CONTINUITY OF THE EXPLOSION TIME IN STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Stochastic ordinary differential equations may have solutions that explode in finite time. In this article we prove the continuity of the explosion time with respect to the different parameters appearing in the equation, such as the initial datum, the drift and the diffusion.

1. INTRODUCTION

In this paper we consider the following stochastic differential equation (SDE):

$$dx = b(x)\,dt + \sigma(x) \circ dw, \tag{P}$$

with $x(0) = x_0 \in \mathbb{R}_{>0}$. Here b and σ are smooth positive functions (C^1 will be enough for our calculations) and w is a (one dimensional) Wiener process defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets, [7]).

It is well known that stochastic differential equations like (P) may explode in finite time. That is, trajectories may diverge to infinity as t goes to some finite time T that in general depends on the particular path.

The *Feller Test for explosions* ([7, 8]) gives a precise description in terms of b and σ of whether explosions in finite time occur with probability zero, positive or one. For example, if b and σ behave like powers at infinity, i.e., $b(s) \sim s^p$ and $\sigma(s) \sim s^q$ as $s \to \infty$, applying the Feller test one obtains that solutions to (P) explode with probability one if $p > \max\{2q, 1\}$. We use $f(s) \sim g(s)$ to mean that there exist constants $0 < c < C$ such that $cg(s) \leq f(s) \leq Cg(s)$ for large enough s. The intuition behind this condition is that $p > 2q$ ensures that the asymptotic behavior of the solutions is governed by the drift term while $p > 1$ impose the solution to grow up so fast that explodes in finite time, as happens in the deterministic case ($\sigma = 0$).

Stochastic differential equations with explosions have been considered, for example, in fatigue cracking (fatigue failures in solid materials) with b and σ of power type, see [12], where solutions may explode in finite time. This explosion time is generally random, depends on the particular sample path and corresponds to the time of ultimate damage or fatigue failure in the material.

For deterministic one-dimensional ODEs ($\sigma = 0$), the dependence of the explosion time T with respect to the different parameters entering in the problem is very well understood, thanks

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to the explicit formula

$$T = \int_{x_0}^{\infty} \frac{1}{b(s)}\, ds.$$

In more general situations (N-dimensional deterministic ODEs, SDEs or parabolic PDEs), where no such explicit formula is available, the situation gets a lot more complicated.

In parabolic semilinear PDEs, typically of the form $u_t - \Delta u = u^p$, for example with Dirichlet boundary conditions, the continuous dependence of the explosion time on the initial data has deserved a great deal of attention and effort. See for instance [1, 5, 6, 9, 10] and also [11] for a general result on the continuity of the explosion time in a general semiflow context.

For systems of (deterministic) ODEs, or even for nonautonomous one-dimensional ODEs, there is no general result concerning the continuous dependence of the explosion time with respect to the initial data or with respect to parameters. Up to our knowledge, the only result that treats a related issue for SDEs is [4] where the authors analyze the behavior of the explosion time under small stochastic perturbations of a one-dimensional ODE.

This paper consists in an abstract result on continuity of the explosion time under structural hypotheses and, as an application of this result, we get the continuity of the explosion time in stochastic differential equations with respect to the initial datum, the drift and the diffusion.

The main idea used in the proofs is to obtain estimates for the first time where two solutions spread at a fixed distance. This idea was previously used in [3, 5] and [11]. The main results on this paper can be summarized as follows:

(1) *Assume b/σ is nondecreasing, $x(t)$ is a solution to (P) with initial datum x_0 and $x_n(t)$ is a solution to (P) with initial datum x_n. Let T and T_n be the explosion times for $x(t)$ and $x_n(t)$ respectively. If $x_n \to x_0$ then $T_n \to T$ a.s.*

(2) *For additive or multiplicative noise, under adequate hypotheses, if $b_n \to b$ and $\sigma_n \to \sigma$ then $T_n \to T$ a.s.*

Organization of the paper. In Section 2 we prove an abstract result on convergence of explosion times for sequences of functions with explosion; in Section 3 we recall some known results on the relation between SDEs and random differential equations; in Section 4 we study the dependence of T on the initial datum; finally in Section 5 we look at the dependence of T with respect to b and σ in two relevant examples; additive and multiplicative noise.

2. An abstract result

In this section we prove a very general result on convergence of the explosion times. Let T_n, T be real numbers and u_n, u functions with values on a Banach space equipped with norm $\|\cdot\|$ such that the following hypotheses hold:

Continuation property:

$$\text{(H1)} \qquad \lim_{t\to T} \|u(t)\| = \infty, \quad \lim_{t\to T_n} \|u_n(t)\| = \infty.$$

That is, we assume that both u and u_n explode in finite times, T and T_n respectively.

