The one-dimensional KPZ equation and its universality

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Based on collaborations with

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Plan of the talk

For non-experts

- Basics

  What is the KPZ equation?

  The KPZ equation is not really well-defined.

- Explicit formula for height distribution

  Tracy-Widom distributions from random matrix theory

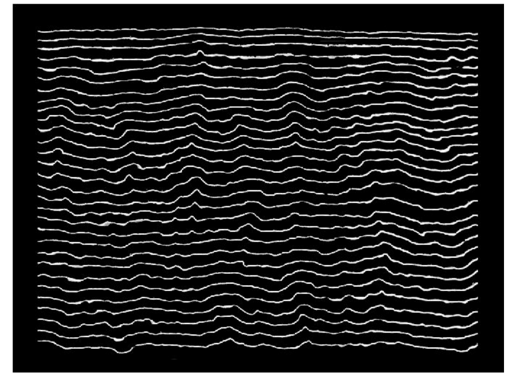
  Behind the tractability $\cdots$ Stochastic integrability

- Universality

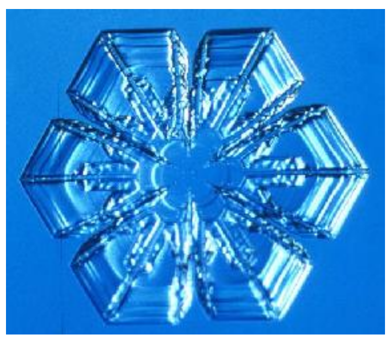
  "KPZ is everywhere"

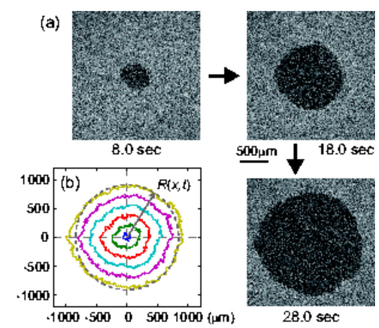
1. Basics of the KPZ equation: Surface growth

- Paper combustion, bacteria colony, crystal growth, etc



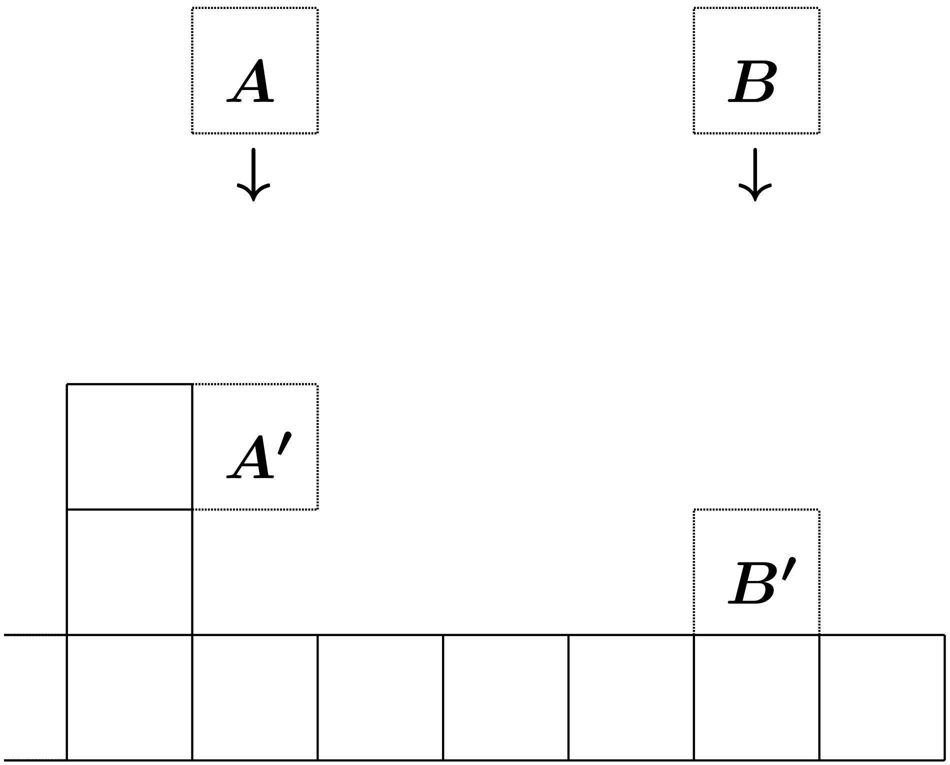
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Connections to integrable systems, representation theory, etc





