Numerical blow-up for the porous medium equation with a source

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Abstract

We study numerical approximations of positive solutions of the porous medium equation with a nonlinear source,

$$\begin{cases} u_t = (u^m)_{xx} + u^p, & (x,t) \in (-L, L) \times (0, T), \\ u(-L,t) = u(L,t) = 1, & t \in [0, T), \\ u(x, 0) = \varphi(x), \geq 1 & x \in (-L, L), \end{cases}$$

where $m > 1$, $p > 0$ and $L > 0$ are parameters. We describe in terms of p, m, and L when solutions of a semidiscretization in space exist globally in time and when they blow up in a finite time. We also find the blow-up rates and the blow-up sets, proving that there is no regional blow-up for the numerical scheme.

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1 Introduction.

In this paper we deal with numerical approximations of the following problem

$$\begin{cases} u_t = (u^m)_{xx} + u^p, & (x,t) \in (-L, L) \times [0,T), \\ u(-L,t) = u(L,t) = 1, & t \in [0,T), \\ u(x,0) = \varphi(x) \geq 1, & x \in (-L, L), \end{cases} \tag{1.1}$$

where $m > 1$ and $p > 0$ are parameters. Problem (1.1) can be thought as a model for nonlinear heat propagation. In this case u stands for the temperature and we are in presence of reaction (giving by the power u^p).

We assume that φ is smooth and compatible with the boundary conditions in order to obtain a regular solution. Also, for simplicity, we assume that φ is a bell shaped function, that is, φ is symmetric, $\varphi(x) = \varphi(-x)$ for $x > 0$, and decreasing in $[0, L]$. These symmetry properties will be preserved by our numerical scheme and make computations easier.

For many differential equations or systems the solutions can become unbounded in finite time, a phenomenon that is known as blow-up. Typical examples where this happens are parabolic problems involving nonlinear reaction terms, like (1.1). The solution of (1.1) only exists for a finite period of time ($T < \infty$, in this case u becomes unbounded in finite time and we say that it has blow-up) or it is defined for all positive t ($T = \infty$, in this case we call it a global solution). These problems have been widely analyzed from a mathematical point of view (see for example [SGKM], [P], [L] and the references therein) but, only few results concerning the numerical approximation of them can be found in the literature (we refer for example to [ALM1], [ALM2], [BB], [BK], [BHR], [DER], [FBR]). Indeed, even for an elementary ordinary differential equation $y' = f(y)$ having blow-up, the usual analysis to obtain error estimates and adaptive step procedures do not apply because they are based on regularity assumptions which are not satisfied in this case.

Here we analyze numerical approximations of blowing up solutions. By a semidiscretization in space we obtain a system of ordinary differential equations which is expected to be an approximation of the original problem. Our objective is to analyze whether this system has a similar behaviour than the original problem.

The nonlinear diffusion that we are considering here, $(u^{m-1}u_x)_x$, degenerates at level $u = 0$, giving only weak solutions if we allow the initial data (or the boundary conditions) to vanish. Since we are interested in blowing up solutions, we avoid this lack of regularity imposing $\varphi \geq 1$ and $u(-L,t) = u(L,t) = 1$. This gives us regular solutions (see [LSU]).

We want to remark that all our results are valid for regular solutions of problems of the form $u_t = (\phi(u))_{xx} + f(u)$ if we impose that $\phi(u) \sim u^m$ and $f(u) \sim u^p$ when u becomes large. To simplify the exposition we state and prove our results for (1.1).

Let us resume what is known for continuos problem (1.1) (see [SGKM] and references therein),

(i) If $p > m$ there exists blowing up solutions for initial data u_0 large enough. The blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{p-1}}$ and the blow-up set is a single point $x = 0$ (this means that u remains bounded away from the origin).

