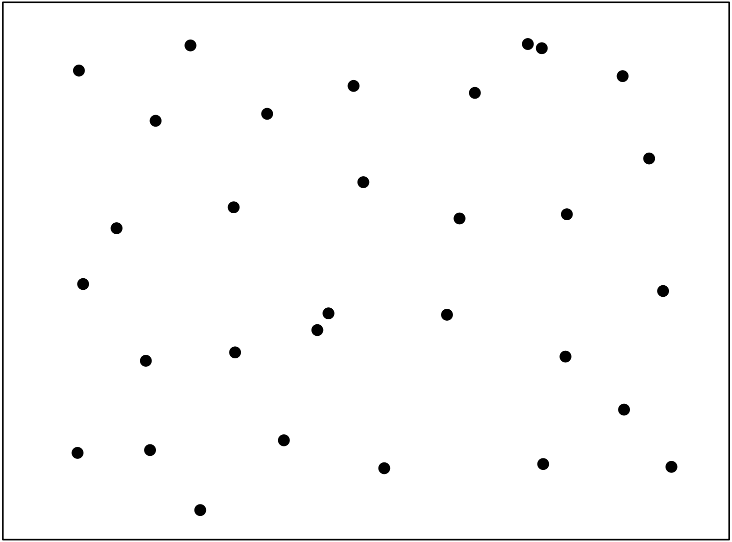
Near-critical percolation and minimal spanning tree in the plane

Christophe Garban
ENS Lyon, CNRS

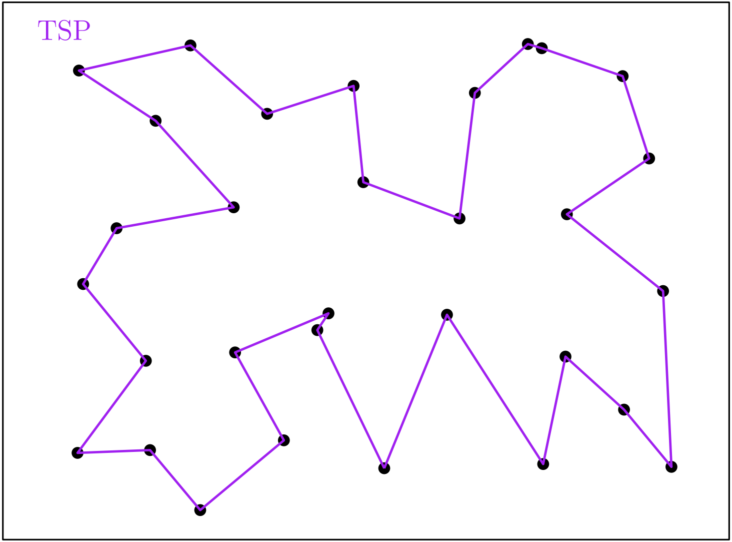
joint work with Gábor Pete and Oded Schramm

37^{th} SPA, Buenos Aires, July 2014

Traveling Salesman Problem

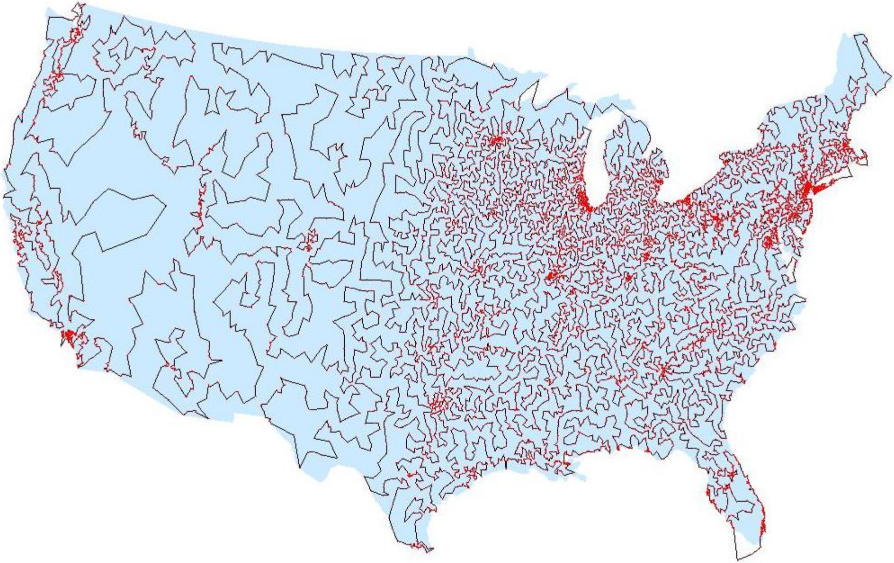


Traveling Salesman Problem



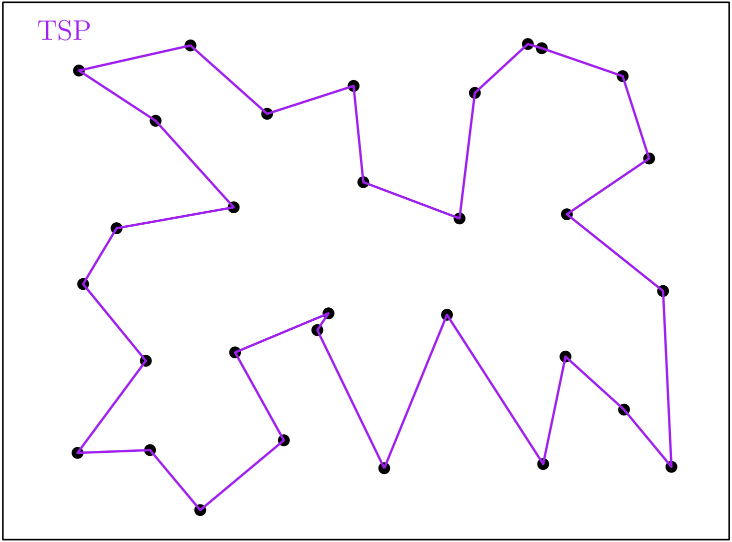
Traveling Salesman Problem

Traveling Salesman Problem

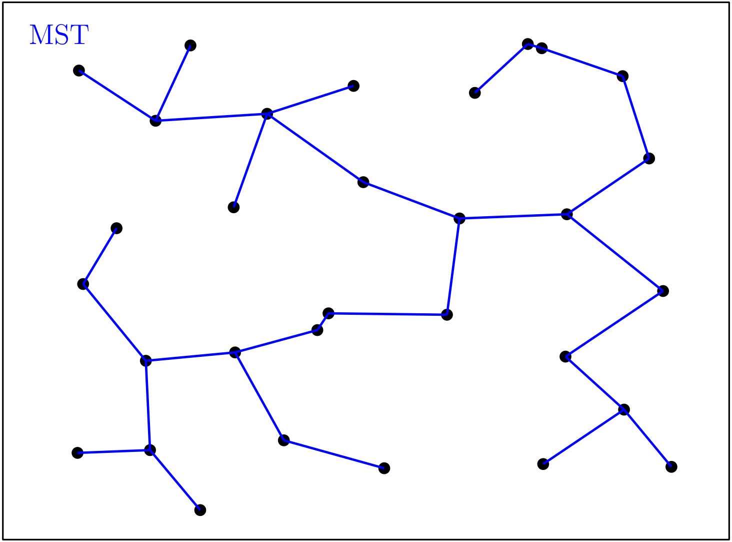


Traveling Salesman Problem

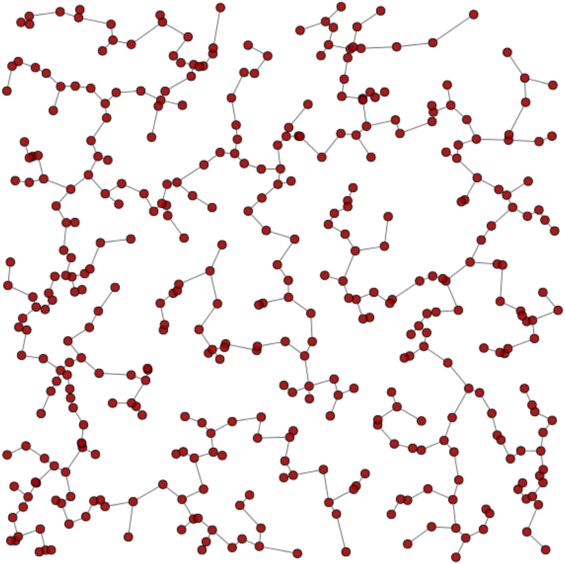
Minimal Spanning Tree (MST)



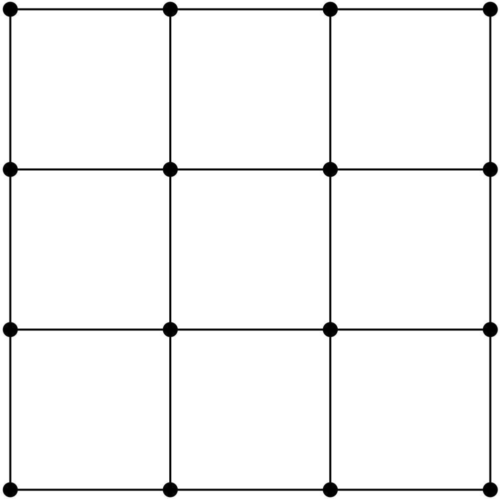
Minimal Spanning Tree (MST)



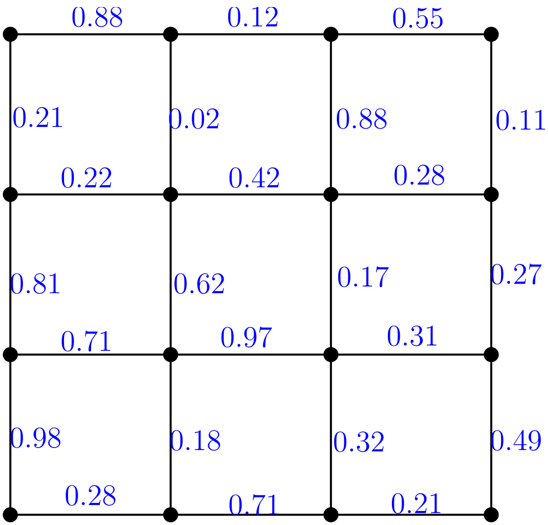
MAIN QUESTION: scaling limit of the planar MST ?

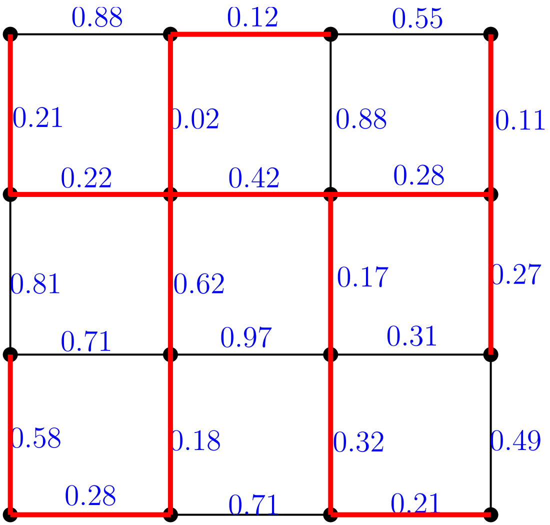


Minimal Spanning Tree on $\mathbb{Z}^2$

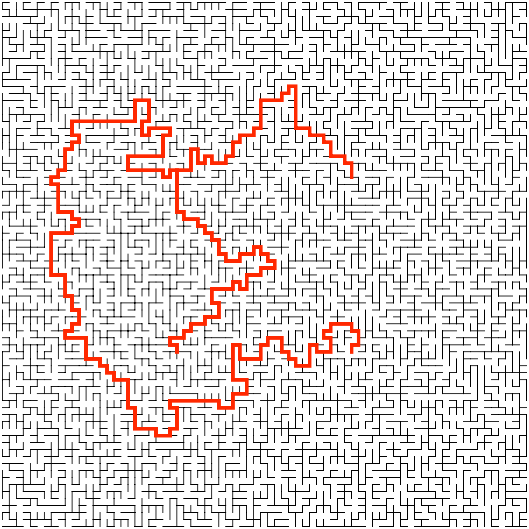


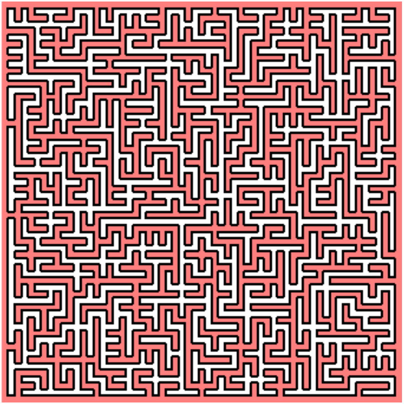
Minimal Spanning Tree on $\mathbb{Z}^2$




0.88
0.12
0.55
0.21
0.02
0.88
0.11
0.22
0.42
0.28
0.81
0.62
0.17
0.27
0.71
0.97
0.31
0.58
0.18
0.32
0.49
0.28
0.71
0.21

MST on $\mathbb{Z}^2$ seen from further away ...