Continuous dependence:

$$\text{For every } t < T \text{ it holds } \lim_{n\to\infty} \sup_{s\in[0,t]} \|u_n(s) - u(s)\| = 0. \tag{H2}$$

That is, we are assuming that u_n approaches u as $n \to \infty$ at times at which u is well defined and bounded.

Uniform upper explosion estimate: There exists a nondecreasing continuous function G, independent of n, such that

$$\|u_n(t)\| \le G\Big(\frac{1}{T_n - t}\Big). \tag{H3}$$

We are assuming that we have a uniform (in n) bound on the explosion rate of the sequence u_n.

The main result of the section, is the following:

Theorem 2.1. *If* (H1)–(H3) *hold, then*

$$\lim_{n\to\infty} T_n = T.$$

We divide the proof of the theorem into two propositions.

Proposition 2.2. *If* (H1)–(H3) *hold, then*

$$\limsup_{n\to\infty} T_n \le T.$$

Proof. It is enough to consider n such that $T_n > T$. Set

$$e_n(t) = \|u_n(t) - u(t)\|.$$

We have $e_n(0) = o(1)$. Assume that for all $t < T$, $e_n(t) < 1$, then $T_n \le T$ due to (H1); but this is impossible. Hence, there exists a first time $t_n < T$ such that $e_n(t_n) = 1$. Hypotheses (H2) implies that $t_n \to T$ since for any subsequence t_{n_k} satisfying $\sup t_{n_k} < T$ we have $1 = e(t_{n_k}) \to 0$. Finally from (H3) we get

$$G\Big(\frac{1}{T_n - t_n}\Big) \ge \|u_n(t_n)\| \ge \|u(t_n)\| - 1 \to \infty, \quad n \to \infty.$$

Recall that, due to (H3), $G(s) \to \infty$ as $s \to 0$. Consequently, $T_n - t_n \to 0$ as $n \to \infty$ and hence

$$\limsup_{n\to\infty} T_n \le \limsup_{n\to\infty} T_n - t_n + T = T,$$

as we wanted to prove. □

The lower semicontinuity is an easy consequence of continuation and continuous dependence properties. We recall the following to complete the proof of the theorem.

Proposition 2.3. *If* (H1) *and* (H2) *hold, then*

$$\liminf_{n\to\infty} T_n \ge T.$$

Proof. We need only consider n such that $T_n < T$. We use the same notation as in the previous proof. By (H1), there is a first time $t_n < T_n$ such that $e_n(t_n) = 1$ and (H2) implies that $t_n \to T$. Since $T_n > t_n$, it follows that $\liminf_{n\to\infty} T_n \ge T$, as we wanted to prove. □

A counterexample. Let us now see that if (H3) fails we can have a sequence verifying (H1) and (H2) but with $T_n \not\to T$. In order to see this fact it suffices to consider a one dimensional deterministic ODE. In fact, let us consider

$$\begin{cases} \dot{u} = u^2, \\ u(0) = 1. \end{cases}$$

The explicit solution is

$$u(t) = \frac{1}{1-t}, \qquad (0 < t < 1).$$

Now take

$$f_n(x) = \begin{cases} x^2 & 0 \le x \le n \\ a_n x^{p_n} + (n^2 - a_n n^{p_n}) & x > n, \end{cases}$$

and consider

$$\begin{cases} \dot{u}_n = f_n(u_n), \\ u_n(0) = 1. \end{cases}$$

Remark that u_n and u coincide until they reach level n, that is,

$$u_n(t) = u(t) \qquad \text{for all } t \le 1 - \frac{1}{n}. \tag{2.1}$$

Therefore u_n is a solution of

$$\begin{cases} \dot{u}_n = a_n u_n^{p_n} + (n^2 - a_n n^{p_n}), \\ u_n(1 + 1/n) = n. \end{cases} \tag{2.2}$$

Assume that $p_n > 1$, then u_n explodes in finite time T_n (hypothesis (H1)) and, from (2.1) it is clear that (H2) holds.

