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# A quantitative theory in stochastic homogenization

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## Random walks in random environments:

## elements of a quantitative homogenization theory

Corrector: Existence of stationary corrector,
identification of covariance structure

Environment viewed from particle:
Quantified ergodicity

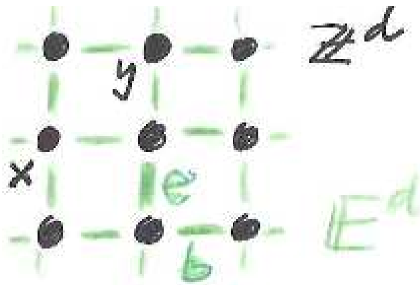
Ingredients of the proof:
Calculus based on spectral gap for Glauber dynamics
and on estimates for elliptic PDE

## My notation: discrete derivatives

**Lattice**,

sites $x$, $y \in \mathbb{Z}^d$

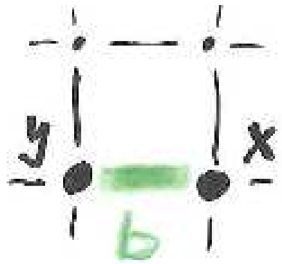
bonds $b$, $e \in \mathbb{E}^d$




**Gradient** $\nabla$.

Scalar field $\zeta\colon \mathbb{Z}^d \to \mathbb{R}$ $\rightsquigarrow$ vector field $\nabla\zeta\colon \mathbb{E}^d \to \mathbb{R}$

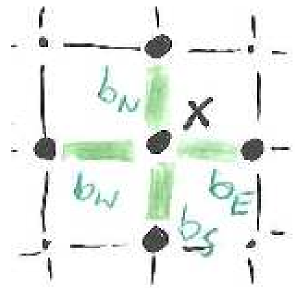
$\nabla\zeta(b) = \zeta(x) - \zeta(y)$




**(negative) Divergence** $\nabla^*$. $\ell^2$-adjoint.

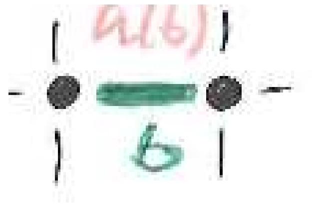
Vector field $g\colon \mathbb{E}^d \to \mathbb{R}$ $\rightsquigarrow$ scalar field $\nabla^* g\colon \mathbb{Z}^d \to \mathbb{R}$

$\nabla^* g(x) = g(b_E) + g(b_N) - g(b_W) - g(b_S)$



# Discrete elliptic operator and its interpretation

**Coefficients** $a$.

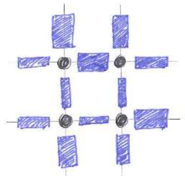



Tensor field $a\colon \mathbb{E}^d \to \mathbb{R}$,

Uniformly elliptic: $\exists\, \lambda > 0 \quad \forall b \in \mathbb{E}^d \quad \lambda \le a(b) \le 1$.

**Elliptic operator**: $\nabla^* a \nabla$ maps scalar fields on scalar fields

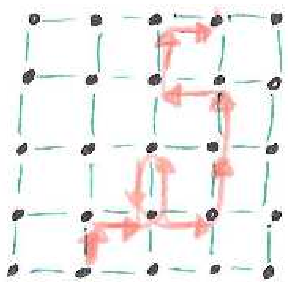
## Network of resistors



$a$ conductances, $a^{-1}$ resistance, $u$ potential,

$j = a\nabla u$ current, stationary iff $\nabla^* j = 0$

## Random walk in heterogen. environm.



generator given by $\nabla^* a \nabla$

reversible w. r. t. uniform distribution,

time continuous

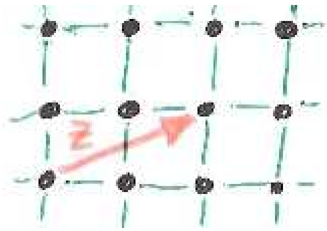
# Random discrete elliptic operator

Field of coefficients $a$ is random variable,

ensemble average $\langle \cdot \rangle$

Simplest setting: $\{a(b)\}_{b \in \mathbb{E}^d}$ are **i**ndependent and **i**dentically **d**istributed