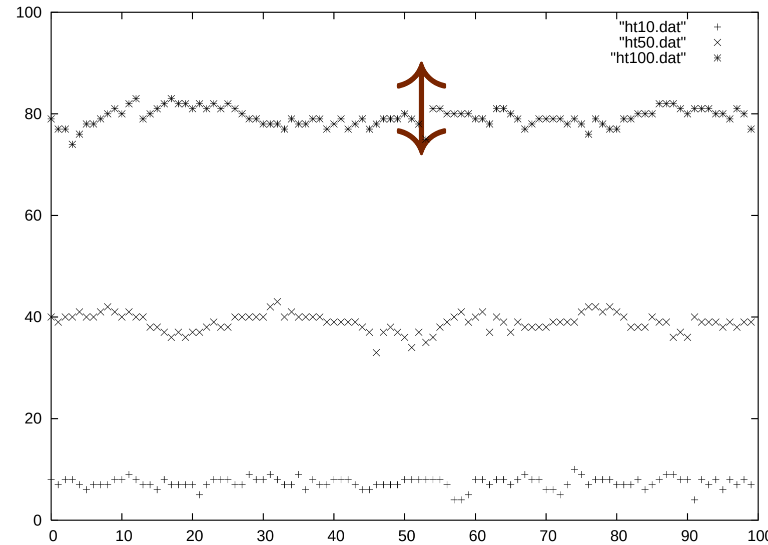
Simulation models

Ex: ballistic deposition



Height fluctuation

$$O(t^{\beta}), \beta = 1/3$$



Flat

KPZ equation

$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

By a simple scaling we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

The KPZ equation now looks like

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

"Derivation"

- Diffusion $$\partial_t h(x,t) = \frac{1}{2}\partial_x^2 h(x,t)$$

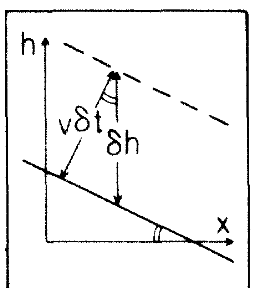
  Not enough: no fluctuations in the stationary state

- Add noise: Edwards-Wilkinson equation

$$\partial_t h(x,t) = \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

  Not enough: does not give correct exponents

- Add nonlinearity $(\partial_x h(x,t))^2 \Rightarrow$ KPZ equation



$$\begin{aligned}\partial_t h &= v\sqrt{1+(\partial_x h)^2}\\ &\simeq v + (v/2)(\partial_x h)^2 + \dots\end{aligned}$$

Dynamical RG analysis: $\to \beta = 1/3$ (KPZ class)

If we set

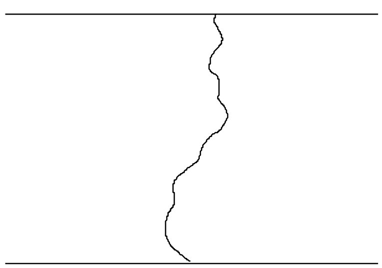
Cole-Hopf transformation

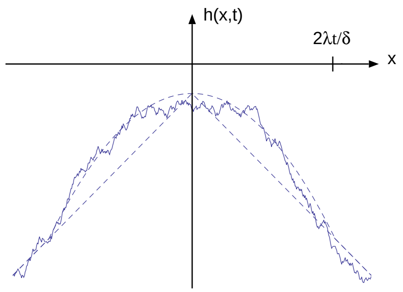
$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity (formally) satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.





The polymer from the origin: $Z(x,0) = \delta(x) = \lim_{\delta\to 0} c_\delta e^{-|x|/\delta}$ corresponds to narrow wedge for KPZ.

The KPZ equation is not well-defined

- With $\eta(x,t)$" $=$ "$dB(x,t)/dt$, the equation for Z can be written as (Stochastic heat equation)

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t) \times dB(x,t)$$

  Here $B(x,t)$ is the cylindrical Brownian motion with covariance $dB(x,t)dB(x',t) = \delta(x-x')dt$.

- Interpretation of the product $Z(x,t) \times dB(x,t)$ should be Stratonovich $Z(x,t) \circ dB(x,t)$ since we used usual calculus. Switching to Ito by $Z(x,t) \circ dB(x,t) = Z(x,t)dB(x,t) + dZ(x,t)dB(x,t)$, we encounter $\delta(0)$.

- On the other hand SHE with Ito interpretation from the beginning

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t)dB(x,t)$$

  is well-defined. For this Z one can define the "Cole-Hopf" solution of the KPZ equation by $h = \log Z$.

  So the well-defined version of the KPZ equation may be written as

$$\partial_t h(x,t) = \tfrac{1}{2}(\partial_x h(x,t))^2 + \tfrac{1}{2}\partial_x^2 h(x,t) - \infty + \eta(x,t)$$

- Hairer found a way to define the KPZ equation without but equivalent to Cole-Hopf (using ideas from rough path and renormalization).

