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# WALKS WITH LONG MEMORY:

# DIFFUSIVE AND SUPER-DIFFUSIVE LIMITS

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**"True" self-avoiding random walk (TSAW), discrete time:**

$$n \mapsto X(n) \in \mathbb{Z}^d,$$

It's local time (occupation time measure):

$$\ell(n,x) := \ell(0,x) + |\{0 < m \leq n : X(m) = x\}|$$

Self-interaction function:

$$w : \mathbb{Z} \to (0,\infty) \quad \text{increasing}$$

The law of the walk:

$$\mathbf{P}\Big( X(n+1) = y \,\Big|\, \mathcal{F}_n, X(n) = x \Big) =$$

$$1\!\!1_{\{\{|x-y|=1\}\}} \frac{w(\ell(n,x) - \ell(n,y))}{\sum_{z:|z-x|=1} w(\ell(n,x) - \ell(n,z))}$$

**TSAW, continuous time:**

$$t \mapsto X(t) \in \mathbb{Z}^d$$

Local time

$$\ell(t,x) := \ell(0,x) + |\{0 < s \le t : X(s) = x\}\,|$$

Rate function:

$$w : \mathbb{R} \to (0,\infty), \qquad \inf_u w(u) = \gamma > 0 \quad \text{(unif. ellipticity)}$$

$$r(u) = \frac{w(u) - w(-u)}{2} \quad \text{increasing}, \qquad s(u) = \frac{w(u) + w(-u)}{2}$$

The law of the walk:

$$\mathbf{P}\Big( X(t+dt) = y \,\Big|\, \mathcal{F}_t, X(t) = x \Big) = 1\!\!1_{\{\{|x-y\,|=1\}\}} w(\ell(t,x) - \ell(t,y))dt$$

**Self-repelling Brownian polymer (SRBP):**

$$t \mapsto X(t) \in \mathbb{R}^d$$

Local time (occupation time measure):

$$\ell(t, A) := \ell(0, A) + |\{0 < s \leq t : X(s) \in A\}|$$

$V : \mathbb{R}^d \to \mathbb{R}$, approximate $\delta$: $C^\infty$, fast decay, positive type:

$$\widehat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x) dx \geq 0 \qquad (*)$$

E.g. $V(x) = e^{-|x|^2}$

The driving force:

$$F : \mathbb{R}^d \to \mathbb{R}^d, \qquad F(x) := -\text{grad}\, V(x).$$

The law of the process:

$$X(t) = B(t) + \int_0^t \int_0^s F(X(s) - X(u))du\, ds,$$

or:

$$dX(t) = dB(t) + \left( \int_0^t F(X(t) - X(u))du \right) dt.$$

or:

$$dX(t) = dB(t) - \text{grad} \left( V * \ell(t, \cdot) \right) (X(t))dt$$

Note: **the position process is pushed by the negative gradient of its own occupation time measure**.

**Question:**
Scaling and (super)diffusive asymptotics of $X(t)$ as $t \to \infty$?

**Roots, history:**

TSAW, physics:

[D. Amit, G. Parisi, L. Peliti (1983)],
[S. Obukhov, L. Peliti (1983)],
[L. Peliti, L. Pietronero (1987)]
...

SRBP, probability:

[J. Norris, C. Rogers, D. Williams (1987)]
[R. Durrett, C. Rogers (1992)],
[M. Cranston, Y. Le Jan (1995)],
[M. Cranston, T. Mountford (1996)],
...

**Conjectures**, based on RG and scaling arguments ("physics"):

- $\mathbf{d = 1}:$ $X(t) \sim t^{2/3}$, intricate, non-Gausssian scaling limit.
  (Limit distributions not identified.)

- $\mathbf{d = 2}:$ $X(t) \sim t^{1/2}(\log t)^{\zeta}$, Gaussian scaling limit.
  (Controversy about the value of $\zeta$.)

- $\mathbf{d \geq 3}:$ $X(t) \sim t^{1/2}$, Gaussian scaling limit.