From (2.2), we obtain

$$T_n = 1 - \frac{1}{n} + \int_n^{+\infty} \frac{ds}{a_n s^{p_n} + (n^2 - a_n n^{p_n})},$$

that is, changing variables, $nu = s$,

$$T_n = 1 - \frac{1}{n} + \int_1^{+\infty} \frac{du}{n^{p_n - 1} a_n (u^{p_n} - 1) + n}.$$

Let us choose

$$a_n = \frac{1}{n^{p_n - 1}},$$

we obtain,

$$T_n = 1 - \frac{1}{n} + \int_1^{+\infty} \frac{du}{(u^{p_n} - 1) + n} \ge 1 - \frac{1}{n} + \int_{(n+1)^{1/p_n}}^{+\infty} \frac{du}{2u^{p_n}}.$$

Therefore, if we choose

$$p_n = 1 + \frac{1}{n}$$

we get

$$T_n \ge 1 - \frac{1}{n} + \frac{n}{2(n+1)^{\frac{1}{n+1}}} \to +\infty, \qquad n \to \infty.$$

Therefore

$$\lim_{n\to\infty} T_n = +\infty \neq T = 1.$$

It is clear that we can modify this example in such a way that $(f_n)_n \subset C^\infty$ (here they are only continuous) and moreover for any $K > 1$ we can select p_n, a_n in such a way that

$$T_n \to K, \qquad n \to \infty.$$

The main obstruction that prevents convergence of the explosion times in this example is the fact that the sources f_n and f are far away at infinity for every n in such a way that f_n grows very slowly making the blow-up times increase (and even go to infinity) with n. In the one dimensional autonomous deterministic case, $\dot{u} = f(u)$ and $\dot{u}_n = f_n(u_n)$ with the same initial datum $u(0) = u_n(0) = u_0$, a sufficient condition to assure (H3) is $f_n \geq g$ with $\int^\infty 1/g < \infty$. This fact plus pointwise convergence of f_n to f implies the convergence of the blow-up times, $T_n \to T$. This can be easily proved using the explicit formula for the explosion times

$$T_n = \int_{u_0}^{+\infty} \frac{1}{f_n(s)}\, ds$$

and the Dominated Convergence Theorem.

3. From SDE to Random Differential Equations

As proved in [13] (see also [7] pp. 295–297), one can get a solution to (P) by means of a random differential equation. Let us define $\phi(t,z)$ the flux associated to the ODE

$$\dot{\phi}(t,z) = \sigma(\phi(t,z)), \qquad \phi(0,z) = z.$$

Observe that if σ is globally Lipschitz, $\phi(t,z)$ is globally defined. Let

$$H(z,t) := \frac{b(\phi(t,z))\,\sigma(z)}{\sigma(\phi(t,z))}. \tag{3.3}$$

By simple computation one can check that

$$H(z,t) = \frac{b(\phi(t,z))}{\partial_z \phi(t,z)}.$$

Now, for any $\omega \in \Omega$ fixed, such that the path of the Wiener process $w(\cdot,\omega)$ is continuous, we consider $z(t)$ to be the solution of the (deterministic, non autonomous) ODE:

$$\dot{z} = H(z(t), w(t,\omega)). \tag{3.4}$$

This type of equations are known as *random differential equations* since the dependence on ω is just on the coefficients of an ODE.

The process $x(t)$ given by $x(t,\omega) := \phi(w(t,\omega), z(t))$ is a solution to (P) in the *Stratonovich sense* with initial datum $x(0) = x_0$. In fact

$$\begin{cases} dx(t) = \dfrac{\partial}{\partial t}\phi(w(t), z(t)) \circ dw(t) + \dfrac{\partial}{\partial z}\phi(w(t), z(t))\, dz(t) = \sigma(x(t)) \circ dw(t) + b(x(t))\, dt, \\ x(0) = x_0. \end{cases}$$

Observe that, as ϕ is globally defined, $x(t)$ explodes if and only if $z(t)$ does, and both variables explode at the same time.

If a SDE is given in Itô form, we can apply this result thanks to the well known conversion formula ([7]). In fact, $x(t)$ solves $dx = f(x)\,dt + g(x)\,dw$ if and only if it solves (P) with $b = f - \frac{1}{2}\sigma'\sigma$, $\sigma = g$.

4. Continuous dependence with respect to initial data

Now we combine the results of the previous sections to prove the continuity of the explosion time with respect to the initial data in stochastic differential equations.