2. Explicit formula for the 1D KPZ equation

Thm (2010 TS Spohn, Amir Corwin Quastel)

For the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ)

$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \det(1 - K_{s,t})$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$ is

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

Explicit formula for the height distribution

Thm

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

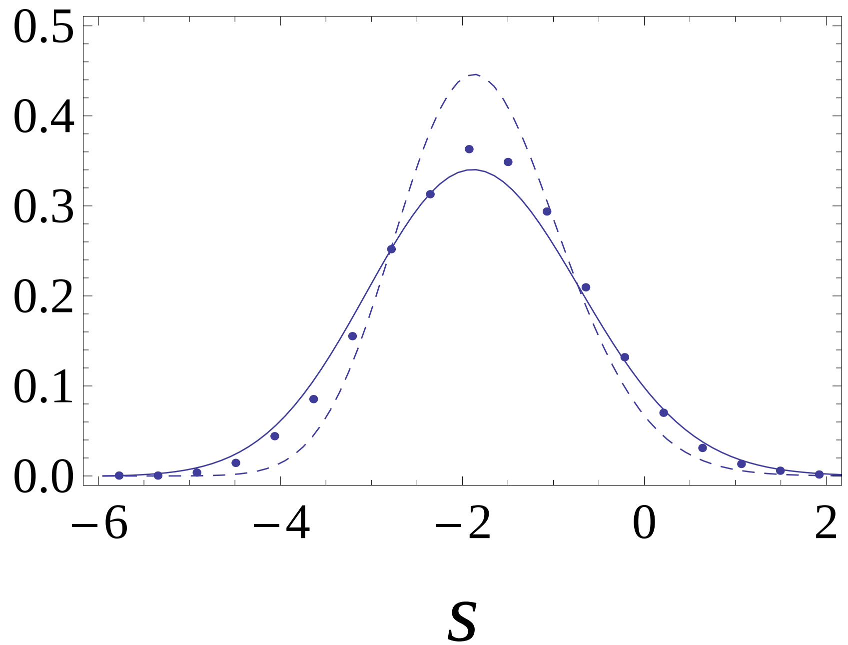
where $\gamma_t = (t/2)^{1/3}$. The distribution function of ξ_t is

$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big]$$
$$\times \Big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\Big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t \lambda} - 1}$$

Finite time KPZ distribution and TW



—: exact KPZ density $F_t'(s)$ at $\gamma_t = 0.94$

— —: Tracy-Widom density

- In the large t limit, F_t tends to the GUE Tracy-Widom distribution F_2 from random matrix theory.

Tracy-Widom distributions

For GUE (Gaussian unitary ensemble) with density $P(H)dH \propto e^{-\mathrm{Tr}H^2}dH$ for H: $N \times N$ hermitian matrix, the joint eigenvalue density is (with $\Delta(x)$ Vandelmonde)

$$\frac{1}{Z}\Delta(x)^2 \prod_i e^{-x_i^2}$$

GUE Tracy-Widom distribution

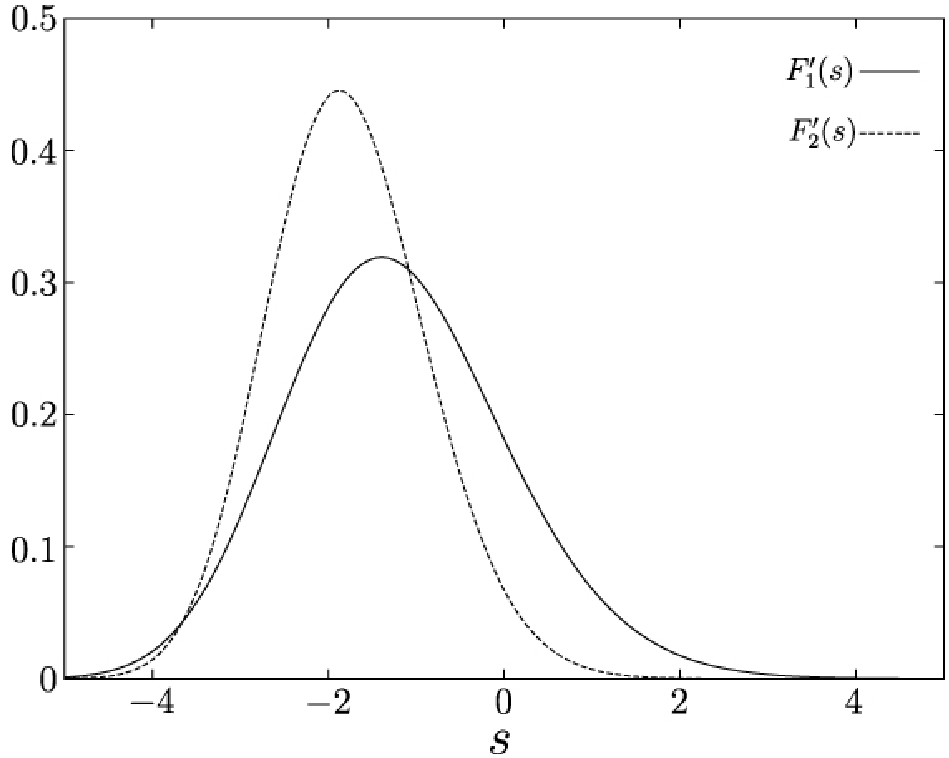
$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max} - \sqrt{2N}}{2^{-1/2}N^{-1/6}} < s\right] = F_2(s) = \det(1 - P_s K_2 P_s)$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

There is also GOE TW (F_1) for GOE (Gaussian orthogonal ensemble, real symmetric matrices).