**Some results: . . .**

- $\mathbf{d = 1}$ : ◦ **Limit thm.** in some particular cases [B. Tóth (AP, 1995)], [B. Tóth, B. Vető (ALEA, 2009)]:

$$\frac{X(t)}{t^{2/3}} \Rightarrow \mathcal{X}.$$

◦ Construction of the **scaling limit process** (TSRM, the Brownian Web, . . . ) [B. Tóth, W. Werner (PTRF, 1998)]

$$t \mapsto \mathcal{X}(t)$$

◦ **"Robust" superdiffusive bounds** [P. Tarrès, B. Tóth, B. Valkó (AP, 2012)]:

$$C_1 t^{5/4} \le \mathbf{E}\Big( X(t)^2 \Big) \le C_2 t^{3/2}.$$

(and more bounds for more general self-interactions)

◦ **Missing:** fully robust proofs.

- $\mathbf{d = 2}$ : ∘ **Super diffusive lower bounds**
  [B. Tóth, B. Valkó (JSP, 2012)]:

$$C_1 t \log\log t \le \mathbf{E}\Big( X(t)^2 \Big) \le C_2 t \log t.$$

  ∘ Expected order:

$$\mathbf{E}\Big( X(t)^2 \Big) \sim t\sqrt{\log t}$$

- $\mathbf{d \ge 3}$ : ∘ **CLT under diffusive scaling**
  [I. Horváth, B. Tóth, B. Vető (PTRF, 2012)]:

$$\frac{X(t)}{t^{1/2}} \Rightarrow N(0, \sigma).$$

## Random walks and diffusions in div-free drift field

**Notation:**

| | |
|---|---|
| $(\Omega, \pi, \tau_z : z \in \mathbb{Z}^d)$ | probability space with ergodic $\mathbb{Z}^d$-action |
| $\mathcal{E} = \{k \in \mathbb{Z}^d : \lvert k \rvert = 1\}$ | possible steps of the rw |
| $v_k : \Omega \to [-1, 1], \quad k \in \mathcal{E}$ | |

- $v_k(\omega) + v_{-k}(\tau_k \omega) \equiv 0$ — vector field
- $\sum_{k \in \mathcal{E}} v_k(\omega) \equiv 0$ — divergence-free
- $\int_\Omega v_k(\omega) d\pi(\omega) = 0,$ — no overall drift

**Lift it** to a stationary and divergence free vector field over $\mathbb{Z}^d$:

$$V_k(\omega, x) := v_k(\tau_x \omega)$$

$$V_{-k}(x+k) + V_k(x) \equiv 0, \qquad \sum_{k \in \mathcal{E}_d} V_k(x) \equiv 0, \qquad \mathbf{E}\Big( V_k(x) \Big) = 0.$$

**The random walk:**

$$\mathbf{P}_\omega\Big( X(t+dt) = x+k \,\Big|\, X(t) = x \Big) = (1 + V_k(\omega, x))\, dt + \mathcal{O}((dt)^2).$$

**The diffusion analogue:**
$V : \mathbb{R}^d \to \mathbb{R}^d$ stationary, divergence-free vector field,

$$dX(t) = dB(t) + V(X(t))dt,$$

**Question:**
Scaling and (super)diffusive asymptotics of $X(t)$ as $t \to \infty$?

**Drift field and its covariances:**

$$\varphi(\omega) := \sum_{k\in\mathcal{E}_d} k v_k(\omega), \qquad \Phi(\omega, x) := \sum_{k\in\mathcal{E}_d} k V_k(\omega, x) = \varphi(\tau_x \omega).$$

$$C_{i,j}(x) := \mathbf{E}\Big( \Phi_i(x)\Phi_j(0) \Big), \qquad \widehat{C}_{i,j}(p) := \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}p\cdot x} C_{i,j}(x)$$