(ii) If $p = m$ the existence of blowing up solutions depends on the length of the interval. Let $\lambda_1(L) = \pi/2L$ be the first eigenvalue of the Laplacian in $[-L, L]$. Then, every solution blows up if and only if $\lambda_1(L) < 1$. That is, if $L > \pi/2$ every positive solution blows up and the blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{m-1}}$. In this case it is known that the blow-up set is given by

$$B(u) = \begin{cases} [-L, L] & \text{if } \pi/2 < L \le m\pi/(m-1), \\ (-m\pi/(m-1), m\pi/(m-1)) & \text{if } L > m\pi/(m-1). \end{cases}$$

Moreover, there exists a self-similar profile, $z(x)$, that gives the asymptotic behaviour for u near T in the form,

$$\lim_{t \nearrow T} (T-t)^{\frac{1}{m-1}} u(x,t) = z(x).$$

Let us remark that if $L > m\pi/(m-1)$ the support of z is the interval

$$\left(-\frac{m\pi}{(m-1)}, \frac{m\pi}{(m-1)} \right).$$

Therefore we have regional blow-up for large values of L. On the other hand, if $L \le \pi/2$ every solution is global.

(iii) If $p < m$ every solution is global.

Now we introduce the numerical scheme. We discretize using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable (it is well known that this discretization in space coincides with the classic central finite difference second order scheme), see [Ci]. Mass lumping is widely used in parabolic problems with blow-up, see for example [ALM1], [ALM2], [Ch], [DER], [N], [NU].

We denote with $U(t) = (u_{-N}(t), ..., u_N(t))$ the values of the numerical approximation at the nodes $x_i = ih$ $(h = L/N)$ at time t. Then $U(t)$ verifies the following equation:

$$\begin{cases} MU'(t) = -AU^m(t) + MU^p(t), \\ u_{-N}(t) = u_N(t) = 1, \\ U(0) = \varphi^I, \end{cases} \tag{1.2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and φ^I is the Lagrange interpolation of the initial datum, φ. Writing this equation explicitly we obtain the following ODE system,

$$
\begin{cases}
u_{-N}(t) = 1, \\
\\
u_k'(t) = \dfrac{1}{h^2}(u_{k+1}^m(t) - 2u_k^m(t) + u_{k-1}^m(t)) + u_k^p(t), \\
\\
u_N(t) = 1, \\
\\
u_k(0) = \varphi(x_k), \qquad -N+1 \le k \le N-1.
\end{cases} \tag{1.3}
$$

First we state a convergence result that says that for any positive τ, the method converges uniformly in sets of the form $[-L, L] \times [0, T-\tau]$. Let us observe that, since solutions are blowing up in L^∞ norm, it is natural to consider uniform convergence. Since the solution develops a singularity at time $t = T$ we cannot expect that the convergence result extends up to T.

Theorem 1.1 *Let $u(x,t) \in C^{4,1}([-L, L] \times [0, T-\tau])$ be a positive solution of* (1.1) *and $U(t)$ the numerical approximation given by* (1.3). *Then, there exists a constant C, that depends on the $C^{4,1}([-L, L] \times [0, T-\tau])$ norm of u, such that for every h small enough it holds*

$$
\max_{t\in[0,T-\tau]} \max_k |u(x_k, t) - u_k(t)| \le Ch^2.
$$

Next we begin with the analysis of the asymptotic behaviour of solutions of (1.3). Next theorem describes when (1.3) has solutions with blow-up. To do this, we introduce $\lambda_1(L, h)$ that stands for the first discrete eigenvalue for the Laplacian, that is the first eigenvalue of the problem

$$
\begin{cases}
A\Psi = \lambda M \Psi, \\
\Psi_{-N} = \Psi_N = 0.
\end{cases} \tag{1.4}
$$

Let us recall that

$$
\lambda_1(L, h) = \left(\frac{\pi}{2L}\right)^2 + O(h^2),
$$

see [FG].