Theorem 4.1. *Assume σ is globally Lipschitz and (b/σ) is nondecreasing. Let $x(t)$ and $x_n(t)$ be solutions to (P) with initial data x_0 and x_n respectively and assume that $x_n \to x_0$. If $x(t)$ explodes at a random time T, then $x_n(t)$ explodes at a random time T_n and*

$$T_n \to T \quad a.s.\ in\ \Omega.$$

Proof. Thanks to the previous section we can think of (P) as a random differential equation. To apply our general result proved in Section 2, we just need to show that (H1), (H2) and (H3) are verified. To this end observe first that (H1) holds since for one-dimensional ODEs with regular coefficients the existence (or not) of explosion does not depend on the initial datum, and (H2) is a consequence of the very well known result on continuous dependence with respect to initial data for (nonautonomous) ODEs. To prove (H3) we consider

$$S := \sup_{n\geq 1}\{T_n; T\}.$$

The monotonicity of the explosion time with respect to initial data implies $\mathbb{P}(S < \infty) = 1$. Now let $M, K > 0$ be two large constants and define

$$A_{K,M} := \{\omega \in \Omega \ : \ S(\omega) \leq K \text{ and } |w(t,\omega)| \leq M, \text{for } t \in [0, K+1]\}. \tag{4.1}$$

Observe that

$$\mathbb{P}\left(\bigcup_{K,M\geq 1} A_{K,M}\right) = 1.$$

Now note that b/σ nondecreasing is equivalent to $H(z,t)$ being nondecreasing in t, hence for $\omega \in A_{K,M}$, we have $H(z, w(t,\omega)) \geq H(z, -M)$ for every $z \in \mathbb{R}, 0 \leq t \leq S(\omega)$.

Let $z(t)$, $z_n(t)$ be solutions to (3.4) with initial data x_0, x_0^n respectively. As mentioned previously, $\phi(w(t), z(t))$, $\phi(w(t), z_n(t))$ solve (P) and the explosion times of $z(t)$ and $z_n(t)$ are T and T_n respectively.

We have

$$\dot{z}_n(t) = H(z_n(t), w(t)) \geq H(z_n(t), -M).$$

Integrating we obtain

$$\int_t^{T_n} \frac{\dot{z}_n(t)}{H(z_n(s), -M)} \geq T_n - t,$$

and changing variables,

$$\int_{z_n(t)}^{+\infty} \frac{du}{H(u, -M)} \geq T_n - t.$$

Let

$$g(\xi) := \left(\int_{\xi}^{+\infty} \frac{du}{H(u,-M)} \right)^{-1}.$$

Since g is increasing, its inverse $G := g^{-1}$ is also increasing and then we have

$$z_n(t) \leq G\left(\frac{1}{T_n - t} \right).$$

Hence (H3) is also verified and the result follows. □

From this theorem we obtain the following corollaries.

Corollary 4.2. *Under the same hypotheses of Theorem 4.1, if x_0 and x_0^n are random variables (with the usual assumptions that guarantee existence and uniqueness) such that $x_0^n \to x_0$ a.s. in Ω, then $T_n \to T$ a.s. in Ω.*

Proof. This result follows exactly as in the proof of Theorem 4.1. Just observe that the arguments used there works for $\omega \in \Omega$ fixed, and so it is irrelevant if the initial datum is deterministic or not as far as $x_0^n \to x_0$ for almost every ω. □

Corollary 4.3. *Under the same hypotheses of Theorem 4.1, if $x_0^n \to x_0$ in probability, then $T_n \to T$ in probability.*

Proof. The proof of this corollary is just an application of Corollary 4.2, taking into account the following lemma. □

Lemma 4.4. *Let $x_n, x : \Omega \to \mathbb{R}$ be random variables. Then $x_n \to x$ in probability if and only if for every subsequence x_{n_k} there exists a sub-subsequence $x_{n_{k_j}}$ such that $x_{n_{k_j}} \to x$ a.s. in Ω.*

Proof. It is clear that if $x_n \to x$ in probability, then the conclusion of the Lemma follows.

To prove the converse we argue by contradiction. Then there exists $\varepsilon > 0$, $\delta > 0$ and a subsequence x_{n_k}, such that

$$\mathbb{P}(|x_{n_k} - x| > \varepsilon) > \delta. \tag{4.2}$$

By hypothesis there exists a sub-subsequence, $x_{n_{k_j}}$, such that $x_{n_{k_j}} \to x$ a.s. Hence, by Egoroff's Theorem, there exists a set $A \subset \Omega$ with $\mathbb{P}(A) < \delta/2$ such that $x_{n_{k_j}} \to x$ uniformly in $\Omega \setminus A$. Therefore, for j large (depending on ε), we have $|x_{n_{k_j}} - x| < \varepsilon$ in $\Omega \setminus A$. This contradicts (4.2) and the proof is complete. □

Remark 4.1. It is worth noticing that both in the additive noise and the multiplicative noise cases the hypotheses of Theorem 4.1 are verified if and only if the drift satisfies $\int 1/b < +\infty$. The explosion of the solutions with probability one can be checked by means of the Feller Test or by direct computation. Also observe that in these cases the hypotheses of Theorem 4.1 are verified if the equation is considered in Itô or in Stratonovich sense.