$H_{-1}$**-condition:**

$$(2\pi)^{-d}\int_{[-\pi,\pi]^d} \frac{\sum_{i=1}^d \widehat{C}_{i,i}(p)}{\sum_{i=1}^d (1-\cos(p\cdot e_i))} dp \quad \begin{cases} <\infty & \mathbf{H_{-1}}\checkmark \\ =\infty & \not{\mathbf{H}}_{-1} \end{cases}$$

Equivalently:

$$\lim_{T\to\infty} T^{-1}\mathbf{E}\Big( \Big(\int_0^T \Phi(S(t))dt\Big)^2 \Big) \quad \begin{cases} <\infty & \mathbf{H_{-1}}\checkmark \\ =\infty & \not{\mathbf{H}}_{-1} \end{cases}$$

**Helmholtz's Theorem, stream field:** $\mathbb{Z}^d_* := \mathbb{Z}^d + (1/2, \dots, 1/2)$

$\mathbf{d} = 2$:
There exists a scalar field (*height function*): $H : \Omega \times \mathbb{Z}^2_* \to \mathbb{R}$ with stationary increments such that

$$V = \mathrm{curl}\, H, \qquad V_k(x) = H(x + \frac{k + \widetilde{k}}{2}) - H(x + \frac{k - \widetilde{k}}{2})$$

$\mathbf{d} = 3$:
There exists a vector field (*stream field*) $H_k : \Omega \times \mathbb{Z}^3_* \to \mathbb{R}$, $k \in \mathcal{E}$, with stationary increments such that

$$V = \mathrm{curl}\, H, \qquad V_k(\omega, x) = \dots \text{explain in plain words}$$

The $H_{-1}$ condition equiv.: The height function/stream field is **stationary** (not just of stationary increments!) and $\mathcal{L}^2$.

**Roots, history:**

[Papanicolaou, Varadhan (1981)] diffusion problem formulated

[Osada (1983)] diffusion, CLT with $\mathcal{L}^{\infty}$ stream-field

[Kozlov (1985)] RW, partly incomplete proof, CLT with $\mathcal{L}^{\infty}$ stream-field

[Oelschläger (1988)] diffusion, CLT with $\mathcal{L}^{2}$ stream-field

[Komorowski, Olla (2003)] RW, strong sector condition, CLT with $\mathcal{L}^{\infty}$ stream-field

[Komorowski, Landim, Olla (2012)] RW, CLT with $\mathcal{L}^{\max\{2+\delta,d\}}$ stream-field

- $\mathbf{d=2}$ : ∘ **Super diffusive bounds**
  [B. Tóth, B. Valkó (JSP, 2012)]: $V:\mathbb{R}^2\to\mathbb{R}^2$ curl of (locally smoothed) GFF, $dX(t)=dB(t)+V(X(t))dt$.

$$C_1 t\log\log t\le \mathbf{E}\Big(X(t)^2\Big)\le C_2 t\log t.$$

  ∘ Expected order:

$$\mathbf{E}\Big(X(t)^2\Big)\sim t\sqrt{\log t}$$

- $\mathbf{d\ge 2}$ : ∘ **CLT under $H_{-1}$:**
  [G. Kozma, B. Tóth (preprint, 2014)]: If $H_{-1}$ holds then

$$\frac{X(t)}{t^{1/2}}\Rightarrow N(0,\sigma).$$

**Environment seen from the position of the walker, SRBP:**

$$\eta(t,x) := -\text{grad}\Big(V * \ell(t,\cdot)\Big)(X(t)+x).$$

$t \mapsto \eta(t,\cdot)$ is a **Markov process** with continuous sample path in

$$\Omega := \Big\{\omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}^d) \,:\, \omega \ \text{ grad-field, } \ ||\omega||_{k,m,r} < \infty\Big\}$$

$$||\omega||_{k,m,r} := \sup_{x\in\mathbb{R}^d} \Big(1+|x|\Big)^{-1/r} \left|\partial^{|m|}_{m_1,\dots,m_d}\omega_k(x)\right|$$

**Stationary measure**: by some "miracle", gradient of (mollified) GFF:

$$\langle\omega_k(x)\omega_l(y)\rangle = -\partial^2_{kl}V*\Delta^{-1}(y-x) =: K_{kl}(y-x), \qquad \hat{K}_{kl}(p) = \frac{p_k p_l}{|p|^2}\hat{V}(p).$$

Proof 1: Itô-calculus. Proof 2: Functional analytic.

