# Extremal Processes of Branching Brownian Motions

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SFB 1060

PROBABILISTIC STRUCTURES
IN EVOLUTION
DFG SPP 1590

# Plan

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PSE PROBABILISTIC STRUCTURES IN EVOLUTION DFG SPP 1590

# Plan

1. Gaussian processes on trees
2. Branching Brownian motion
3. The extremal process of BBM
4. Variable speed BBM
5. Universality

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This is too hard in general, but we will look at a setting where these questions have a chance to be answered. Branching Brownian motion is at the heart of this setting.

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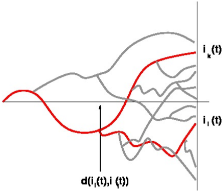

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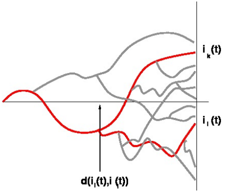
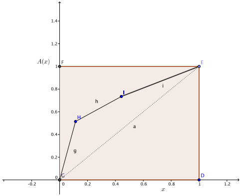
- For fixed time horizon $t$, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.





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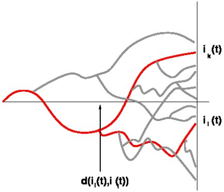


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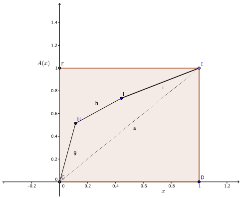


Can be constructed as time change of branching Brownian motion

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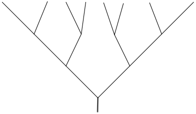

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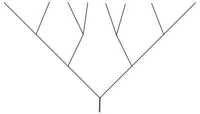

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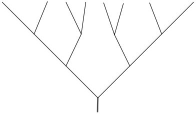

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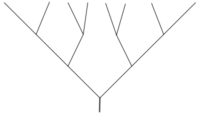

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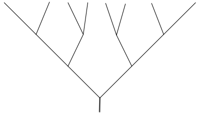

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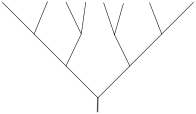

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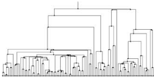

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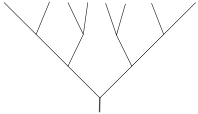

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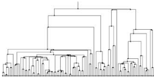

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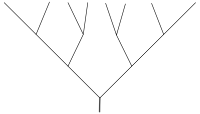

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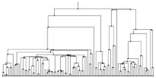

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- General $A$: variable speed BBM [Derrida-Spohn '88, Fang-Zeitouni '12]

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- Is there a limiting extremal process, $\mathcal{P}$, such that

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where PPP($\mu$) denotes the Poisson Point Process with intensity $\mu$.

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Note in particular that as long as $A(s) \le s$, for all $s \le 1$, then $\bar{A}(s) = s$, and the order of the maximum is the same as in the REM.

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Note the special role of the linear function $A(s) = s$

# Branching Brownian motion


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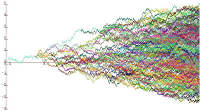

Picture by **Matt Roberts**, Bath

BBM is the canonical model of a spatial branching process.

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One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

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Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: $v$,
- death: $-v^2$,
- diffusive migration: $\partial_x^2 v$.

# F-KPP equation and BBM

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**Lemma** (McKeane '75, Ikeda, Nagasawa, Watanabe '69)

*Let* $f : \mathbb{R} \to [0,1]$ *and* $\{x_k(t) : k \leq n(t)\}$ *BBM.*

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

*Then* $v \equiv 1 - u$ *is the solution of the F-KPP equation with initial condition* $v(0,x) = 1 - f(x)$.

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**Theorem** (Bramson '78)

*The equation*

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' - \omega^2 + \omega = 0.$$

*has a unique solution satisfying* $0 < \omega(x) < 1$, $\omega(x) \to 0$, *as* $x \to +\infty$, *and* $\omega(x) \to 1$, *as* $x \to -\infty$, *up to translation, i.e. if* $\omega, \omega'$ *are two solutions, then there exists* $a \in \mathbb{R}$ *s.t.* $\omega'(x) = \omega(x + a)$.

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*For suitable initial conditions,*

$$u(t, x + m(t)) \to \omega(x),$$

*where* $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$, *where* $\omega$ *is one of the stationary solutions.*

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In particular, it gives Bramson's celebrated result

$$\lim_{t\to\infty} \mathbb{P}(\max_{k\leq n(t)} x_k(t) - m(t) \leq x) = \omega(x)$$

# The derivative martingale

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$Z \overset{(d)}{=} \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale,

$$Z(t) = \sum_{k \leq n(t)} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

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The form $(*)$ seems universal, but $Z$ is particular.
For the REM on the GW tree $(*)$ holds with $Z$ a standard exponential.