Probability densities of Tracy-Widom distributions



F_2'(GUE), F_1'(GOE)

Derivation of the formula by replica approach

Dotsenko, Le Doussal, Calabrese

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for narrow wedge case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the (attractive) δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

We are interested not only in the average $\langle h \rangle$ but the full distribution of h. We expand the quantity of our interest as

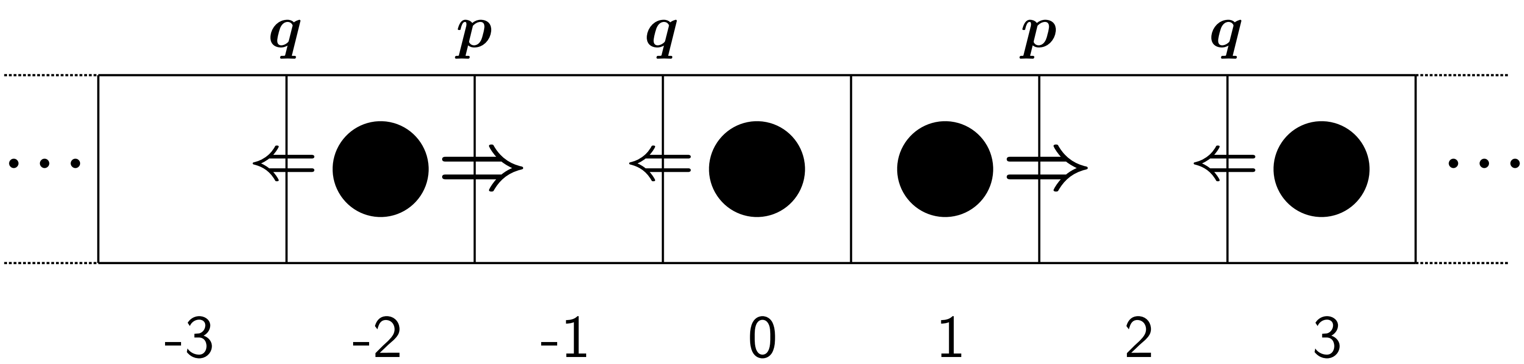
$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N\frac{\gamma_t^3}{12}}$$

Using the integrability (Bethe ansatz) of the δ-Bose gas, one gets explicit expressions for the moment $\langle Z^n \rangle$ and see that the generating function can be written as a Fredholm determinant. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$!

One should consider regularized discrete models.

ASEP

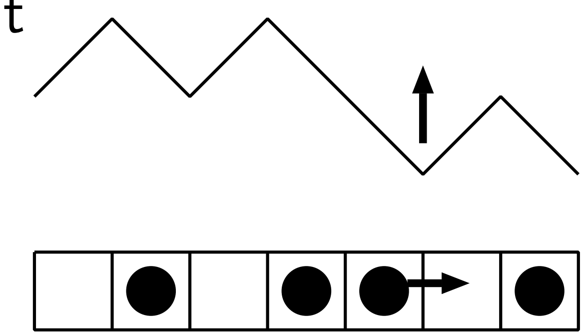
ASEP $=$ asymmetric simple exclusion process



- TASEP(Totally ASEP, $p = 0$ or $q = 0$)
- $N(x,t)$: Integrated current at $(x, x+1)$ upto time t

  $\Leftrightarrow$ height for surface growth
- In a certain weakly asymmetric limit

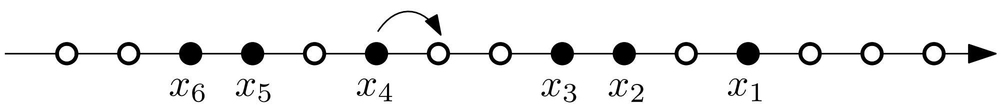
  ASEP $\Rightarrow$ KPZ equation

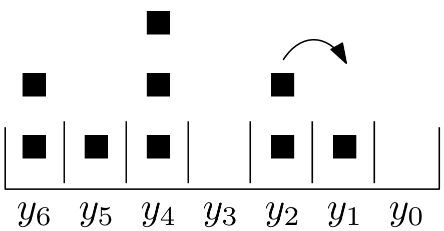


q-TASAEP and q-TAZRP

- q-TASEP 2011 Borodin-Corwin

  A particle i hops with rate $1 - q^{x_{i-1}-x_i-1}$.





- q-TAZRP 1998 TS Wadati

  The dynamics of the gaps $y_i = x_{i-1} - x_i - 1$ is a version of totally asymmetric zero range process in which a particle hops to the right site with rate $1 - q^{y_i}$. The generator of the process can be written in terms of q-boson operators.

