\delta(l_2\alpha - l_3\gamma)}{\eta}, l_1 + l_2\alpha)$$

$$RT(0,0,y) = \frac{1}{3}(\frac{\delta(l_2\alpha - l_3\gamma)}{\eta}, \frac{-\delta(l_3\delta - l_2\beta)}{\eta}, l_2\beta)$$

$$RT(0,0,z) = \frac{1}{3}(\frac{l_2\alpha - 2l_3\gamma}{3} + x, -l_3\delta + y, z)$$

Now, the estimate (5.7) follows easily for $\mathbf{w} \in B$ computing $\|RT\mathbf{w} - \mathbf{w}\|$ from these expressions and using the conditions of Lemma 5.9. As an example we show that (5.7) holds for $\mathbf{w} = (0,0,y)$. The other cases can be treated analogously. We have

$$RT\mathbf{w} - \mathbf{w} = \frac{1}{3}(\frac{\delta}{\eta}(\alpha l_2 - \gamma l_3), -\frac{\delta}{\eta}(\delta l_3 - \beta l_2), l_2\beta - 3y) \tag{5.8}$$

now, since $\|\nabla\mathbf{w}\|_{0,K} = |K|^{\frac{1}{2}}$, in order to verify (5.7) it is enough to see that

$$|RT\mathbf{w} - \mathbf{w}| \leq h_K \tag{5.9}$$

From Lemma 5.9 we have that δ/η is bounded by a constant depending on $\overline{\psi}$ and so, (5.9) follows immediately from the expression (5.8) taking into account that α, β, δ and γ have absolute value less than or equal to one.

Now, for a general $\mathbf{w} \in \mathcal{P}_1(K)$, calling $\mathbf{w}_i$, the elements of the basis B, we have $\mathbf{w} = \sum_i a_i\mathbf{w}_i$ and therefore,

$$\|\mathbf{w} - RT\mathbf{w}\|_{0,K}^2 \leq C\sum_i a_i^2\|\mathbf{w}_i - RT\mathbf{w}_i\|_{0,K}^2 \leq Ch_K^2\sum_i \|a_i\nabla\mathbf{w}_i\|_{0,K}^2 = Ch_k^2\|\nabla\mathbf{w}\|_{0,K}^2$$

where we have used the orthogonality between $\nabla\mathbf{w}_i$ and $\nabla\mathbf{w}_j$, $i \neq j$ for the last equality. $\square$

REMARK 9. *The "maximum angle condition" is also necessary to have the estimate (5.7). In fact, by using the expressions given in the proof of Theorem 5.10 it is* *easy to construct counterexamples when each of the conditions 1),2) and 3) of Lemma 5.9 does not hold.*

REMARK 10. *We could try to obtain an estimate like (5.1) under the maximum angle condition. Following the steps of the proof of Theorem 5.10 this would be a succesful task until we have to deal with the field* $\mathbf{w} = (0,0,y)$. *For this choice of* $\mathbf{w}$, *it is easily seen that an estimate like (5.1) does not hold with a constant independent of* β *and* l_3 *if we take* $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (\sqrt{1-\beta^2}, \beta, 0)$, $\mathbf{v}_3 = (0, 1/\sqrt{2}, 1/\sqrt{2})$, $l_1 = l_2 = 1$.

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