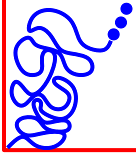




→

SLE_κ process with $\kappa = 8$

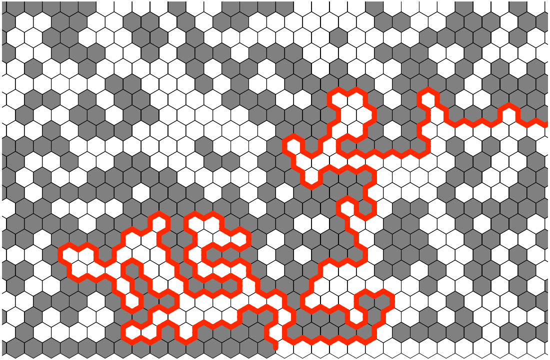
Theorem by Lawler Schramm Werner (2003)



Scaling limit of percolation

Theorem (Smirnov, 2001)

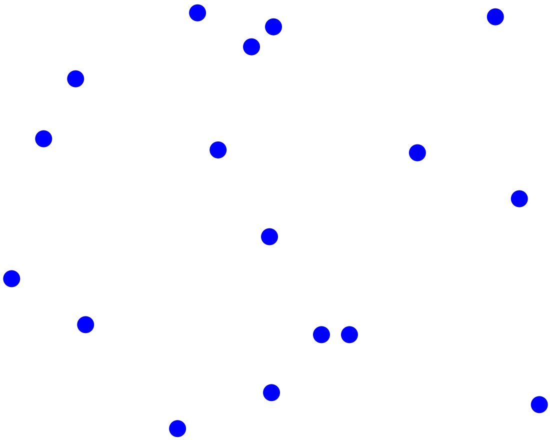
Critical site percolation on $\eta\mathbb{T}$ is asymptotically (as $\eta \searrow 0$) conformally invariant.



Convergence to SLE_6

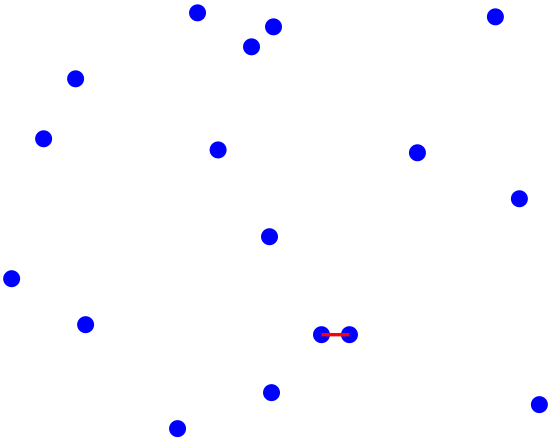
A "greedy" algorithm to compute the MST

Kruskal's algorithm:



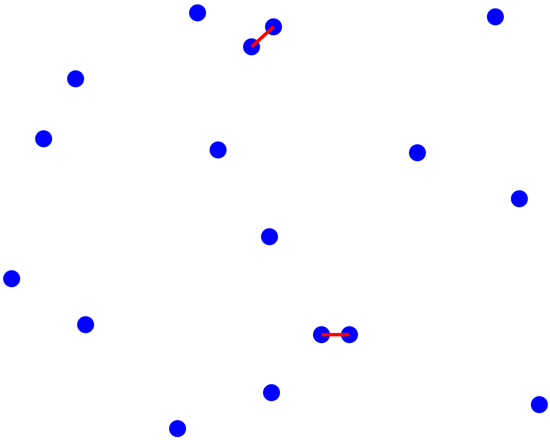
A "greedy" algorithm to compute the MST

Kruskal's algorithm:



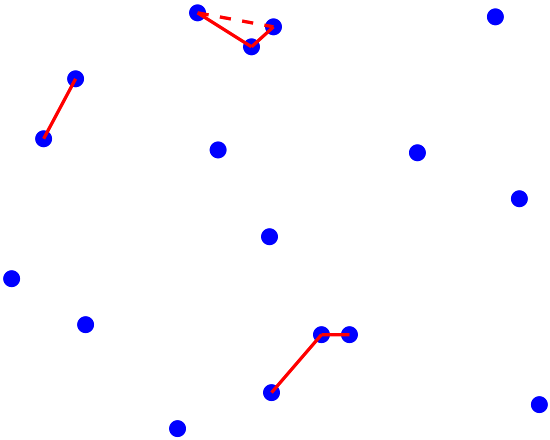
A "greedy" algorithm to compute the MST

Kruskal's algorithm:



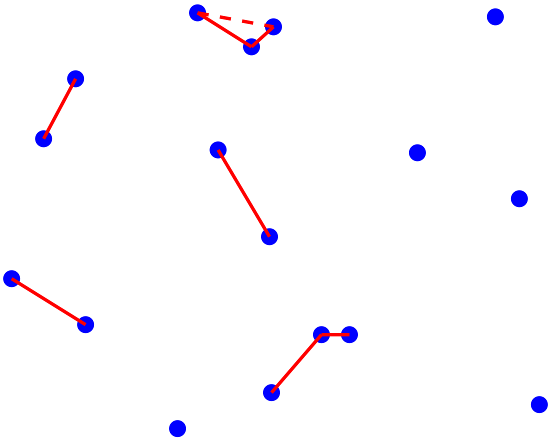
A "greedy" algorithm to compute the MST

Kruskal's algorithm:



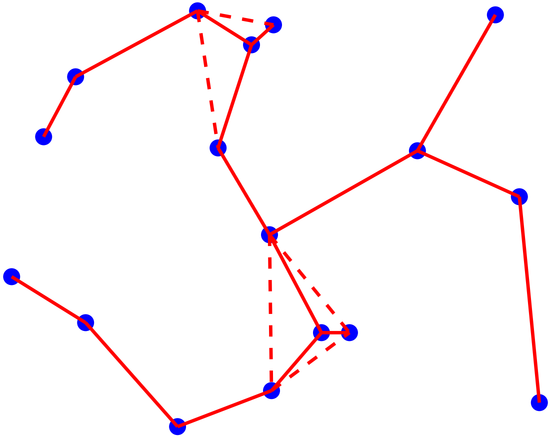
A "greedy" algorithm to compute the MST

Kruskal's algorithm:



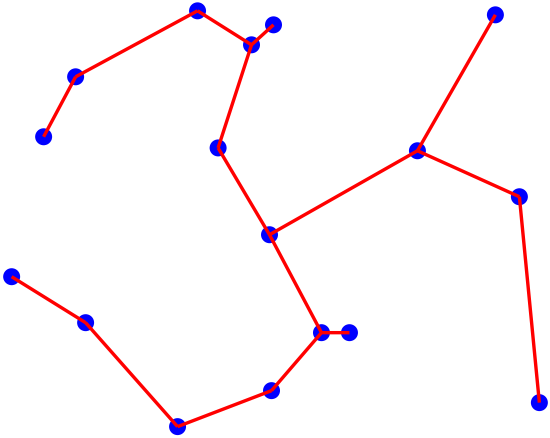
A "greedy" algorithm to compute the MST

Kruskal's algorithm:

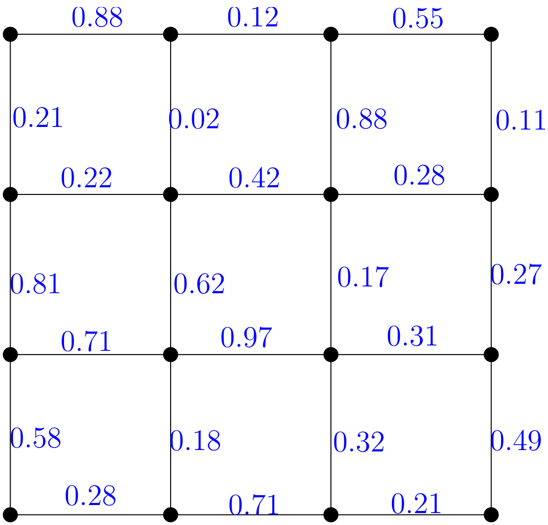


A "greedy" algorithm to compute the MST

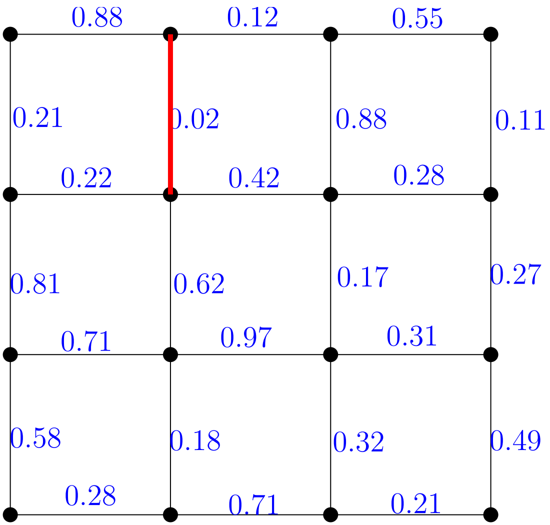
Kruskal's algorithm:



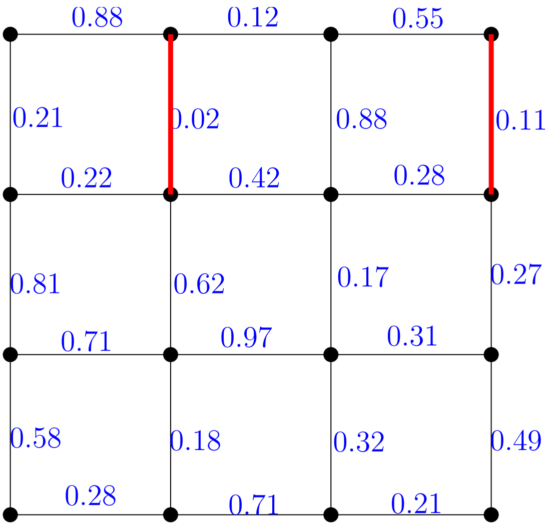
Kruskal's algorithm on $\mathbb{Z}^2$



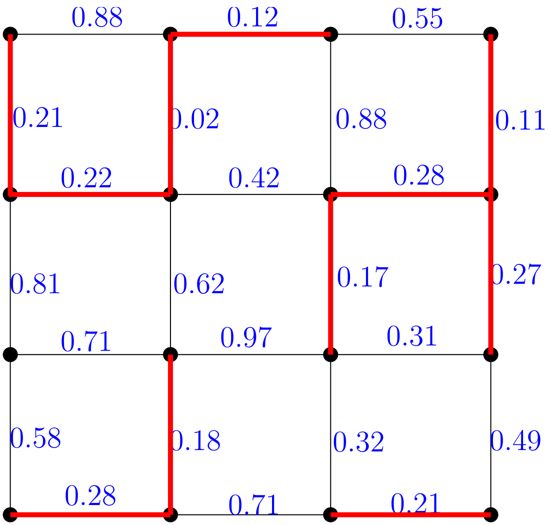
Kruskal's algorithm on $\mathbb{Z}^2$



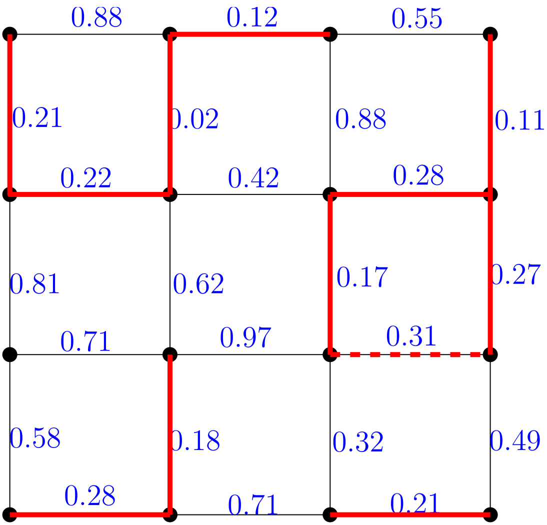
Kruskal's algorithm on $\mathbb{Z}^2$



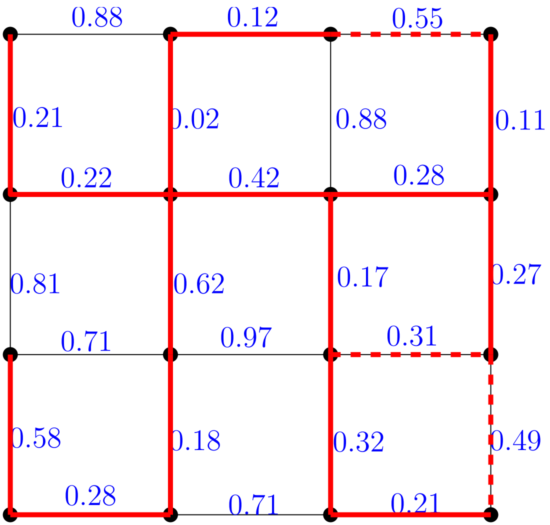
Kruskal's algorithm on $\mathbb{Z}^2$



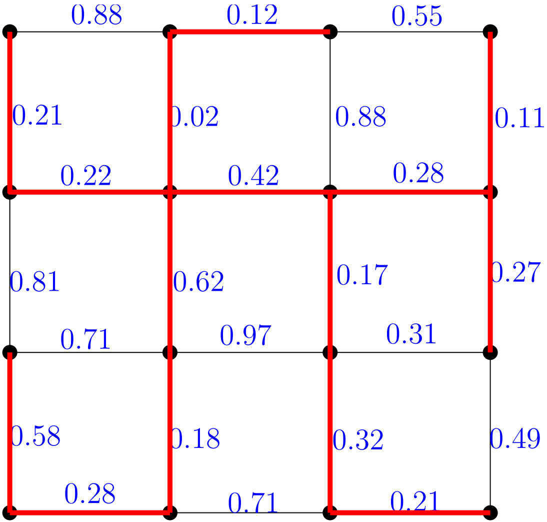
Kruskal's algorithm on $\mathbb{Z}^2$



Kruskal's algorithm on $\mathbb{Z}^2$

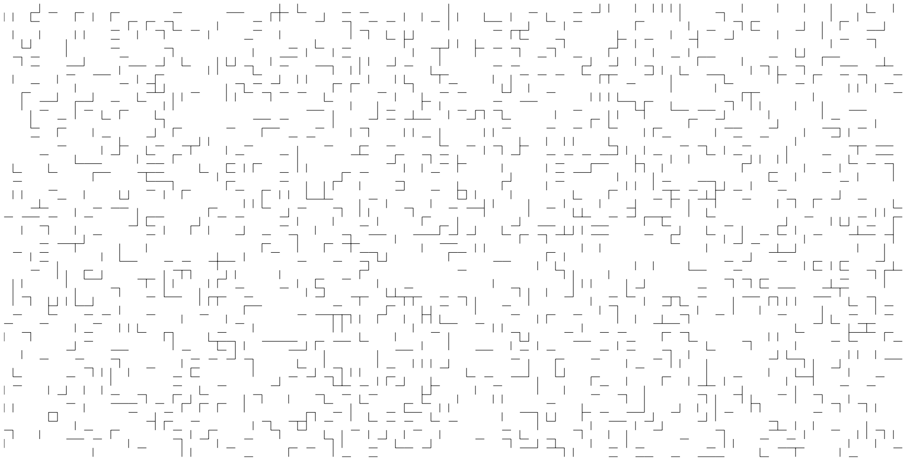


Kruskal's algorithm on $\mathbb{Z}^2$



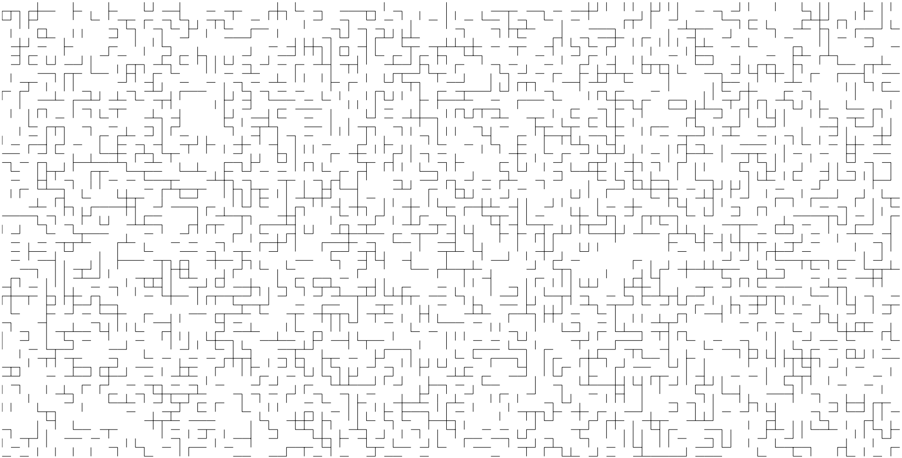
Percolation Model on $\mathbb{Z}^2$

$\omega_p,\ p = 0.16666$



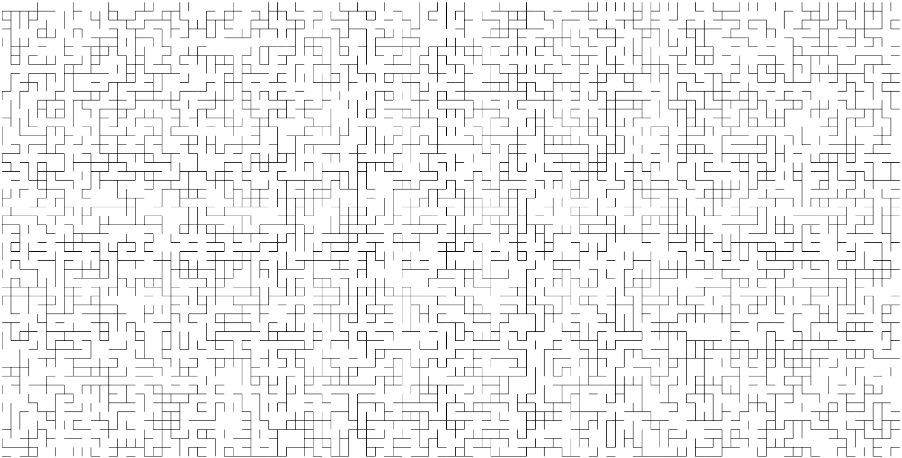
Percolation Model on $\mathbb{Z}^2$

ω_p, $p = 0.33333$



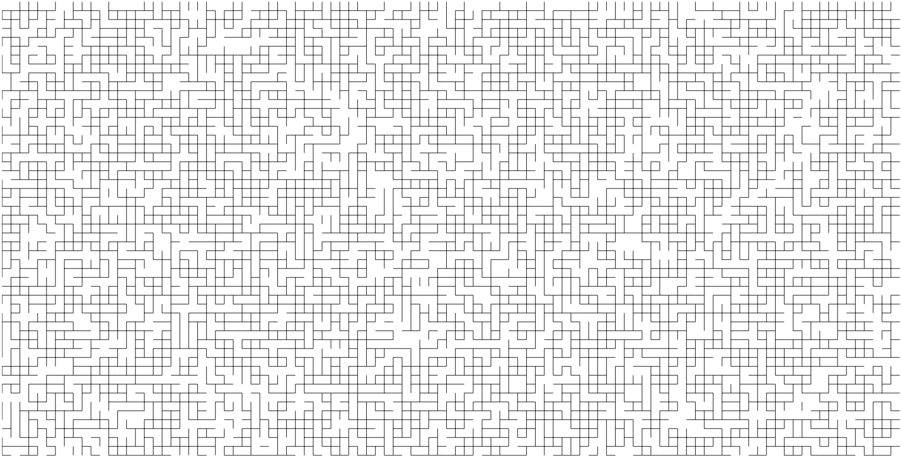
Percolation Model on $\mathbb{Z}^2$

$\omega_p,\ p = 0.50000$



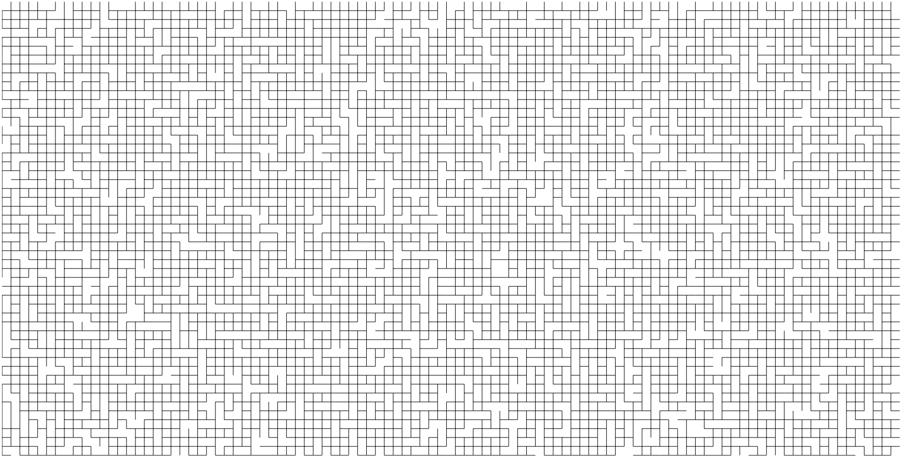
Percolation Model on $\mathbb{Z}^2$

$\omega_p, \ p = 0.66666$

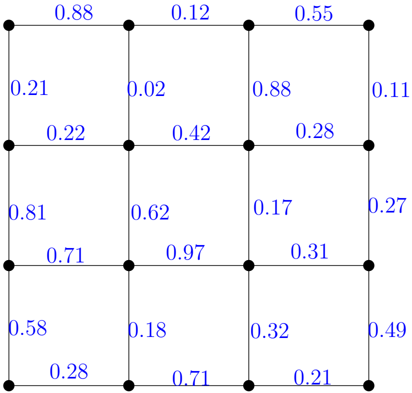


Percolation Model on $\mathbb{Z}^2$

$\omega_p,\ p = 0.83333$



Monotone coupling in percolation



Definition (Standard coupling)

For all $e \in \mathbb{Z}^2$, sample $u_e \sim \mathcal{U}([0,1])$.
For any fixed $p \in [0,1]$, let

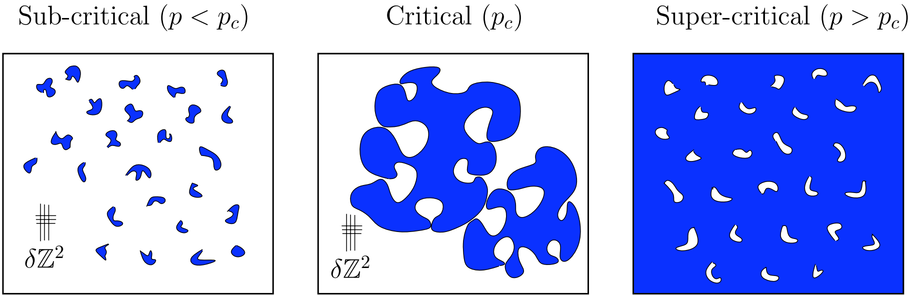
$$\omega_p(e) := 1_{u_e \le p}$$

As such $\omega_p \sim \mathbb{P}_p$ for all p and

$$\omega_p \le \omega_{p'} \quad \text{if } p \le p'$$

"Abrubt" phase transition

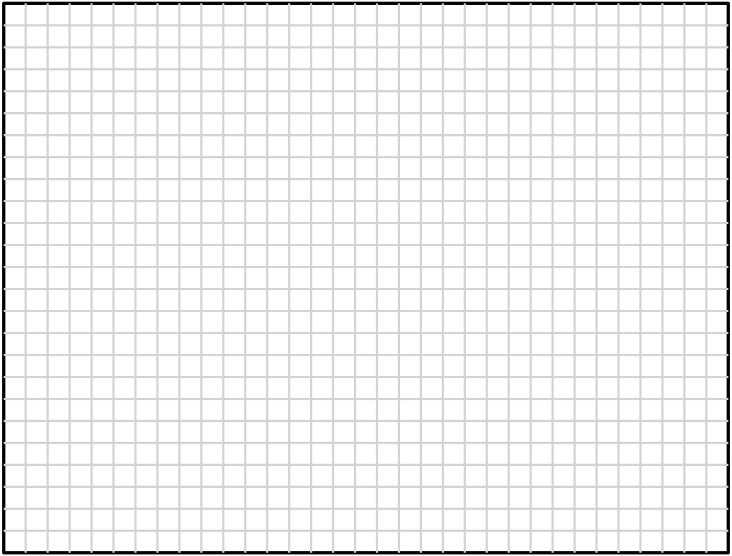
Seen from far away it looks as follows:



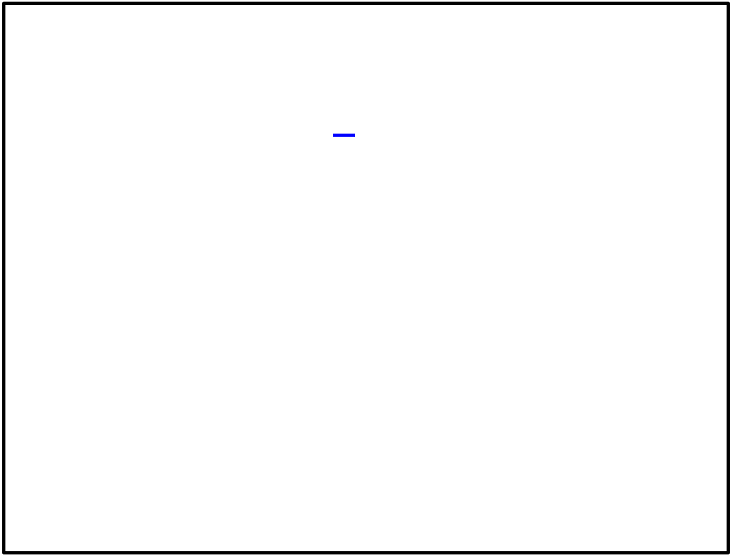
Theorem (Kesten, 1980)