5. Continuous dependence with respect to b and σ

In this section we show in the two most important examples, additive and multiplicative noise, how the abstract result of Section 2 can be applied to deal with perturbations of the drift and the diffusion.

The idea is as follows: First, one constructs H and H_n as in Section 3. Then one has to verify (H1)–(H3) for $z(t)$ and $z_n(t)$ the solutions to

$$\dot{z} = H(z, w(t)) \qquad \text{and} \qquad \dot{z}_n = H_n(z_n, w(t))$$

respectively, for almost every $\omega \in \Omega$.

5.1. **Additive noise: σ constant.**

Assume that σ is a positive constant. In this case we have

$$\phi(t, z) = z + \sigma t,$$

and

$$H(z, t) = b(z + \sigma t).$$

Therefore H is increasing in both variables if b is increasing and $\sigma > 0$.

Assume that we have $\sigma_n \to \sigma$ and $b_n \to b$, uniformly on compact sets with

$$b_n(s),\ b(s) \geq h(s), \qquad \text{and} \int^{\infty} \frac{1}{h(s)}\, ds < \infty.$$

It follows that (H1) holds for almost every ω by direct computation. Also (H2) holds by the uniform convergence on compact sets of b_n to b for $\omega \in A_{K,M}$, where $A_{K,M}$ is given by (4.1).

To check (H3) we observe that

$$T - t = \int_t^T \frac{\dot{u}_n}{b_n(u_n + \sigma_n w)} \leq \int_t^T \frac{\dot{u}_n}{b_n(u_n - \sigma_n M)} = \int_{u_n(t)-\sigma_n M}^{\infty} \frac{ds}{b_n(s)} \leq \int_{u_n(t)-\sigma_n M}^{\infty} \frac{ds}{h(s)}.$$

Hence, if we call

$$g(x) = \int_x^{\infty} \frac{ds}{h(s)},$$

we have, for large n,

$$u_n(t) \leq g^{-1}(T - t) + 2\sigma M =: G\left(\frac{1}{T - t}\right),$$

and so $T_n \to T$.

5.2. **Multiplicative noise: σ linear.**

Assume that $\sigma(s) = as$ with $a > 0$. Recall that ϕ is the solution of

$$\begin{cases} \dot{\phi} = \sigma(\phi) = a\,\phi, \\ \phi(0, z) = z. \end{cases}$$

Therefore in this case we have $\phi(t, z) = ze^{a\,t}$ and hence

$$H(z, t) = \frac{b(ze^{a\,t})}{e^{a\,t}}.$$

Assume that we have $a_n \to a$ and $b_n \to b$, uniformly on compact sets. It follows that (H1) and (H2) hold (by the uniform convergence on compact sets of b_n to b) for $\omega \in A_{K,M}$.

Now we look for (H3). As before, we impose that

$$b_n(s),\, b(s) \geq h(s), \qquad \text{with} \int^{\infty} \frac{1}{h(s)}\, ds < \infty.$$

Proceeding as before, we obtain, for n large enough,

$$\begin{aligned} T - t \ &= \int_t^T \frac{\dot{u}_n\, e^{a_n w}}{b_n(u_n\, e^{a_n w})} \leq \int_t^T \frac{\dot{u}_n e^{a_n M}}{b_n(u_n\, e^{-a_n M})} \\ &= e^{2a_n M} \int_{u_n(t)e^{-a_n M}}^{\infty} \frac{ds}{b_n(s)} \leq e^{2a_n M} \int_{u_n(t)e^{-a_n M}}^{\infty} \frac{ds}{h(s)} \\ &\leq e^{4aM} \int_{u_n(t)e^{-\frac{a}{2}M}}^{\infty} \frac{ds}{h(s)}. \end{aligned}$$

Hence, if we call

$$g(x) = \int_x^{\infty} \frac{ds}{h(s)},$$

we have

$$u_n(t) \leq e^{\frac{a}{2}M} g^{-1}\left(\frac{T-t}{e^{4aM}}\right) =: G\left(\frac{1}{T-t}\right),$$

and $T_n \to T$.

Example. Both for additive and multiplicative noise, just consider $b_n(s) = \alpha_n s^{p_n}$, $b(s) = \alpha s^p$, with $\alpha_n \to \alpha > 0$, and $p_n \to p > 1$.

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