Most general setting: $\mathbb{Z}^d$ acts on space of $a$'s by translation

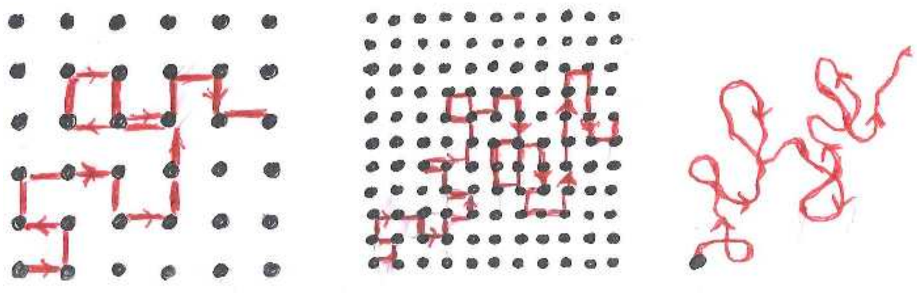



Stationarity: $\forall\, z \in \mathbb{Z}^d \quad a$ and $a(\cdot + z)$ have same distribution

Ergodicity: If $\forall\, z \in \mathbb{Z}^d \;\; \zeta(a(\cdot + z)) = \zeta(a)$ then $\zeta = \langle \zeta \rangle$ a. s.

# Qualitative homogenization = invariance principle

Provided $\langle\cdot\rangle$ is stationary and ergodic,



rescaled random walk
$\overset{\text{law}}{\to}$ Brownian motion
with covariance $a_{hom}$
for $\langle\cdot\rangle$-almost every $a$

[Kozlov '79, Papanicolaou & Varadhan '79, ...
Sidoravicius & Sznitman '04]

## Two opposite research directions ...

**Probability community**:

Push *qualitative* theory to *borderline* situations

(e. g. supercritical percolation clusters

[Biskup&Berger '07, Mathieu&Piatnitski '07])

**Applied Math community**:

Develop *quantitative* theory in *standard* situation

(e. g. homogenization error,

error in "representative volume element method")

[Yurinskii '86, Naddaf&Spencer '98, Conlon&Naddaf '00,

Mourrat '11, Armstrong&Smart '14]

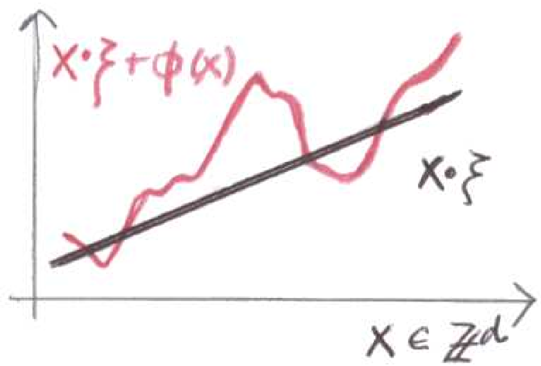
**... this talk is about the latter**

## Corrector

**Existence of stationary corrector**

**Identification of covariance structure**

# Corrector: incomplete definition




Given affine function $x \cdot \xi$, ($\xi \in \mathbb{R}^d$, $|\xi| = 1$),

find $\phi$ s. t. $\phi(x) + x \cdot \xi$ is $a$-harmonic:

$$\nabla^* a(\nabla \phi + \xi) = 0.$$

Boundary conditions for $\phi$ at $|x| \uparrow \infty$?

(almost sure sublinear growth ok for existence and uniqueness [Benjamini & Duminil-Copin & Kozma & Yadin])

Merit of corrector for probabilists:

In these **"harmonic coordinates"**, RW is Martingale

## Notion of stationary corrector

Given direction $\xi \in \mathbb{R}^d$, for every coefficient field $a(x)$,
seek $\phi(a;x)$ with $\nabla^* a(\nabla\phi + \xi) = 0$.