$$p_c(\mathbb{Z}^2) = \frac{1}{2}$$

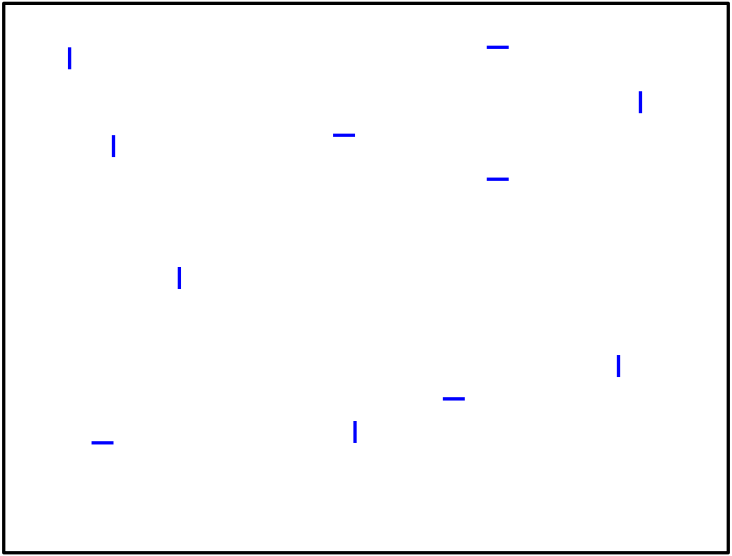
Kruskal's algorithm on $\mathbb{Z}^2$



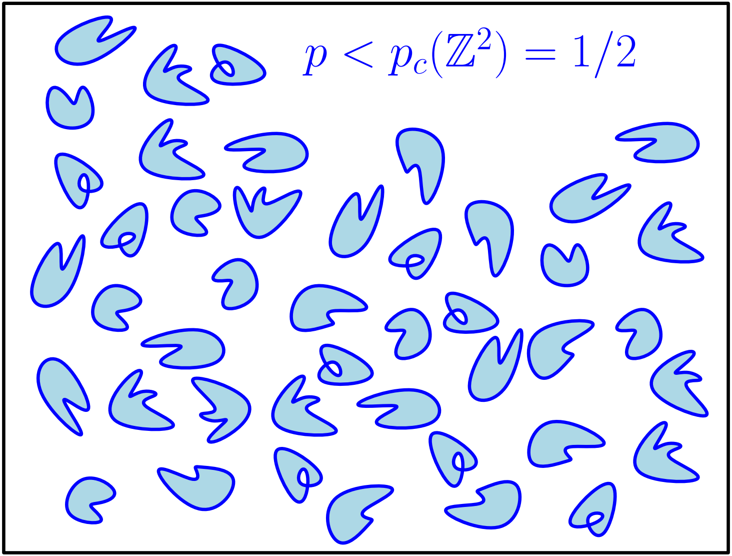
Kruskal's algorithm on $\mathbb{Z}^2$



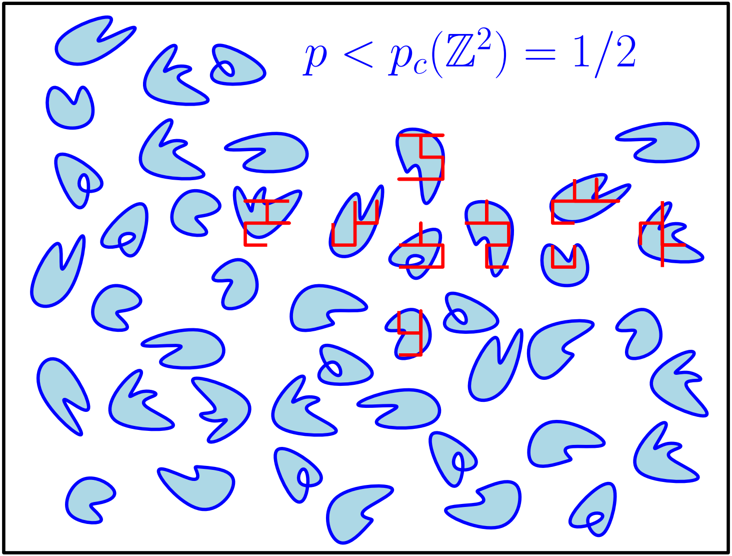
Kruskal's algorithm on $\mathbb{Z}^2$



Kruskal's algorithm on $\mathbb{Z}^2$

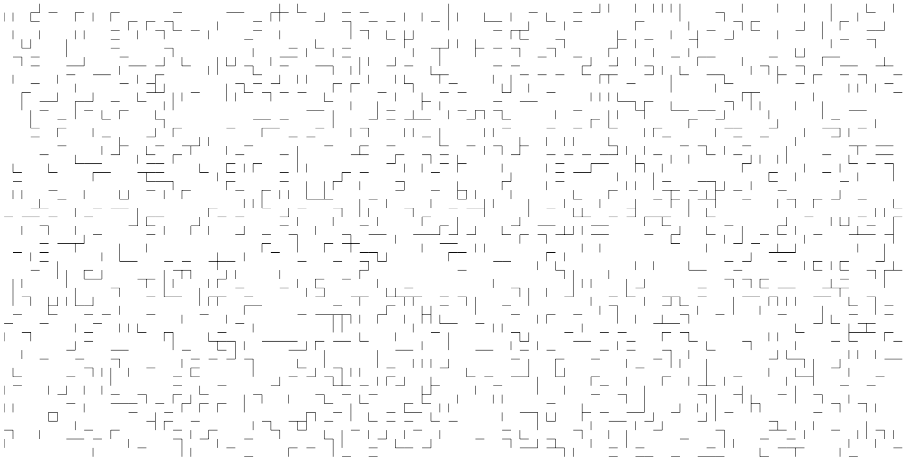


Kruskal's algorithm on $\mathbb{Z}^2$



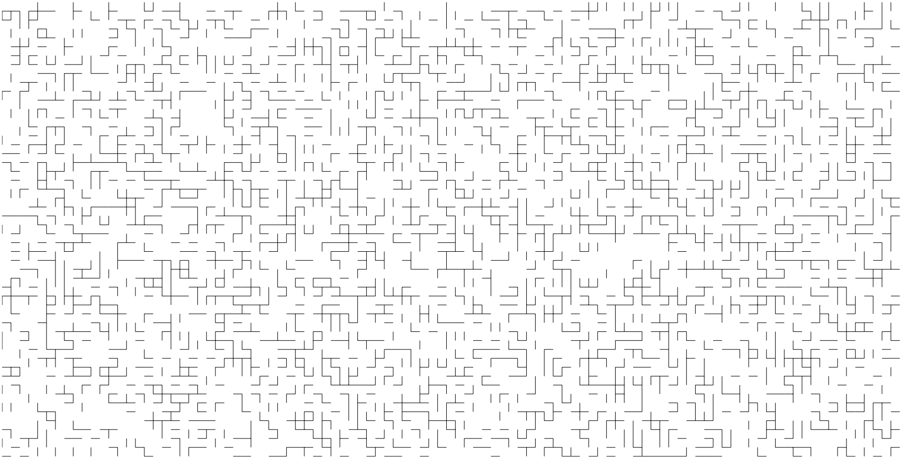
Kruskal's algorithm on $\mathbb{Z}^2$

$\omega_p, \ p = 0.16666$



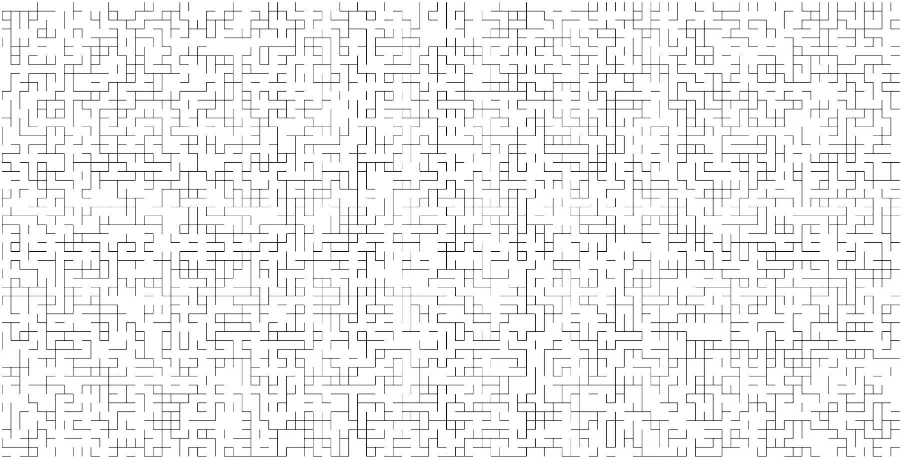
Kruskal's algorithm on $\mathbb{Z}^2$

$\omega_p, \ p = 0.33333$



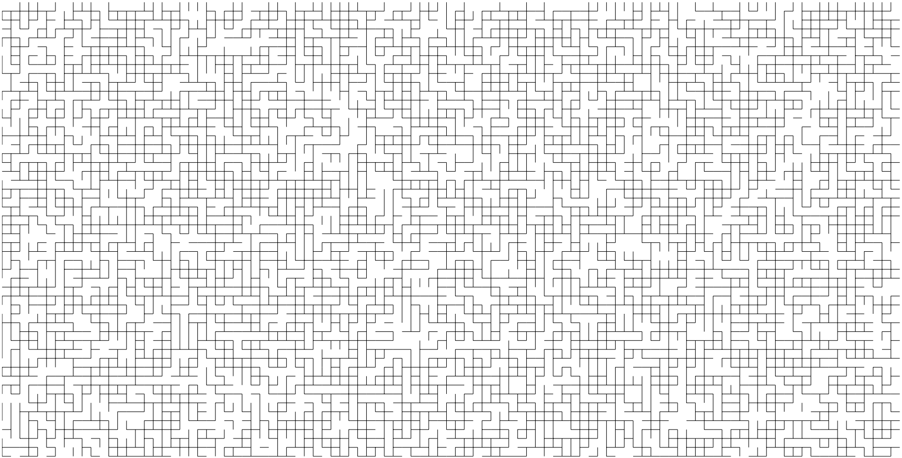
Kruskal's algorithm on $\mathbb{Z}^2$

$\omega_p,\ p = 0.50000$



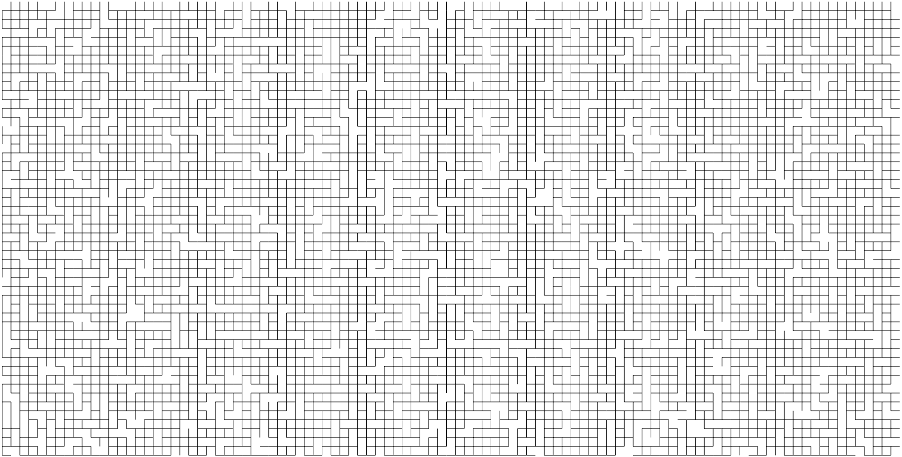
Kruskal's algorithm on $\mathbb{Z}^2$

ω_p, $p = 0.66666$



Kruskal's algorithm on $\mathbb{Z}^2$

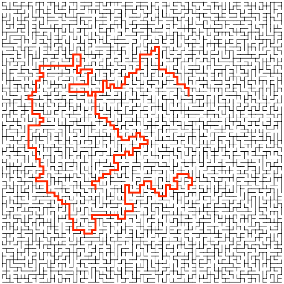
$\omega_p,\ p = 0.83333$



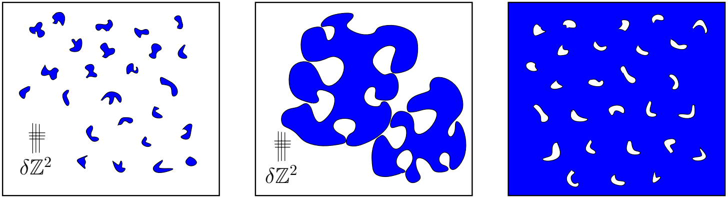
Minimal Spanning Tree in the plane

Theorem (Aizenman, Burchard, Newman, Wilson, 1999)

*The Minimal Spanning Tree on $\eta\mathbb{Z}^2$ is **tight** as $\eta \to 0$ (for a metric on the space of planar spanning trees inspired by the Hausdorff distance)*



- On the triangular lattice, we will prove the **convergence** as $\eta \to 0$
- This requires a detailed analysis of **near-critical percolation:**



References

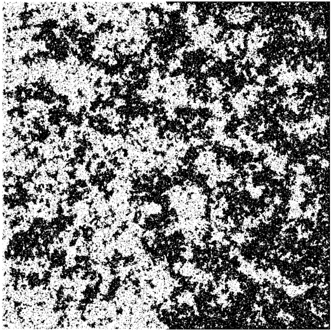
A) 2010, Pivotal, cluster and interface measures for critical planar percolation, G., Pete, Schramm, *J.A.M.S. 2013*.

B) 2013, The scaling limits of near-critical and dynamical percolation, G., Pete, Schramm, arXiv:1305.5526

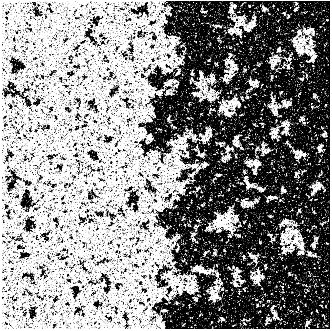
C) 2013, The scaling limits of the Minimal Spanning Tree and Invasion Percolation in the plane, G., Pete, Schramm, Arxiv:1309.0269

Near-critical geometry in general

Ising model near its critical point:



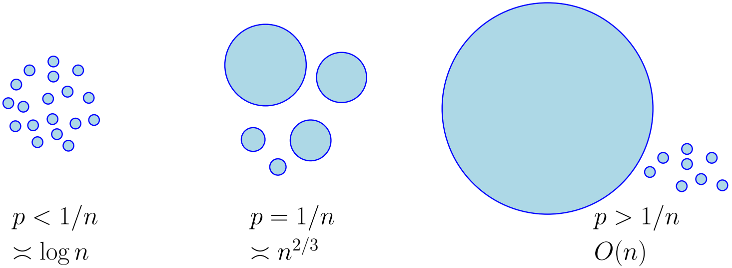
$T = T_c$



$T = T_c - \delta T$

Near-critical percolation (mean-field case)

- Erdos-Renyi random graphs $G(n,p)$, $p \in [0,1]$ (p-percolation on the **complete graph** Δ_n).



- It is well-known that "everything happens" in the near-critical window

$$p = \frac{1}{n} + \lambda \frac{1}{n^{4/3}}$$

where λ is the near-critical parameter

A quotation and a Theorem

Alon and Spencer (2002):

> " *With* $\lambda = -10^6$, *say we have feudalism. Many components (castles) are each vying to be the largest. As* λ *increases the components increase in size and a few large components (nations) emerge. An already large France has much better chances of becoming larger than a smaller Andorra. The largest components tend to merge and by* $\lambda = 10^6$ *it is very likely that a giant component, the Roman Empire, has emerged. With high probability this component is nevermore challenged for supremacy but continues absorbing smaller components until full connectivity – One World – is achieved.* "

Theorem (Addario-Berry, Broutin, Goldschmidt, Miermont, 2013)

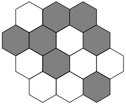
Let MST_n *be the Minimal Spanning Tree on* Δ_n

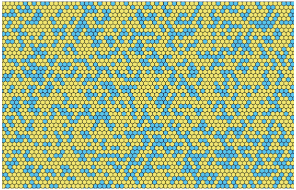
$$(\mathrm{MST}_n, \frac{1}{n^{1/3}} d_{graph}) \xrightarrow{law} \mathrm{MST}_\infty$$

where the convergence in law holds under the Gromov-Hausdorff topology.

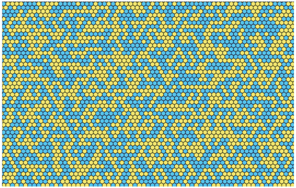
Near-critical percolation in the plane

Site percolation on the triangular lattice $\mathbb{T}$:

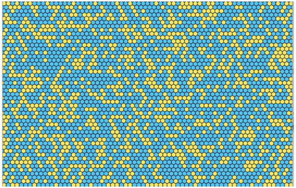




"feudalism"
$p < 1/2$



$p = 1/2$



"Roman empire"
$p > 1/2$

Renormalise the lattice as follows: $\eta\mathbb{T}$ where η corresponds to the mesh of the rescaled lattice.

$\eta \to 0$??

looking for the right ZOOMING

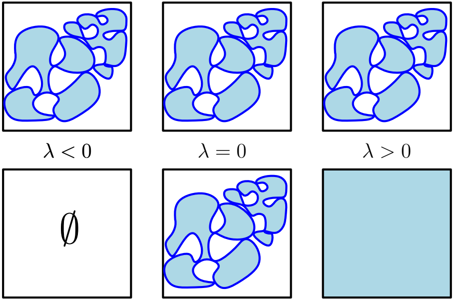
We shall now zoom around p_c as follows:

$$\boxed{p = p_c + \lambda\, r(\eta)}$$

looking for the right ZOOMING

We shall now zoom around p_c as follows:

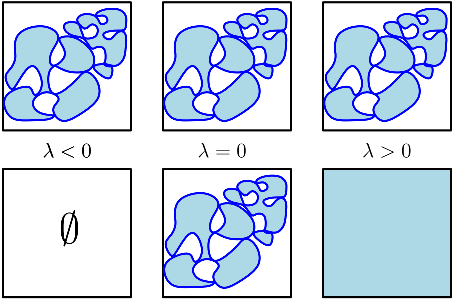
$$p = p_c + \lambda\, r(\eta)$$



looking for the right ZOOMING

We shall now zoom around p_c as follows:

$$\boxed{p = p_c + \lambda\, r(\eta)}$$



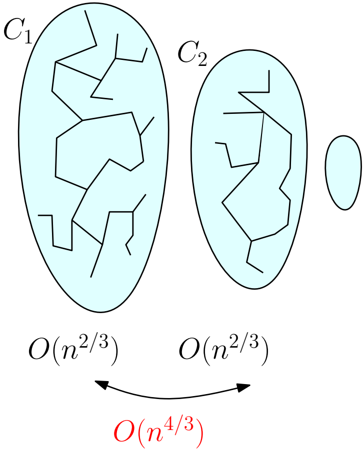
Theorem (Kesten, 1987)

The right zooming factor is given by

$$\begin{aligned} r(\eta) &:= \eta^2 \alpha_4(\eta, 1)^{-1} \\ &= \eta^{3/4+o(1)} \end{aligned}$$

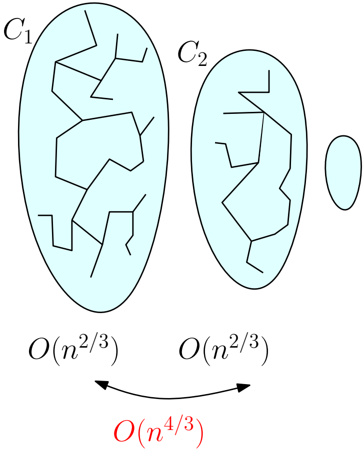
Heuristics behind these scalings

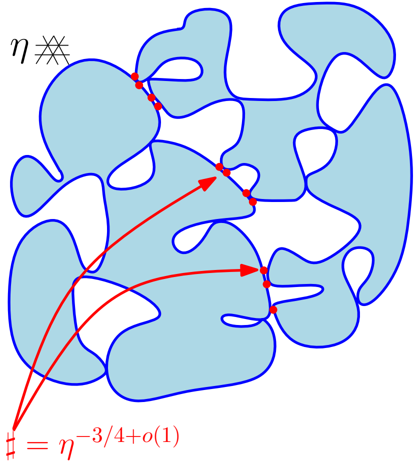
$p = 1/n + \lambda n^{-4/3}$ versus $p_c + \lambda \eta^{3/4+o(1)}$



Heuristics behind these scalings

$p = 1/n + \lambda n^{-4/3}$ versus $p_c + \lambda \eta^{3/4+o(1)}$





Scaling limit ?

Definition

Define $\omega_\eta^{\mathrm{nc}}(\lambda)$ to be the percolation configuration on $\eta\mathbb{T}$ of parameter

$$\boxed{p = p_c + \lambda\, r(\eta)}$$

For all $\eta > 0$, we define this way a monotone càdlàg process

$$\lambda \in \mathbb{R} \mapsto \omega_\eta^{\mathrm{nc}}(\lambda) \in \{0,1\}^{\eta\mathbb{T}}$$

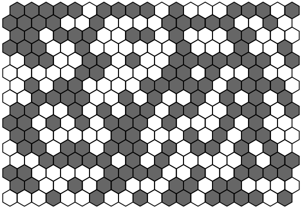
Question

Does the process $\lambda \in \mathbb{R} \mapsto \omega_\eta^{\mathrm{nc}}(\lambda)$ converge (in law) as $\eta \searrow 0$ to a limiting process

$$\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda) \;\; ?$$

- For which **topology** ?? Find an appropriate Polish space (E, d) whose points $\omega \in E$ are naturally identified to **percolation configurations**.