- $N(x, t)$: Integrated current for q-TAZRP

Rigorous replica

2012 Borodin-Corwin-TS

- For ASEP and q-TAZRP, the n-point function like $\langle \prod_i q^{N(x_i,t)} \rangle$ satisfies the n particle dynamics of the same process (Duality). This is a discrete generalization of δ-Bose gas for KPZ. One can apply the replica approach to get a Fredholm det expression for generating function for $N(x,t)$.

- Rigorous replica: the one for KPZ (which is not rigorous) can be thought of as a shadow of the rigorous replica for ASEP or q-TAZRP.

- BCS+Petrov Plancherel theorem, more generalizations to come soon!

Various generalizations and developments

- Flat case (replica) (Le Doussal, Calabrese)

  The limiting distribution is GOE TW F_1 (Geometry dependence)

- Multi-point case (replica) (Dotsenko)

- Stochastic integrability...Connections to quantum integrable systems

  quantum Toda lattice, XXZ chain, Macdonald polynomials...

  ($\Rightarrow$ Talk by Ivan Corwin)

Stationary 2pt correlation

Not only the height/current distributions but correlation functions show universal behaviors.

- For the KPZ equation, the Brownian motion is stationary.

$$\boldsymbol{h(x,0) = B(x)}$$

where $\boldsymbol{B(x), x} \in \mathbb{R}$ is the two sided BM.

- Two point correlation

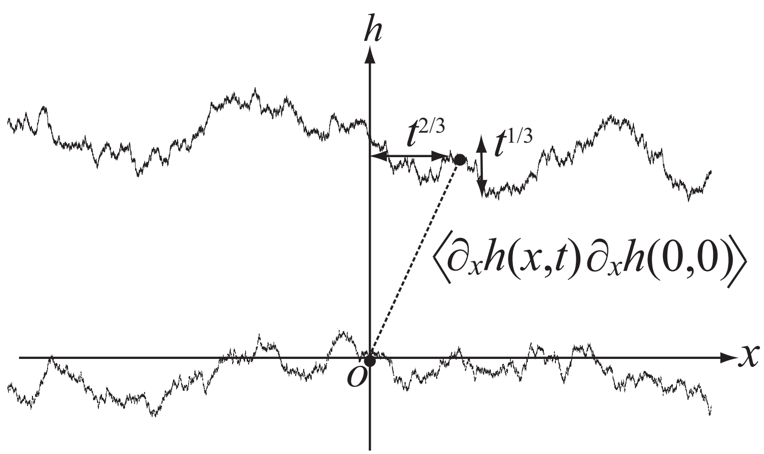
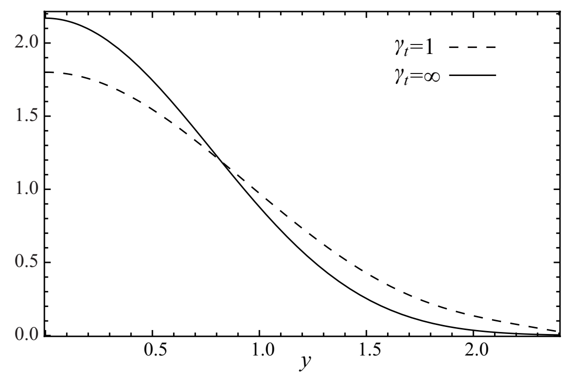


Figure by exact solution

Imamura TS (2012)

$$\langle \partial_x h(x,t) \partial_x h(0,0) \rangle = \frac{1}{2}(2t)^{-2/3} g_t''(x/(2t)^{2/3})$$

The figure can be drawn from the exact formula (which is a bit involved though).

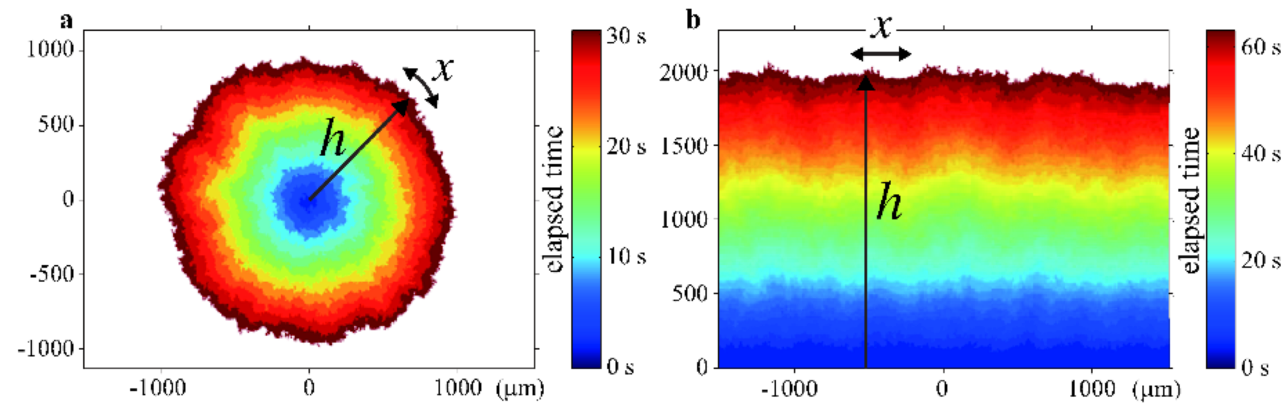


Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t := (\frac{t}{2})^{\frac{1}{3}} = 1$.
The solid curve is the scaling limit $g''(y)$.