Want $\phi(a;x)$ to be *stationary*,
i. e. $\phi(a(z+\cdot),x) = \phi(a;z+x)$ for all shifts $z \in \mathbb{Z}^d$

**Regularize by "massive" term**:
Seek $\phi_T(a;x)$ with $T^{-1}\phi_T + \nabla^* a(\nabla\phi_T + \xi) = 0$.

$T < \infty$: $\forall\ a(\cdot)\ \exists!\ \phi_T(a;\cdot) \Longrightarrow$ stationarity & $\langle \phi_T \rangle = 0$.
**Goal: Uniform estimates** for $T \uparrow \infty$.

**... stationarity much stronger than sublinear growth**

# Quantification of ergodicity of $\langle\cdot\rangle$: Spectral Gap

**Definition**. An ensemble $\langle\cdot\rangle$ has spectral gap $\rho > 0$ iff

$$\forall\, \zeta(a) \quad \Big\langle \big(\zeta - \langle\zeta\rangle\big)^2 \Big\rangle \le \frac{1}{\rho} \Big\langle \sum_{e\in\mathbb{E}^d} \Big(\frac{\partial\zeta}{\partial a(e)}\Big)^2 \Big\rangle.$$

Spectrum of $\sum_e \frac{\partial}{\partial a(e)}^* \frac{\partial}{\partial a(e)}$ (=Generator of Glauber dynamics on $\ell^2_{\langle\cdot\rangle}(\mathbb{E}^d)$) $\subset \{0\} \cup [\rho, \infty)$.

Morally speaking: $\langle (a(x) - \langle a\rangle)(a(0) - \langle a\rangle)\rangle \in \ell^1(\mathbb{Z}^d_x)$
(Dobrushin-Shlosman)

Weakening: $\big(\frac{\partial\zeta}{\partial a(e)}\big)^2 \rightsquigarrow \Big(\zeta - \big\langle \zeta \big| \{a(b)\}_{b\neq e}\big\rangle\Big)^2$
allows for singular (single-site) distributions

... Naddaf&Spencer '97/'98 $\rightsquigarrow$ thermally fluctuating surfaces

## Moment bounds on corrector ...

Recall $T^{-1}\phi_T + \nabla^* a(\nabla\phi_T + \xi) = 0$

**Theorem 1**[Gloria&O. *AP* 2011, &Neukamm *Invent.* 2014]

Suppose $\langle\cdot\rangle$ stationary, spectral gap $\rho$. Then for all $p < \infty$

$$\begin{aligned}\langle|\nabla\phi_T|^{2p}\rangle &\le C(d,\lambda,\rho,p)\\ \langle|\phi_T|^{2p}\rangle &\le C(d,\lambda,\rho,p)\left\{\begin{array}{cc} 1 & \text{for } d>2 \\ (\ln T)^p & \text{for } d=2\end{array}\right\}\end{aligned}$$

Consequence: stationary corrector exists for $d > 2$

Interpretation: Gaussian free field-behavior

## ... optimal in $T$

## Large-scale behavior $\neq$ Gaussian free-field

Note: Corrector $\phi$ depends linearly on direction $\xi$: $\nabla^* a(\nabla\phi+\xi)=0$

**Theorem 2** [Mourrat&O. *submitted* 2013]
Suppose $\langle\cdot\rangle$ is i. i. d. and $d>2$. Then

$$\langle \phi(x)\phi(0)\rangle \;\approx\; \mathrm{cov}(x) \quad \text{for } x\in\mathbb{Z}^d,\ |x|\gg 1,$$

with cov$(x)$ defined via Fourier transform

$$\mathcal{F}\mathrm{cov}(k) \;=\; \frac{Q(\xi,k;\xi,k)}{(k\cdot a_{hom}k)^2}\left\{\begin{array}{l}\geq 0\\ \sim \frac{1}{|k|^2}\\ \neq \frac{1}{k\cdot a_{hom}k}\end{array}\right\} \quad \text{for } k\in\mathbb{R}^d$$

for some four-linear form $Q(\xi_1,\xi_2;\xi_3,\xi_3)$ on $\mathbb{R}^d$.