The first natural idea which comes to mind

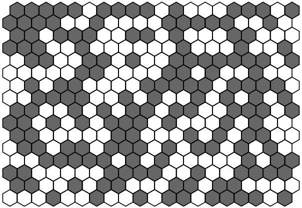


This configuration on $\eta\mathbb{T}$ may be coded by the distribution

$$X_\eta := \eta \sum_{x \in \eta\mathbb{T}} \sigma_x \, \delta_x$$

$\{X_\eta\}_\eta$ is **tight** in $\mathcal{H}^{-1-\varepsilon}$ and converge to the **Gaussian white noise** on $\mathbb{R}^2$.

The first natural idea which comes to mind



This configuration on $\eta\mathbb{T}$ may be coded by the distribution

$$X_\eta := \eta \sum_{x \in \eta\mathbb{T}} \sigma_x \, \delta_x$$

$\{X_\eta\}_\eta$ is **tight** in $\mathcal{H}^{-1-\varepsilon}$ and converge to the **Gaussian white noise** on $\mathbb{R}^2$.

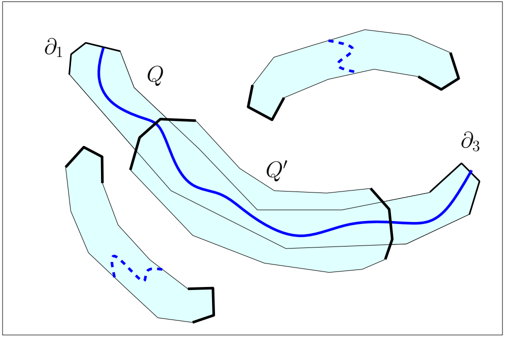
Theorem (Benjamini, Kalai, Schramm, 1999)

*This setup is **NOT** appropriate to handle percolation: natural observables for percolation are highly discontinuous under the topology induced by $\| \cdot \|_{\mathcal{H}^{-1-\varepsilon}}$ and in fact are not even measurable in the limit.*

Some other historical approaches

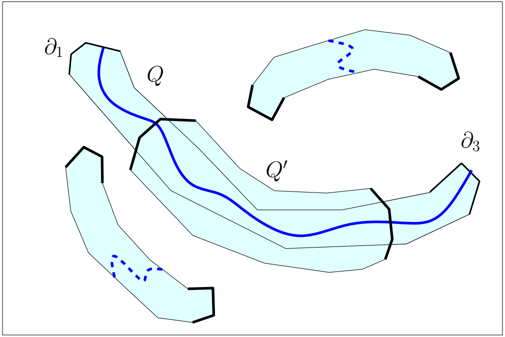
1. Aizenman 1998 and Aizenman, Burchard 1999.
2. Camia, Newman 2006.
3. The topological space $(\mathscr{H}, \mathcal{T})$ of **Schramm-Smirnov**, 2011

The Schramm-Smirnov space $\mathscr{H}$



- Let $(\mathcal{Q}, d_{\mathcal{Q}})$ be the space of all **quads**.
- On might consider the space $\{0,1\}^{\mathcal{Q}}$

The Schramm-Smirnov space $\mathscr{H}$



- Let $(\mathcal{Q}, d_{\mathcal{Q}})$ be the space of all **quads**.
- On might consider the space $\{0,1\}^{\mathcal{Q}}$
- In fact, one considers instead $\mathscr{H} \subset \{0,1\}^{\mathcal{Q}}$ which preserves the **partial order** on $\mathcal{Q}$: $Q > Q'$
- Schramm-Smirnov prove that $\mathscr{H}$ can be endowed with a natural topology $\mathcal{T}$ ($\approx$ Fell's topology) for which, $(\mathscr{H}, \mathcal{T})$ is compact, Hausdorff and metrizable

The "critical slice" $\omega_\infty \sim \mathbb{P}_\infty$

Definition ($\lambda = 0$)

For each mesh $\eta > 0$, one may view $\omega_\eta \sim \mathbb{P}_\eta$ as a random point in the compact space $(\mathscr{H}, d_{\mathscr{H}})$.

Theorem (Smirnov 2001, CN 2006, GPS 2013)

$\omega_\eta \sim \mathbb{P}_\eta$ *converges in law in* $(\mathscr{H}, d_{\mathscr{H}})$ *to a* **continuum percolation**

$$\omega_\infty \sim \mathbb{P}_\infty$$

$\Rightarrow$ this handles the case $\lambda = 0$

Away from the "critical slice"

Recall:

Question

Let $\lambda > 0$ *be fixed.*

$$p = p_c + \lambda\, r(\eta)$$

Does $\omega_\eta^{\mathrm{nc}}(\lambda)$ *converge in law in* $\mathscr{H}$ *to a limiting object ?*

Main results

Theorem (G., Pete, Schramm 2013)

Fix $\lambda \in \mathbb{R}$.

$$\omega_\eta^{\mathrm{nc}}(\lambda) \xrightarrow{(d)} \omega_\infty^{\mathrm{nc}}(\lambda)$$

The convergence in law holds in the space $(\mathscr{H}, d_{\mathscr{H}})$.

Theorem (G., Pete, Schramm 2013)

The càdlàg process $\lambda \mapsto \omega_\eta^{\mathrm{nc}}(\lambda)$ converges in law to $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$ for the ***Skorohod topology*** *on $\mathscr{H}$.*

Main results

Theorem (G., Pete, Schramm 2013)

Fix $\lambda \in \mathbb{R}$.

$$\omega_\eta^{\mathrm{nc}}(\lambda) \xrightarrow{(d)} \omega_\infty^{\mathrm{nc}}(\lambda)$$

The convergence in law holds in the space $(\mathscr{H}, d_{\mathscr{H}})$.

Theorem (G., Pete, Schramm 2013)

The càdlàg process $\lambda \mapsto \omega_\eta^{\mathrm{nc}}(\lambda)$ *converges in law to* $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$ *for the* ***Skorohod topology*** *on* $\mathscr{H}$.

Theorem (Nolin, Werner 2007)

Fix $\lambda \neq 0$. *All the subsequential scaling limits of* $\omega_{\eta_k}^{\mathrm{nc}}(\lambda) \overset{(d)}{\to} \tilde{\omega}_\infty(\lambda)$ *are such that their* **interfaces** *are* **singular** *w.r.t the* SLE_6 *curves !*

Two possible approaches

Recall the case $\lambda = 0$ (critical case). One has $\omega_\eta \sim \mathbb{P}_\eta$ and we wish to prove a scaling limit result.

- tightness, ✓
- **uniqueness** ??
- main ingredient for uniqueness: **Cardy/Smirnov's formula** !

Two possible approaches

Recall the case $\lambda = 0$ (critical case). One has $\omega_\eta \sim \mathbb{P}_\eta$ and we wish to prove a scaling limit result.

- tightness, ✓
- **uniqueness** ??
- main ingredient for uniqueness: **Cardy/Smirnov's formula** !

1. This suggests the following approach to handle the case $\lambda \neq 0$: for all $p \neq p_c(\mathbb{T}) = 1/2$, find a **massive harmonic observable** F_p:

$$\boxed{\Delta F_p(x) \approx m(p) F_p(x)}$$

The "mass" $m(p)$ should then scale as $|p - p_c|^{8/3}$.

Two possible approaches

Recall the case $\lambda = 0$ (critical case). One has $\omega_\eta \sim \mathbb{P}_\eta$ and we wish to prove a scaling limit result.

- tightness, ✓
- **uniqueness** ??
- main ingredient for uniqueness: **Cardy/Smirnov's formula** !

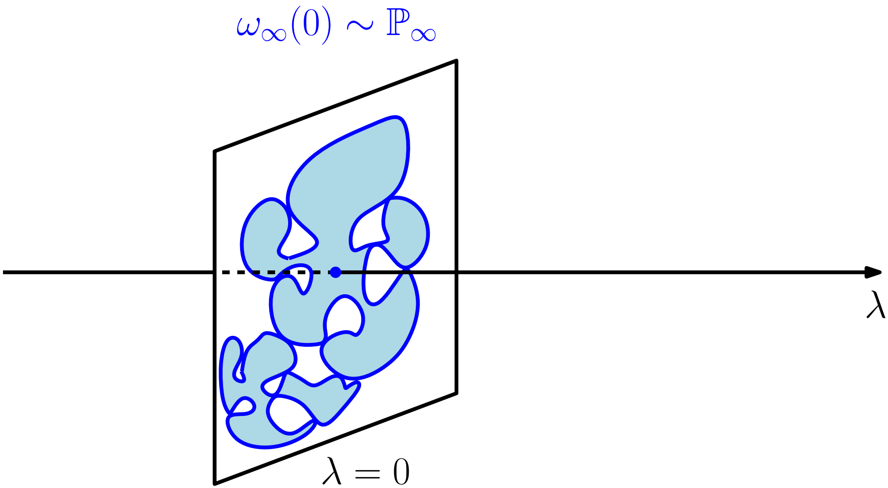
1. This suggests the following approach to handle the case $\lambda \neq 0$: for all $p \neq p_c(\mathbb{T}) = 1/2$, find a **massive harmonic observable** F_p:

$$\boxed{\Delta F_p(x) \approx m(p) F_p(x)}$$

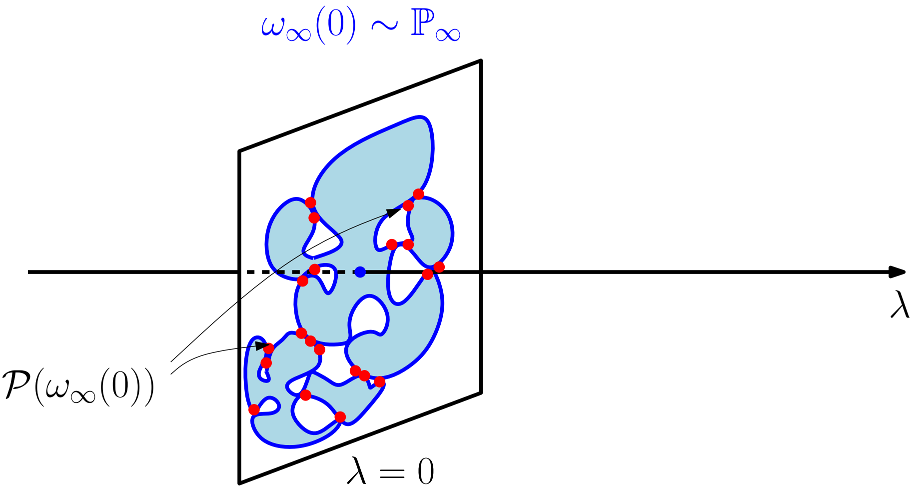
The "mass" $m(p)$ should then scale as $|p - p_c|^{8/3}$.