3. Universality

- The Tracy-Widom distributions appear in various contexts (Universality).
- A simplest example of universality is the central limit theorem. For any independent random variables with moment conditions CLT holds.
- Understanding of universality of TW distributions from the context of random matrix theory has been developed.
- Its universality from the context of surface growth or directed polymer has been much less well understood.

Universality 1: Expeirments by Takeuchi-Sano



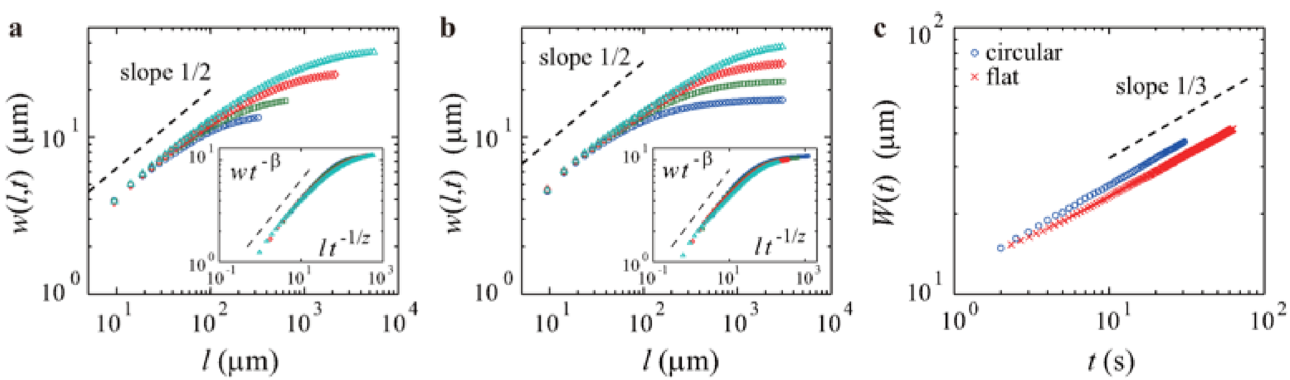


Figure 2 | **Family-Vicsek scaling.** a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

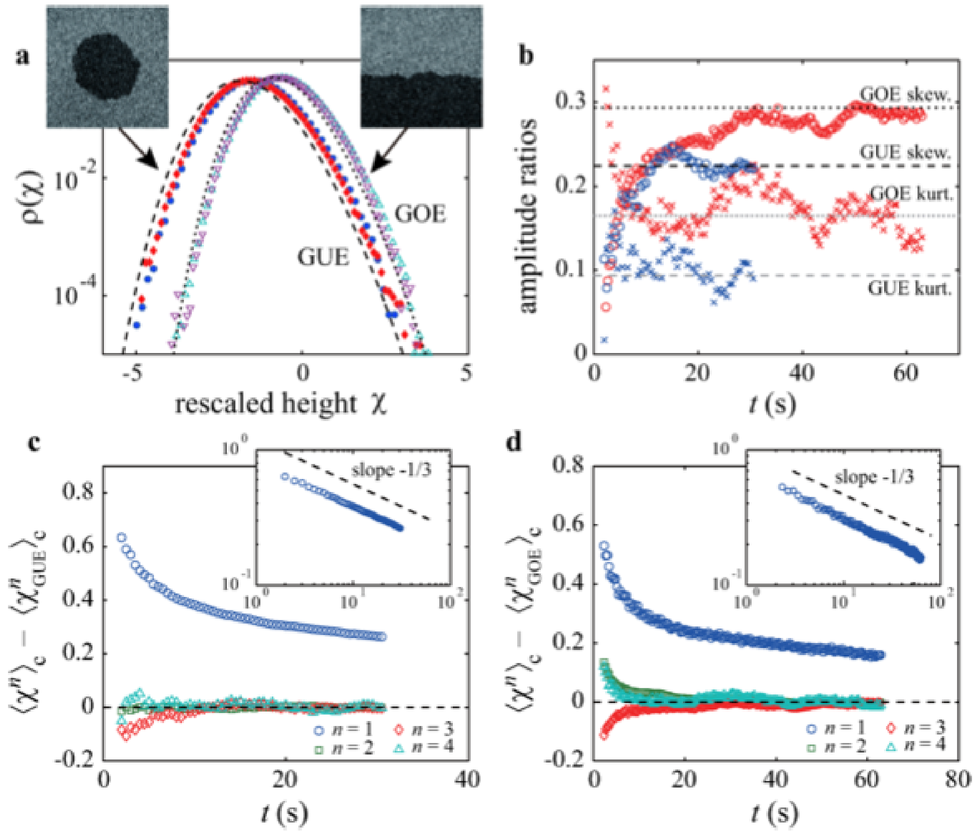


Figure 3 | Universal fluctuations. a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle \chi^n \rangle_c$ and the corresponding TW distributions $\langle \chi^n_{\mathrm{GUE}} \rangle_c$ for the ciruclar interfaces (c) and $\langle \chi^n_{\mathrm{GOE}} \rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