## Explicit formula for covariance ...

Recall $\mathcal{F}\mathrm{cov}(k) = \frac{Q(\xi,k;\xi,k)}{(k\cdot a_{hom}k)^2} \quad \xi, k \in \mathbb{R}^d.$

Four-linear form $Q(\xi_1, \xi_2; \xi_3, \xi_4)$ on $\mathbb{R}^d$ given by

$$\sum_{i=1}^{d} \Big\langle (\nabla\phi_1+\xi_1)(e_i)(\nabla\phi_2+\xi_2)(e_i)\, \mathcal{L}\, (\nabla\phi_3+\xi_3)(e_i)(\nabla\phi_4+\xi_4)(e_i) \Big\rangle,$$

$\phi_j$ is corrector for direction $\xi_j$, i. e. $\nabla^* a(\nabla\phi_j + \xi_j) = 0$.

Operator $\mathcal{L} = \mathcal{L}^* \geq 0$ on $\ell^2(\mathbb{E}^d)$ characterized by

$$\langle(\zeta_1 - \langle\zeta_1\rangle)(\zeta_2 - \langle\zeta_2\rangle)\rangle = \sum_e \Big\langle \frac{\partial\zeta_1}{\partial a(e)} \mathcal{L} \frac{\partial\zeta_2}{\partial a(e)} \Big\rangle \quad \text{Helffer-Sjöstrand}$$

## ... no mystery but proof requires strong estimates

Representation $\frac{\partial\phi(x)}{\partial a(e)} = -\nabla G(x,e)(\nabla\phi+\xi)(e)$ and

homogenization $\nabla G(x,e) \approx -\sum_{i=1}^{d} \frac{\partial G_{\mathrm{hom}}}{\partial x_i}(x - x(e))(\nabla\phi_i + e_i)(e)$

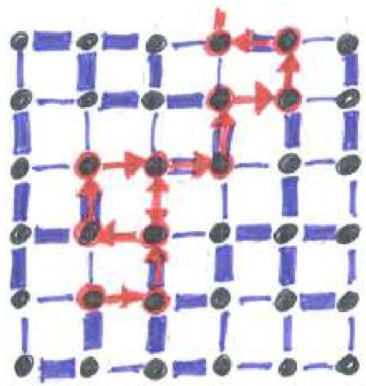
# Environment viewed from particle

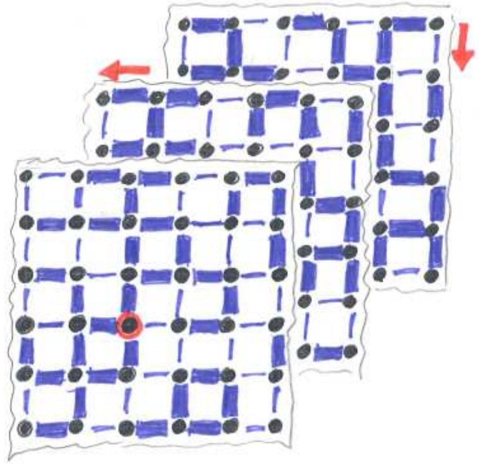
## Optimal ergodicity properties

# Environment viewed from particle

Environment
as seen from
a random walker

process on space
of coefficient fields



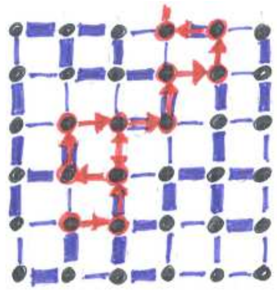


**... quantify ergodicity of this process**

# Math. description of process & its ergodicity

Generator of random walk:

$$\partial_t u + \nabla^* a \nabla u = 0$$



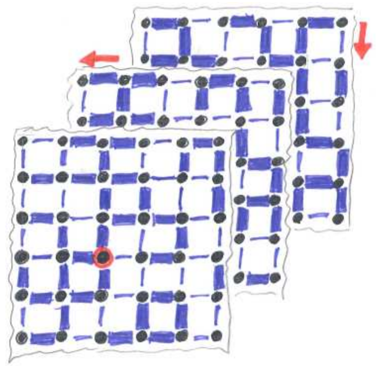
preserves stationarity,

i. e. $u(a(z+\cdot);x) = u(a, z+x)$

lift to $U(a) := u(a;0)$

Generator of
environment viewed from particle:

$$\partial_t U + \sum_{i=1}^d D_i^* a(e_i) D_i U = 0$$



"Horizontal" derivatives $D_1, \cdots, D_d$:

$(D_i U)(a) = U(a(\cdot + e_i)) - U(a)$

**At what rate $\langle |U(t) - \langle U \rangle|^2 \rangle \to 0$ as $t \uparrow \infty$?**

## Optimal decay rate

Of interest (local drift): Initial data in divergence form:

$$\partial_t U + \sum_{i=1}^{d} D_i^* a(e_i) D_i U = 0 \qquad U(t=0) = D^* G$$

**Theorem 3**[Gloria&Neukamm&O. *Invent.* 2014]

$\langle\cdot\rangle$ stationary, spectral gap $\rho$. Then $\forall\, p_0(d,\lambda) \le p < \infty$

$$\langle |U(t)|^{2p} \rangle^{\frac{1}{2p}} \;\le\; C(d,\lambda,\rho,p)\; t^{-\frac{d}{4}-\frac{1}{2}} \sum_{\text{edges } e} \Big\langle \Big(\frac{\partial G}{\partial a(e)}\Big)^{2p} \Big\rangle^{\frac{1}{2p}}.$$

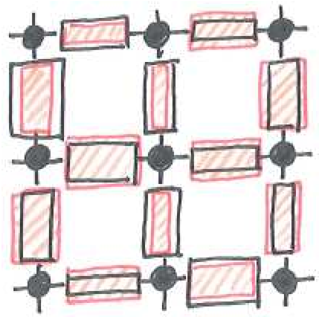
R. h. s. quantifies locality of $G$.

## Relate ergodicity of 2 processes on space of coefficient fields

Glauber dynamics:

Generator $\sum_{\text{edges}\, e} (\frac{\partial}{\partial a(e)})^* \frac{\partial}{\partial a(e)}$

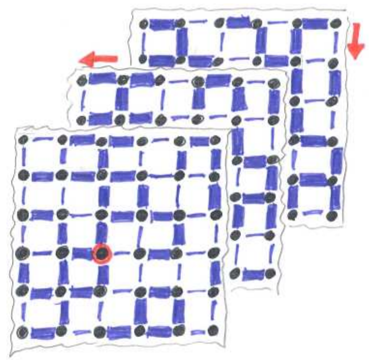
typically has spectral gap



Environment viewed from particle:

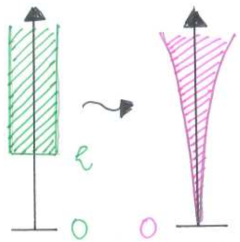
Generator $\sum_{i=1}^{d} D_i^* a(e_i) D_i$

does *not* have spectral gap



spectral gap for $\sum_{\text{edges}\, e} (\frac{\partial}{\partial a(e)})^* \frac{\partial}{\partial a(e)}$

$\rightsquigarrow$ quantified decay of $\exp(-t \sum_{i=1}^{d} D_i^* a(e_i) D_i)$



## Ingredients for efficient proofs

**Calculus based on Spectral Gap**

**and on estimates for elliptic PDE**

## Recall result on (regularized) corrector

**Definition.** $\langle\cdot\rangle$ has spectral gap $\rho > 0$ iff

$$\forall\, \zeta(a) \quad \left\langle(\zeta-\langle\zeta\rangle)^2\right\rangle \le \frac{1}{\rho}\left\langle \sum_{e\in\mathbb{E}^d} \left(\frac{\partial\zeta}{\partial a(e)}\right)^2\right\rangle.$$