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**Poisson Point Process:** $\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right)$

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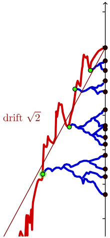
**Cluster process:**

$$\Delta(t) \equiv \sum_k \delta_{x_k(t)-\max_{j\leq n(t)} x_j(t)}.$$

conditioned on the event $\left\{\max_{j\leq n(t)} x_j(t) > \sqrt{2}t\right\}$ converges in law to point process, $\Delta$.

[Chauvin, Rouault '90]




$$\mathcal{E} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}, \qquad \Delta^{(i)} \text{ iid copies of } \Delta$$

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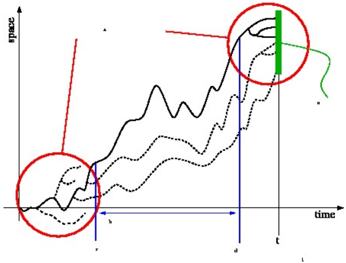


**Interpretation:**

$p_i$: positions of maxima of clusters with recent common ancestors.

$\Delta^{(i)}$: positions of members of clusters seen from their maximal one

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for any $\phi \in \mathcal{C}_c(\mathbb{R})$ non-negative, where

$$C(\phi) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 - u(t, y + \sqrt{2}t)\right) y e^{\sqrt{2}y} dy$$

$u(t, y)$: solution of F-KPP with initial condition $u(0, y) = e^{-\phi(y)}$.

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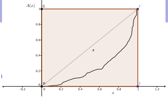
Then show that the limit is the Laplace functional of the process $\mathcal{E}$ described above.

# Variable speed BBM.....below the straight line...



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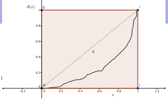

## Theorem (B-Hartung '13,'14)

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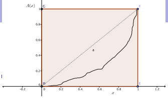

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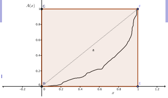


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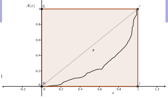

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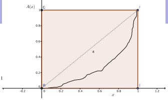

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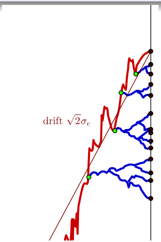
$\tilde{m}(t) \equiv \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.

$p_i$: e the atoms of a PPP$(C(b)Y_a e^{-\sqrt{2}x}dx)$,

$\Delta$: are as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2}bt\}$.



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1) Explicit construction for the case of two speeds:

$$A(s) = \begin{cases} s\sigma_1^2 & 0 \le s < b \\ b\sigma_1^2 + (s-b)\sigma_2^2 & b \le s \le 1 \end{cases}, \quad 0 < b \le 1.$$

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- Asymptotics of solutions of the FKPP equation at very large values ahead of the travelling wave.

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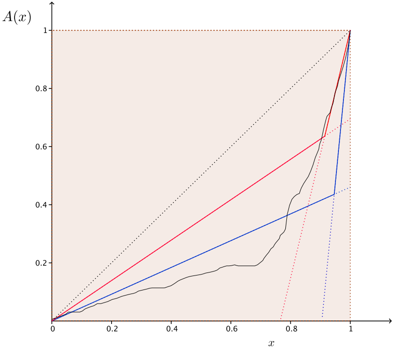
2) Gaussian comparison for general $A$:

Use comparison for Laplace functionals with two-speed process; only good approximation of covariance near 0 and 1 needed.



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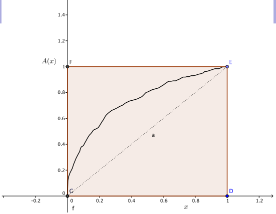


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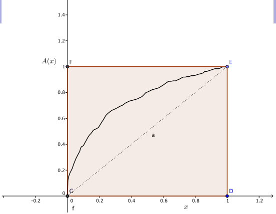


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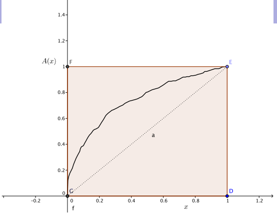


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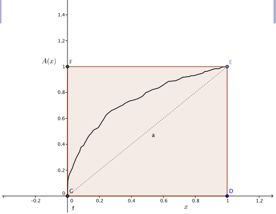


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- If $A$ is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

- If $A$ is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is

$$m(t) = C_\sigma t - D_\sigma t^{1/3} - \sigma^2(1) \ln t$$

(with explicit constants $C_\sigma$ and $D_\sigma$) but there are no explicit limit laws or limit processes available.

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- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
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- Statistics of zeros of Riemann zeta-function [Fyodorov, Keating '12]

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# Thank you for your attention!



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