2. A "perturbative" approach.

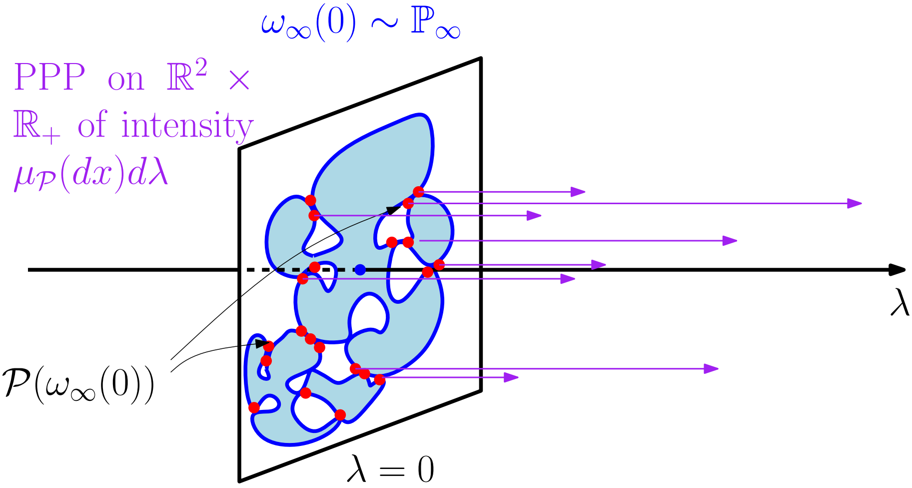
Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$



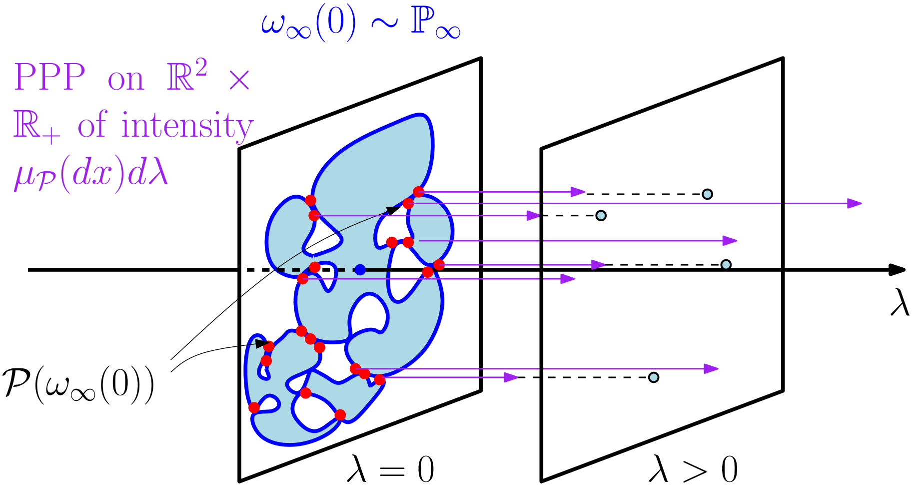
Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$



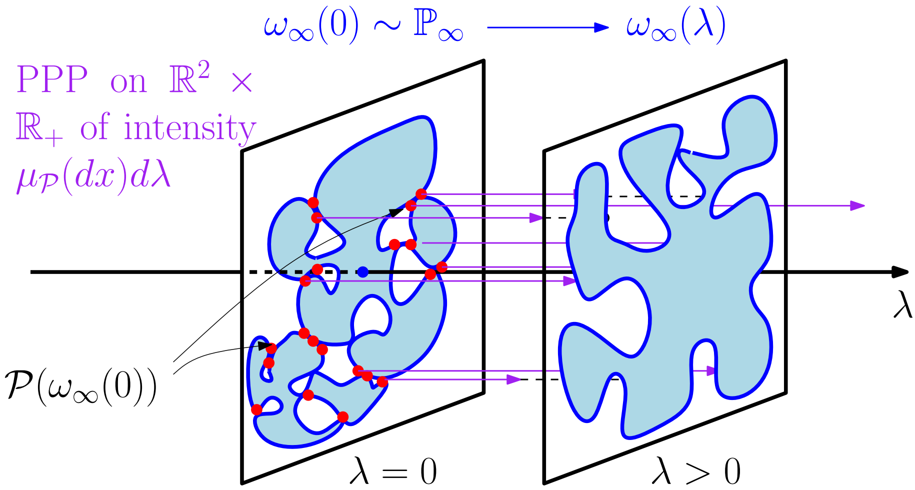
Naïve Strategy to build $\lambda \mapsto \omega_\infty^{nc}(\lambda)$



Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\text{nc}}(\lambda)$

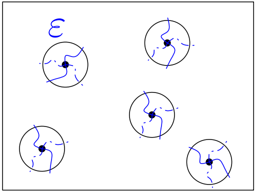


Naïve Strategy to build $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$



Difficulty 1: "too many" pivotal points

The mass measure μ is highly degenerate (∞)

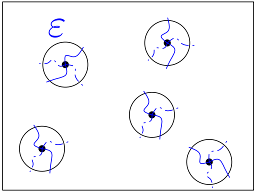


$\Rightarrow$ introduce a cut-off $\varepsilon > 0$ and try to define μ^ε, a mass measure on the pivotal points $\mathcal{P}^\varepsilon$.

Difficulty 1: "too many" pivotal points

The mass measure μ is highly degenerate (∞)

$\Rightarrow$ introduce a cut-off $\varepsilon > 0$ and try to define μ^ε, a mass measure on the pivotal points $\mathcal{P}^\varepsilon$.

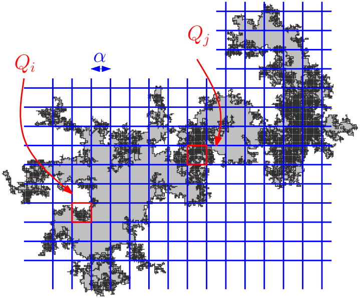


Theorem (GPS 2013)

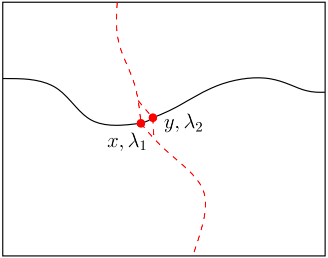
There is a measurable map μ^ε from $\mathscr{H}$ to the space of locally finite measures such that

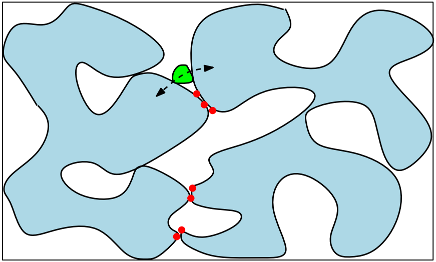
$$(\omega_\eta, \mu^\varepsilon(\omega_\eta)) \xrightarrow{(d)} (\omega_\infty, \mu^\varepsilon(\omega_\infty))$$

as $\eta \searrow 0$



$\lambda \mapsto \omega_\eta^{\mathrm{nc},\varepsilon}(\lambda) \Rightarrow$ STABILITY problem as $\varepsilon \searrow 0$?





Theorem (GPS 2013)

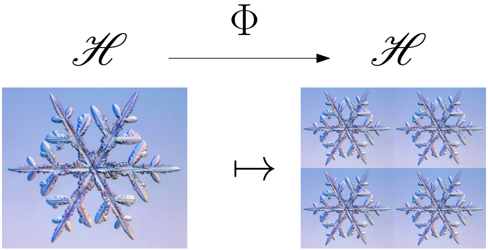
There is a function $\psi : [0,1] \to [0,1]$, *with* $\psi(0) = 0$ *so that unif. in* $0 < \eta < \varepsilon$,

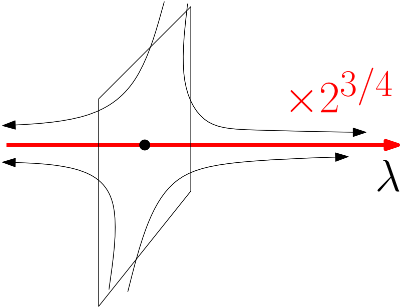
$$\mathbb{E}\left[d_{\mathsf{Sk}}(\omega_\eta(\cdot), \omega_\eta^\varepsilon(\cdot))\right] \leq \psi(\varepsilon)$$

Theorem

Near-critical percolation behaves as follows under the scaling $z \mapsto \alpha \cdot z$:

$$\left(\lambda \mapsto \alpha \cdot \omega_\infty^{\mathrm{nc}}(\lambda)\right) \overset{(d)}{=} \left(\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\alpha^{-3/4}\lambda)\right)$$





Some other properties

1. Conformal covariance

Some other properties

1. Conformal covariance
2. Obtain scaling limits of
   (i) Invasion percolation
   (ii) Gradient percolation
   (iii) **Dynamical percolation**

Some other properties

1. Conformal covariance
2. Obtain scaling limits of
   (i) Invasion percolation
   (ii) Gradient percolation
   (iii) **Dynamical percolation**
3. Two natural Markov processes on $\mathscr{H}$

Theorem

- $t \mapsto \omega_\infty(t)$ *is a reversible Markov process for the measure* $\mathbb{P}_\infty$.
- $\lambda \mapsto \omega_\infty^{\text{nc}}(\lambda)$ *is a non-reversible* ***time-homogeneous*** *Markov process.*

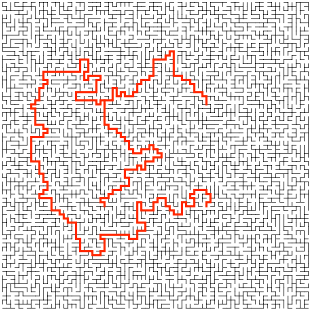
Some other properties

1. Conformal covariance
2. Obtain scaling limits of
   - (i) Invasion percolation
   - (ii) Gradient percolation
   - (iii) **Dynamical percolation**
3. Two natural Markov processes on $\mathscr{H}$

Theorem

- $t \mapsto \omega_\infty(t)$ *is a reversible Markov process for the measure* $\mathbb{P}_\infty$.
- $\lambda \mapsto \omega_\infty^{\mathrm{nc}}(\lambda)$ *is a non-reversible* ***time-homogeneous*** *Markov process.*
- *!! These are NOT Feller processes.*

Main theorem for the scaling limit of the MST

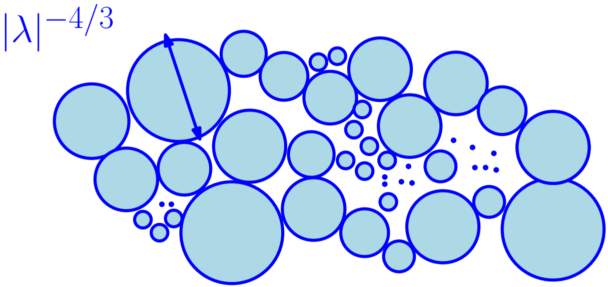


Theorem (GPS 2013)

1. *On the rescaled triangular lattice* $\eta\mathbb{T}$, MST_η *converges in law to* MST_∞ *(under the topology used in ABNW 1999)*
2. *The UNIVERSALITY of this limit only requires the universality of the critical slice of percolation*

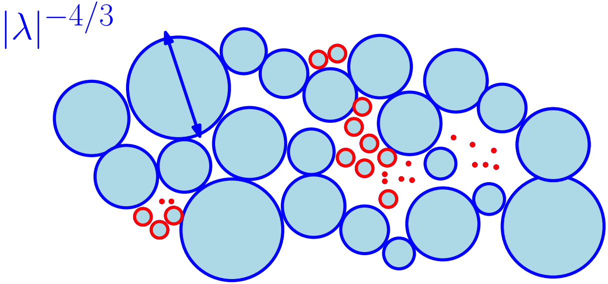
Very rough idea of proof

Take $\lambda \approx -\infty$



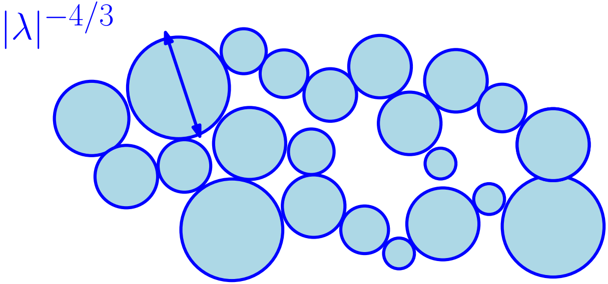
Very rough idea of proof

Take $\lambda \approx -\infty$
Take ϵ small



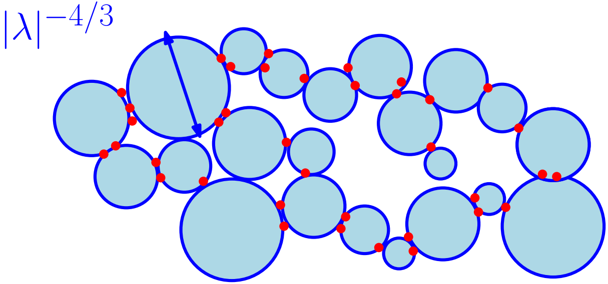
Very rough idea of proof

Take $\lambda \approx -\infty$
Take ϵ small



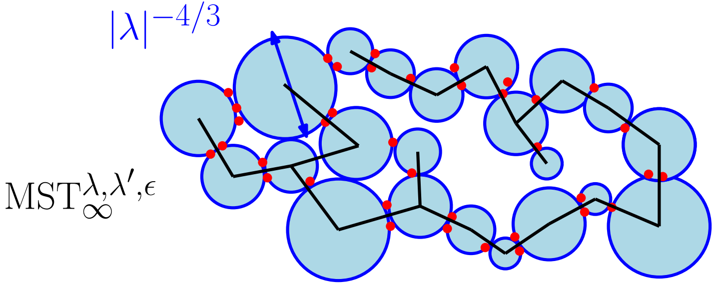
Very rough idea of proof

Take $\lambda \approx -\infty$
Take ϵ small
Take $\lambda' \approx \infty$ small



Very rough idea of proof

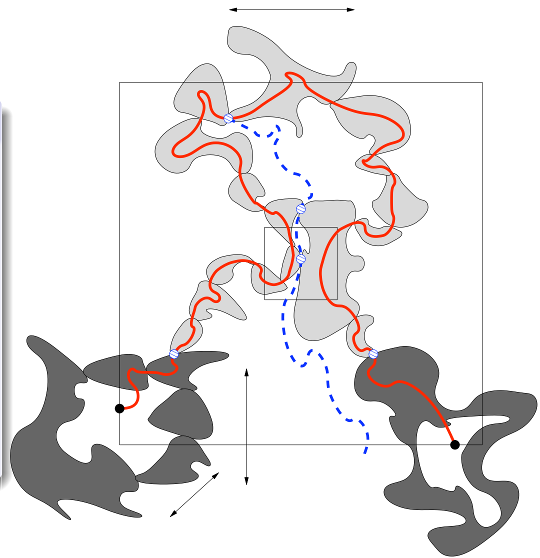
Take $\lambda \approx -\infty$
Take ϵ small
Take $\lambda' \approx \infty$ small