**Theorem 1** $\langle\cdot\rangle$ stationary, spectral gap $\rho$.

Then $\quad T^{-1}\phi_T + \nabla^* a(\nabla\phi_T + \xi) \quad$ satisfies

$$\begin{aligned} \langle|\nabla\phi_T|^{2p}\rangle &\le C(d,\lambda,\rho,p) \\ \langle|\phi_T|^{2p}\rangle &\le C(d,\lambda,\rho,p)\begin{Bmatrix} 1 & \text{for } d>2 \\ (\ln T)^p & \text{for } d=2 \end{Bmatrix} \end{aligned}$$

## Use of spectral gap ...

Split into: $\left\{ \begin{array}{l} \langle|\phi_T|^{2p}\rangle \lesssim \left\{ \begin{array}{ll} 1 & d>2 \\ (\ln T)^p & d=2 \end{array} \right\} \langle|\nabla\phi_T+\xi|^{2p}\rangle, \\ \langle|\nabla\phi_T+\xi|^{2p}\rangle \lesssim 1 \end{array} \right\}.$

Argument for $\langle|\phi_T|^{2p}\rangle \lesssim (\ln T)^p \langle|\nabla\phi_T+\xi|^{2p}\rangle$ in $d=2$.

Use $L^p$-version of SG $\langle|\zeta-\langle\zeta\rangle|^{2p}\rangle \le C(p,\rho) \Big\langle\Big(\sum_{e\in\mathbb{E}^d}(\frac{\partial\zeta}{\partial a(e)})^2\Big)^p\Big\rangle$

Apply to $\zeta=\phi_{T(0)}$, need estimate on $\sum_{e\in\mathbb{E}^d}(\frac{\partial\phi_T(0)}{\partial a(e)})^2$

**... requires deterministic "sensitivity estimate"**

## Sensitivity estimate from single PDE estimate

Get $\frac{\partial \phi_T(0)}{\partial a(e)}$ from $(\frac{1}{T} + \nabla^* a \nabla)\partial\phi_T + \nabla^* \partial a(\nabla \phi_T + \xi) = 0$

$\rightsquigarrow$ For $(\frac{1}{T} + \nabla^* a \nabla) v = -\nabla^* g$ seek control of $v(0)$ by $g$

Standard estimate:

$$|v(0)| \le C(\ln T)^{\frac{1}{2}} \Big( \sum_{e:\, |x(e)| \le \sqrt{T}} |\nabla v(e)|^2 + \frac{1}{T} \sum_{x\,:\, |x| \le \sqrt{T}} |v(x)|^2 \Big)^{\frac{1}{2}}$$

$$\le C(\lambda)(\ln T)^{\frac{1}{2}} \Big( \sum_e \exp(-c_0 \tfrac{|x(e)|}{\sqrt{T}}) |g(e)|^2 \Big)^{\frac{1}{2}}.$$

Need refined version valid for $|q-2| \ll 1$:

$$|v(0)| \le C(\ln T)^{1-\frac{1}{q}} \Big( \sum_{|x(e)| \le \sqrt{T}} |x(e)|^{q-2} |\nabla v|^q + \frac{1}{T} \sum_{|x| \le \sqrt{T}} |v|^q \Big)^{\frac{1}{q}}$$

$$\le C(\lambda)(\ln T)^{1-\frac{1}{q}} \Big( \sum_e \exp(-c_0 \tfrac{|x(e)|}{\sqrt{T}}) |x(e)|^{q-2} |g|^q \Big)^{\frac{1}{q}}$$

## Deterministic sensitivity est. + stationarity + SG

refined PDE estimate for $(\frac{1}{T}+\nabla^* a \nabla)v = -\nabla^* g$ , i. e.

$$|v(0)| \;\le\; C(\lambda)(\ln T)^{1-\frac{1}{q}}\Big(\sum_e \mathrm{exp}(-c_0\tfrac{|x(e)|}{\sqrt{T}})|x(e)|^{q-2}|g|^q\Big)^{\frac{1}{q}}$$