**Environment seen from the walker, RWDFRE:**

$$\eta(t) := \tau_{X(t)}\omega$$

$t \mapsto \eta(t, \cdot)$ is a **Markov process** with bounded jump rates in $(\Omega, \pi)$ $\pi$ is **stationary and ergodic** for $\eta(t)$, due to div-freeness.

All mentioned results valid in the *stationary regime*.

Put ourselves in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$ and apply various resolvent methods . . .

**Martingale decomposition:**

$$X(t) = M(t) + \int_0^t \varphi(\eta(s))ds$$

∘ $M(t)$: $\mathcal{L}^2$-martingale with stationary and ergodic increments

∘ $\varphi : \Omega \to \mathbb{R}^d$

SRBP: $\varphi(\omega) := \omega(0)$ RWDFRE: $\varphi(\omega) := \sum_{k \in \mathcal{E}_d} k v_k(\omega).$

∘ (partial) decorrelation: easy

**Goals:**

$\mathbf{H_{-1}}$✓: diffusive limit (CLT) for the second term on the r.h.s.
— try non-reversible Kipnis-Varadhan theory

~~$\mathbf{H_{-1}}$~~ : superdiffusive bound for var. of the second term on r.h.s.
— try Landim-Quastel-Salmhofer-Yau method

**SRBP: Gaussian Hilbert Space (Fock space / Wiener space):**

$$\mathcal{L}^2(\Omega,\pi) =: \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

The infinitesimal generator acting on $\mathcal{L}^2(\Omega,\pi)$:

$$G = \Delta + \sum_{l=1}^{d} \left(\nabla_l a_l + a_l^* \nabla_l\right) = -S + A_- + A_+,$$

where

$$a_l^* :\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n): \; = \; :\omega_l(0)\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n):$$

$$a_l :\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n): \; = \sum_{m=1}^{n} K_{lk_m}(x_m) :\omega_{k_1}(x_1)\cdots \cancel{\omega_{k_m}(x_m)} \cdots\omega_{k_n}(x_n):$$

Proof: careful use of commutation relations, plus "directional derivative" identity (a la Malliavin calculus).

**SRBP, Proofs:**

**Diffusive limits in $\mathbf{d \geq 3}$:**
Non-reversible Kipnis-Varadhan theory: $H_{-1}$-bound and *graded sector condition*
[S. Sethuraman, S.R.S. Varadhan, H-T. Yau (2000)]
— with improvement on conditions of applicability. □

**Superdiffusive lower bound in $\mathbf{d = 2}$:**
Variational approach of
[C. Landim, J. Quastel, M. Salmhofer, H-T. Yau (2004)]
— with particularities . . . □

**RWDFRE, some details:**

**Some operators** on the Hilbert space $\mathcal{L}^2(\Omega, \pi)$:

$\mathcal{L}^2(\Omega, \pi)$-gradient : $\qquad \nabla_k f(\omega) := f(\tau_k \omega) - f(\omega)$

$$\nabla_k^* = \nabla_{-k}$$

$\mathcal{L}^2(\Omega, \pi)$-Laplacian : $\qquad \Delta f(\omega) := \sum_{k \in \mathcal{E}} \left( f(\tau_k \omega) - f(\omega) \right)$

$$\Delta^* = \Delta \leq 0$$

multiplication ops. : $\qquad M_k f(\omega) := v_k(\omega) f(\omega)$

$$M_k^* = M_k$$

A commutation relation – due to div-freeness of $v$:

$$\sum_{k\in\mathcal{E}} M_k \nabla_k + \sum_{k\in\mathcal{E}} \nabla_{-k} M_k = 0$$