Takeuchi Sano TS Spohn, Sci. Rep. 1,34(2011)

Universality 2: Beijeren-Spohn Conjecture

- The scaled KPZ 2-pt function would appear in rather generic 1D multi-component systems

  This would apply to (deterministic) 1D Hamiltonian dynamics with three conserved quantities, such as the Fermi-Pasta-Ulam chain with $V(x) = \frac{x^2}{2} + \alpha \frac{x^3}{3!} + \beta \frac{x^4}{4!}$.

  There are two sound modes with velocities $\pm c$ and one heat mode with velocity 0. The sound modes would be described by KPZ; the heat mode by $\frac{5}{3}$—Levy.

- Now there have been several attempts to confirm this by numerical simulations. Mendl, Spohn, Dhar, Beijeren, $\cdots$

- If nonlinearity vanishes, can show appearance of Levy modes and/or diffusive modes (Olla), but in general difficult to prove.

Mendl Spohn

MD simulations for shoulder potential

$$V(x) = \infty \ (0 < x < \tfrac{1}{2}), 1(\tfrac{1}{2} < x < 1), \ 0(x > 1)$$

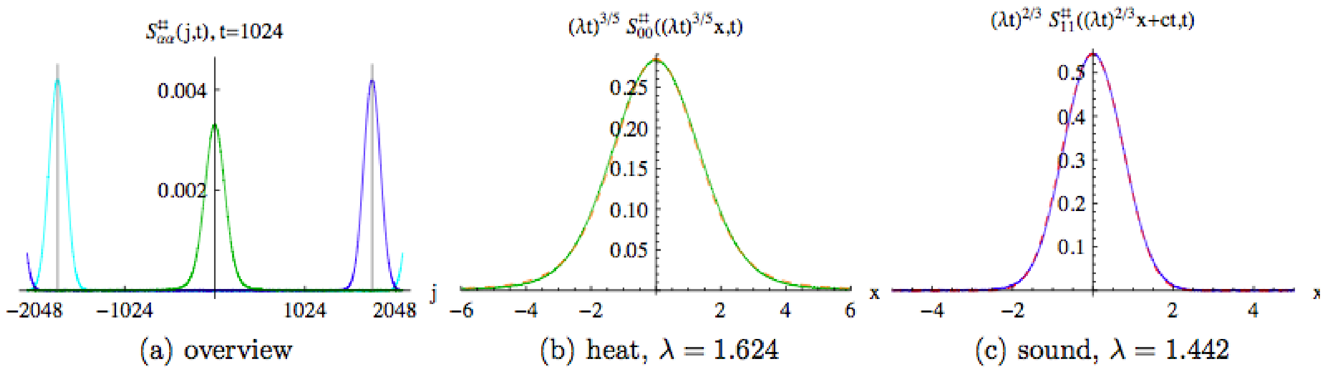


Figure 1: (Color online) MD simulation of an equal mass chain with shoulder potential as defined in Eq. (2.2) and parameters $N = 4096$, $p = 1.2$, $\beta = 2$, at $t = 1024$. (a) Diagonal matrix entries, $S^{\sharp}_{\alpha\alpha}(j,t)$, of the correlator. The gray vertical lines show the sound speed predicted from theory. The tails of the sound peaks reappear on the opposite side due to periodic boundary conditions. (b) Rescaled heat and (c) right sound peak. The theoretical scaling exponents are used and λ is fitted numerically to minimize the L^1-distance between simulation and prediction. The dashed orange curve is the predicted $\frac{5}{3}$-Levy distribution $f_{\mathrm{L},5/3}$ and the dashed red curve shows f_{KPZ}.

Stochastic model

The conjecture would hold also for stochastic models with more than one conserved quantities.

Arndt-Heinzel-Rittenberg(AHR) model (1998)