A.s. properties of MST_∞

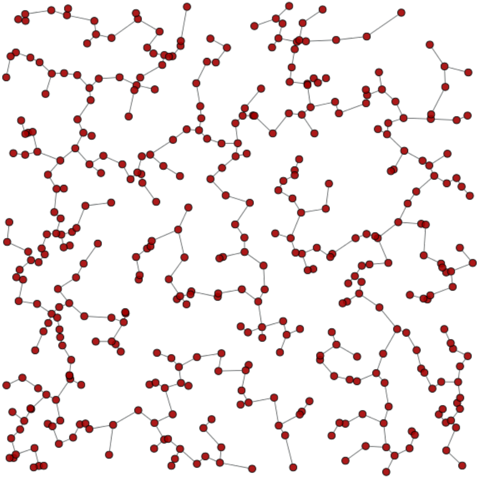
Theorem (GPS 2013)

1. ***Rotational invariance***
2. *The Hausdorff dimension of the branches a.s. lies in* $(1+\varepsilon, 7/4-\varepsilon)$
3. *There are no points of degree* ≥ 5
4. *There are no pinching points*



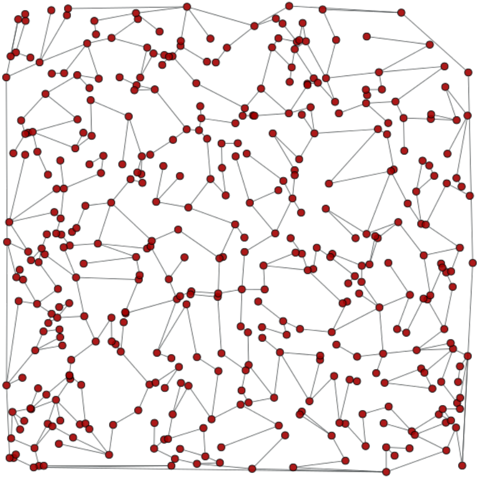
Some open questions

- Show that MST_∞ is not conformally-invariant
- Find the Hausdorff dimension d of branches (d ??)
- Show that $\text{MST}_\infty \neq \text{SLE}_8$!!!



Some open questions

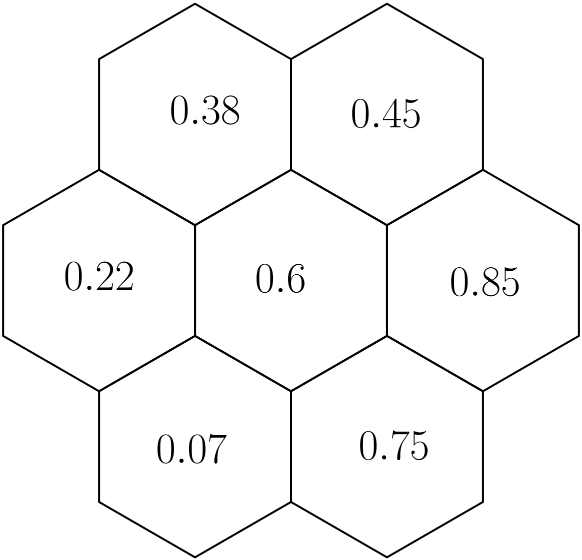
- Show that MST_∞ is not conformally-invariant
- Find the Hausdorff dimension d of branches (d ??)
- Show that $\text{MST}_\infty \neq \text{SLE}_8$!!!



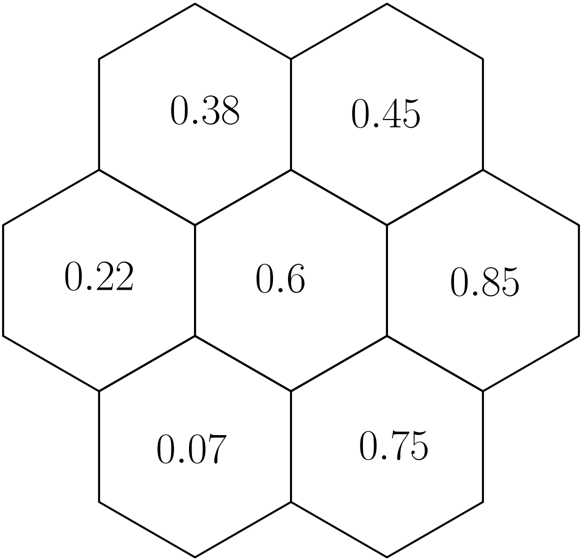
The end

The end

Back to the MST in the plane

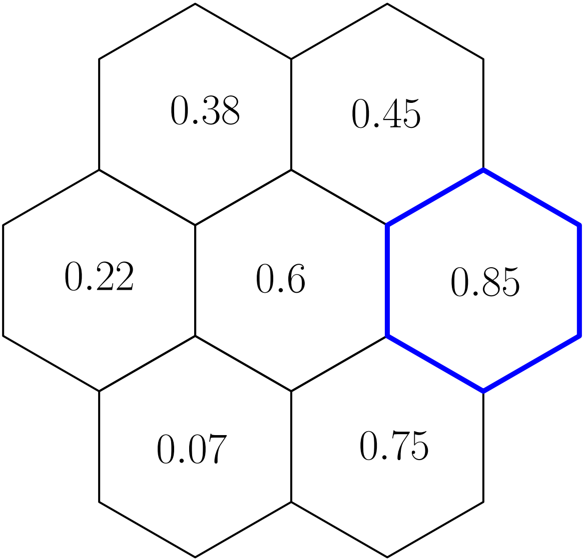


Back to the MST in the plane



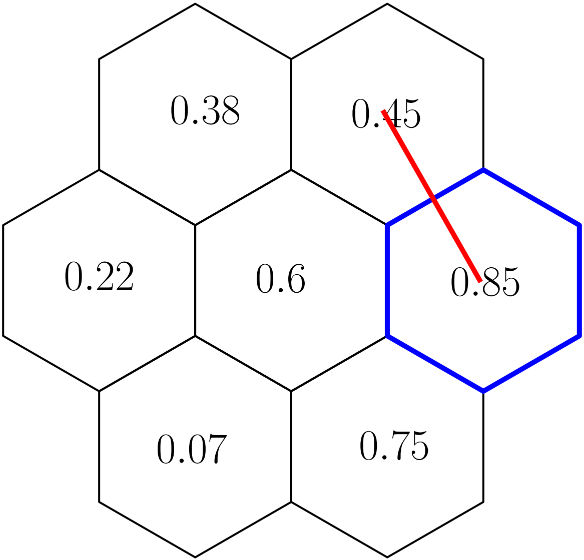
$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



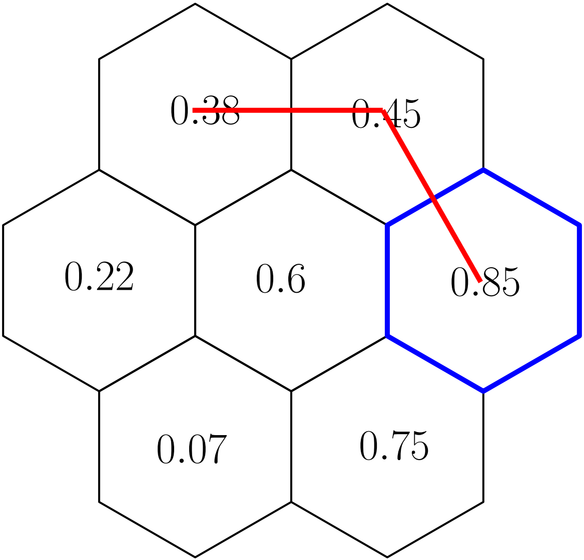
$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



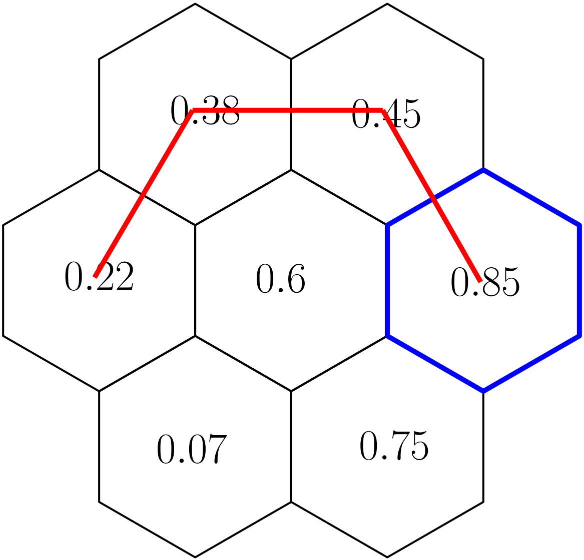
$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



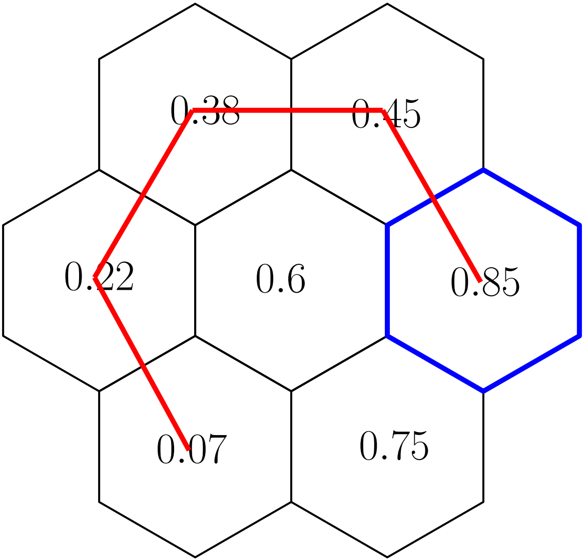
$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



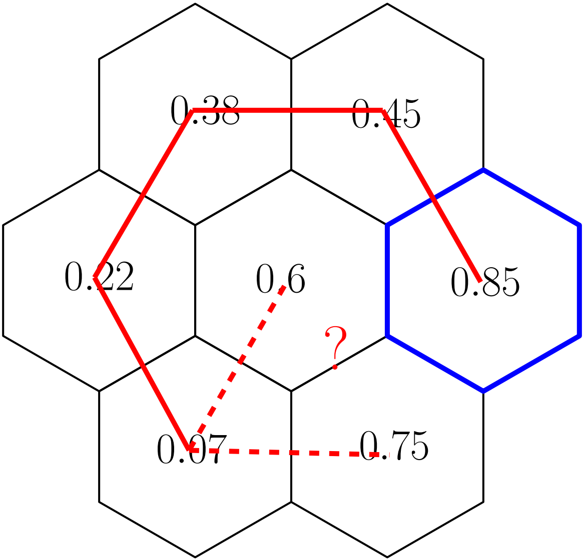
$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



$$u(e) := u_x \wedge u_y$$

Back to the MST in the plane



$$u(e) := u_x \wedge u_y$$