The **infinitesimal generator** of the environment process:

$$L = P - I = \frac{1}{2}\Delta + \sum_{k\in\mathcal{E}} M_k \nabla_k =: -S + A$$

**Relaxed Sector Condition** [I. Horváth, B. Tóth, B. Vető (2012)]

**Theorem:** Efficient martingale approximation (a la Kipnis-Varadhan) holds for $\int_0^t \varphi(\eta_s) ds$ if

| | | |
|---|---|---|
| (1) | " $S^{-1/2} A S^{-1/2}$ " | is *skew self-adjoint* (not just skew symmetric). |
| (2) | $\varphi \in \mathtt{Ran}(S^{-1/2})$ | $H_{-1}$-condition |

**Remarks:**

(1) Extends Varadhan et al.'s *Graded Sector Condition*.

(2) Proof: partly reminiscent of Trotter-Kurtz.

What is missing from skew self-adjointmess of $B = S^{-1/2} A S^{-1/2}$? (defined on an appropriately chosen dense subspace)

**von Neumann's criterion**:

$$\begin{pmatrix} B \ \text{ skew symmetric, and} \\ \overline{\mathtt{Ran}(B \pm I)} = \mathcal{H} \end{pmatrix} \Leftrightarrow \begin{pmatrix} B \ \text{ essentially} \\ \text{skew self-adjoint} \end{pmatrix}$$

Needed:

$$\sum_{k \in \mathcal{E}} M_k \Big( (-\Delta)^{-1/2} \nabla_k \Big) \psi = (-\Delta)^{1/2} \psi \qquad \Rightarrow \qquad \psi = 0.$$

**Warning:** Formal manipulation deceives: $\psi \notin \mathtt{Dom}(-\Delta)^{-1/2}$!

Raise it to the lattice $\mathbb{Z}^d$:

**Wanted:**
**NO nontrivial scalar field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ with stationary increment, and $\mathbf{E}\Big( \Psi \Big) = 0$ solves the PDE**

$$\Delta \Psi = V \cdot \nabla \Psi.$$

Note similarity: No sublinearly growing harmonic function on $\mathbb{Z}^d$.

$\mathbf{d = 2}$ : with bare hands

$\mathbf{d \geq 3}$ : use Nash inequality or [Morris, Peres (2005)]: heat-kernel upper bound

□

## The talk was based on the following papers:


B. Tóth: The 'true' self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems. *Ann. Probab.* **23**: 1523–1556 (1995)

B. Tóth, W. Werner: The true self-repelling motion. *Probab. Theory Related Fields* **111**: 375–452 (1998)

B. Tóth, B. Vető: Continuous time 'true' self-avoiding random walk on $\mathbb{Z}$. *ALEA, Lat. Am. J. Probab. Math. Stat.* **8:** 5975 (2011)

I. Horváth, B. Tóth, B. Vető: Diffusive limits for "true" (or myopic) self-avoiding random walks and self-repellent Brownian polymers in three and more dimensions. *Probab. Theory Rel. Fields* **153**: 691-726 (2012)

I. Horváth, B. Tóth, B. Vető: Relaxed sector condition. *Bull. Inst. Math. Acad. Sin. (N.S.)* **7**: 463-476 (2012)

P. Tarrès, B. Tóth, B.Valkó: Diffusivity bounds for 1d Brownian polymers. *Ann. Probab.* **40**: 695-713 (2012)

B. Tóth, B. Valkó: Superdiffusive bounds on self-repellent Brownian polymers and diffusion in the curl of the Gaussian free field in d=2. *J. Stat. Phys.* **147**: 113-131 (2012)

G. Kozma, B. Tóth: Central limit theorem for random walks in divergence-free random drift field: $\mathcal{H}_{-1}$ suffices. *preprint* (2014)