$\Longleftrightarrow$ sensitivity estimate for $\frac{\partial\phi_T(0)}{\partial a(e)}$ $\quad (p=\frac{q}{2-q}\gg 1)$

$$(\sum_e(\frac{\partial\phi_{T(0)}}{\partial a(e)})^2)^p \le C^p(\lambda)(\ln T)^{p-1}\sum_e \mathrm{exp}(-c_0\tfrac{|x(e)|}{\sqrt{T}})|x(e)|^{-2}|(\nabla\phi_T+\xi)(e)|^{2p}$$

& $\quad$ stationarity of $\nabla\phi_T+\xi$ $\quad$ & $\quad$ $L^p$-spectral gap $\quad\Longrightarrow$

$$\langle|\phi_T|^{2p}\rangle \lesssim \Big\langle\Big(\sum_e(\frac{\partial\phi_T}{\partial a(e)})^2\Big)^p\Big\rangle \lesssim (\ln T)^p\langle|\nabla\phi_T+\xi|^{2p}\rangle$$

## Argument for $\langle|\nabla\phi+\xi|^{2p}\rangle \lesssim 1$

PDE estimate for $\nabla^* a\nabla v = -\nabla^* g$: $\exists\ \alpha(d,\lambda) > 0\ \ \forall\ R < \infty$

$$\sum_{b:|x(b)|\le R} |\nabla v(b)|^2 \le C(d,\lambda)\sum_e\left(\frac{|x(e)|}{R}+1\right)^{-\alpha}|g(e)|^2$$

$\Longrightarrow$ Sensitivity estimate:

$\forall$ linear functionals $F_R g$ with $|F_R g| \le \left(\sum_{|x(b)|\le R}|g(b)|^2\right)^{\frac{1}{2}}$:

$$\sum_e\left(\frac{\partial F(\nabla\phi+\xi)}{\partial a(e)}\right)^2 \le C(d,\lambda)\sup_e\left(\frac{|x(e)|}{R}+1\right)^{-\alpha}|(\nabla\phi+\xi)(e)|^2$$

& Stationarity + $L^p$-version of SG (need $p\alpha > d$) $\Longrightarrow$:

$$\langle|F_R(\nabla\phi+\xi)|^{2p}\rangle \lesssim R^{pd} + R^d\langle|\nabla\phi+\xi|^{2p}\rangle$$

# Argument for $\langle|\nabla\phi+\xi|^{2p}\rangle\lesssim 1$, cont.

Have $\forall$ linear functionals $F_R g$ with $|F_R g|\le(\sum_{|x(b)|\le R}|g(b)|^2)^{\frac{1}{2}}$:

$\langle|F_R(\nabla\phi+\xi)|^{2p}\rangle\lesssim R^{pd}+R^d\langle|\nabla\phi+\xi|^{2p}\rangle$

& "Compactness" for $a$-harmonic gradient fields $\phi+\xi\cdot x$:

$\exists\alpha(d,\lambda)>0\ \forall\ R<\infty\ \exists\ F_1,\cdots,F_{N(d,\lambda)}$ as above s. t.

$|(\nabla\phi+\xi)(e_i)|^2\le C(d,\lambda)R^{-\alpha}\sum_{n=1}^{N}|F_n(\nabla\phi+\xi)|^2$

$\Longrightarrow\quad R^{p\alpha}\langle|\nabla\phi+\xi|^{2p}\rangle\lesssim R^{pd}+R^d\langle|\nabla\phi+\xi|^{2p}\rangle$

— buckles for $R\gg 1$ provided $p\alpha>d$

[& Bella, Armstrong&Smart]

## **Elements of a quantitative homogenization theory**

Corrector: Existence of stationary corrector,
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## Connect to frontier in qualitative homogenization

Supercritical percolation:
existence stationary corrector ok in $d > 2$
[&Lamacz&Neukamm, *submitted*]

with drift (à la Sznitman&Zeitouni):
construction of stationary measure?