- Rules

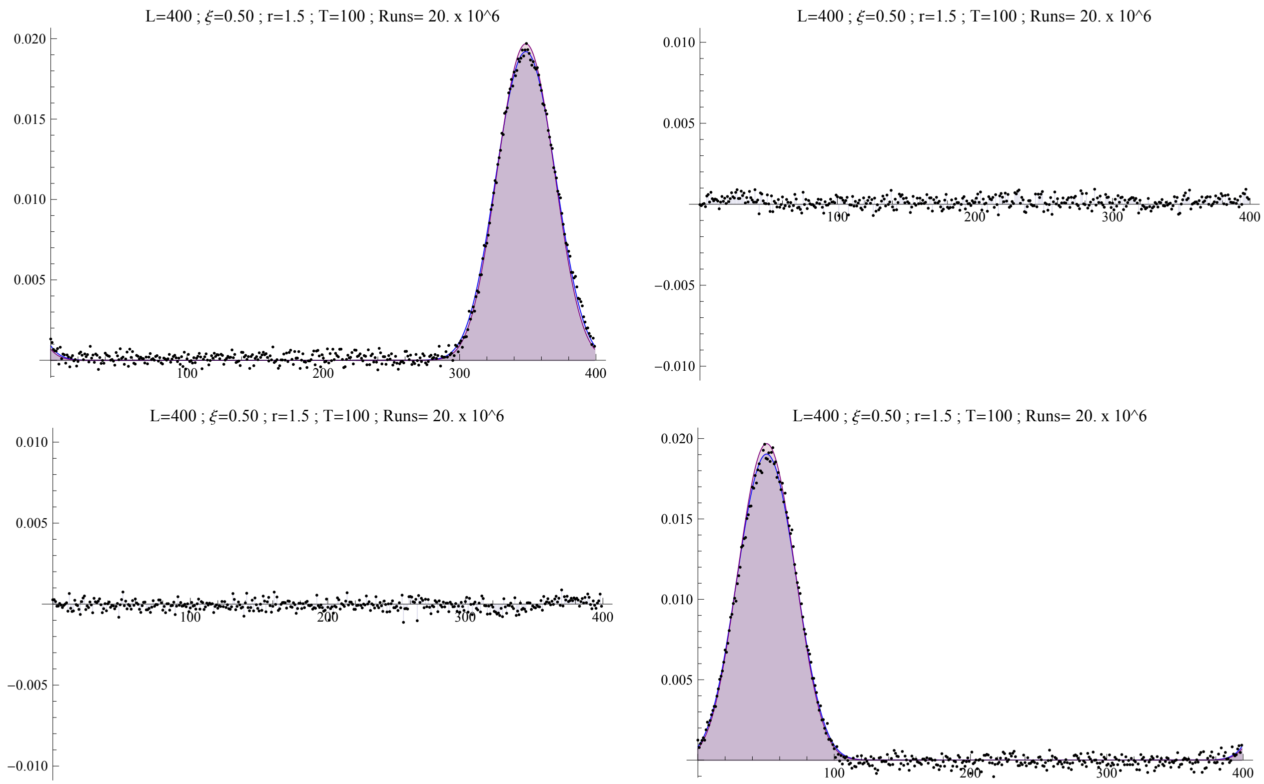
$$+\,\mathbf{0} \xrightarrow{\boldsymbol{\alpha}} \mathbf{0}\,+$$

$$\mathbf{0}\,- \xrightarrow{\boldsymbol{\alpha}} -\,\mathbf{0}$$

$$+\,- \xrightarrow{\mathbf{1}} -\,+$$

- Two conserved quantities (numbers of $+$ and $-$ particles).
- Exact stationary measure is known in a matrix product form.

2013 Ferrari TS Spohn



The KPZ 2pt correlation describes those for the two modes.

Proving the conjecture for this process seems already difficult.

KPZ in higher dimension?

In higher dimensions, there had been several conjectures for exponents. There are almost no rigorous results.

2012 Halpin-Healy

New extensive Monte-Carlo simulations in 2D on the distributions.

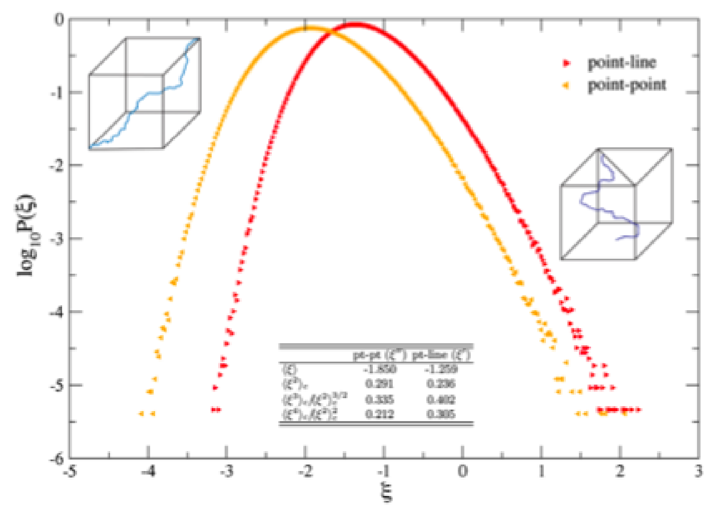


FIG. 4 (color online). Universal PDFs: 2 + 1 DPRM *point-point* and *point-line* geometries. Table inset: Distribution moments.

New universal distributions?

4. Summary

- KPZ equation is a model equation to describe surface growth. It is considered to be of fundamental importance from several points of view.
- One can write down fairy compact explicit formula for its height distribution. This is related to nice algebraic structures behind the equation.
- There is a strong universality associated with the KPZ equation. Understanding its nature is an outstanding challenge for the future.