Theorem 1.2 *There exists blowing up solutions for* (1.3) *if and only if $p > m$ or $p = m$ with $\lambda_1(L, h) < 1$. Moreover, in case $p = m$ with $\lambda_1(L, h) < 1$ every nontrivial solution blows up. In case $p > m$ and φ an initial data such that the solution u of problem* (1.1) *blows up then U also blows up for every h small enough.*

This theorem shows that the numerical scheme reproduces the blow-up cases in a very accurate way. Let us remark that $\lambda_1(L, h) \to \lambda_1(L)$ as $h \to 0$.

Numerical blow-up has been studied before. In [ALM1], [ALM2], [BK], [BHR], [Ch], [N], [GR] the semilinear heat equation is considered. See also [LR] for a time discretization of (1.1).

As a consequence of our blow-up results we can get bounds on the time remaining to achieve the numerical blow-up time in terms of $U(t)$. This allows us to prove that the numerical blow-up time T_h converges to T when h goes to zero.

Corollary 1.1 *Let φ be an initial datum for* (1.1) *such that u blows up, if we call T and T_h the blow-up times for u and u_h respectively, we have*

$$\lim_{h\to 0} T_h = T.$$

Now we describe the blow-up rate for the numerical scheme.

Theorem 1.3 *Let $U(t)$ be a blowing up solution of* (1.3) *and T_h its blow-up time, then there exists two positive constants $C_i = C_i(h)$ such that*

$$C_1(T_h - t)^{-1/(p-1)} \le \|U\|_\infty(t) \le C_2(T_h - t)^{-1/(p-1)}.$$

Moreover, if $p > m$ there holds,

$$\lim_{t\nearrow T_h} (T_h - t)^{1/(p-1)}\|U\|_\infty(t) = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

We remark that this rate coincides with the blow-up rate of the continuous problem (1.1).

With the blow-up rate we can characterize the numerical blow-up set, that is the set of nodes x_k such that $u_k(t) \to +\infty$ when $t \nearrow T_h$.

Theorem 1.4 *Let $U(t)$ be a blowing up solution of* (1.3) *which blows up at time T_h, then*

1. *For $p > m$ the blow-up set is given by a finite number of nodes, i.e.,*

   $$B(U) = [-Kh,\, Kh],$$

   where $K \equiv K(p, m)$ is the only integer that verifies the following expression,

   $$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \le \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}, \quad \textit{that is,} \quad K = \left[\frac{\ln((p-1)/(p-m))}{\ln(m)}\right].$$

2. *For the case $p = m$ we obtain global blow-up, i.e.,*

   $$B(U) = [-L, L].$$

Numerical blow-up sets have been studied in [N], [Ch], where the semilinear heat equation is treated. There, it is proved that the blow-up set can be larger than a single point. In [GR] there is precise characterization of the numerical blow-up set. Up to our knowledge this type of results were not available for the porous medium equation.

The explicit formula for the number of blow-up nodes shows that the non-linear diffusion cames into play.

We observe that in the case $p > m$ the blow-up set of the numerical solution can be larger than a single point, nevertheless, since K does not depend on h, we have that

$$B(U) = [-Kh, Kh] \to \{0\} = B(u), \qquad h \searrow 0.$$

On the other hand, when $p = m$, $B(U) = [-L, L]$ for every h. Therefore we find that regional blow-up is not possible for a numerical scheme with a fixed mesh. However we can recover regional blow-up by looking carefully at $U(t)$ in suitable self-similar variables. To do this we introduce the self-similar variables given by

$$\begin{cases} y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ \\ (T_h - t) = e^{-s}. \end{cases} \tag{1.5}$$

In these new variables $Y = (y_k(s))$, problem (1.3) reads,

$$\begin{cases} y_{-N}(s) = e^{-\frac{1}{p-1}s}, \\ \\ y_k'(s) = \dfrac{1}{h^2} e^{\frac{m-p}{p-1}s} (y_{k+1}^m(s) - 2y_k^m(s) + y_{k-1}^m(s)) - \dfrac{1}{p-1} y_k(s) + y_k^p, \\ \\ y_N(s) = e^{-\frac{1}{p-1}s}. \end{cases} \tag{1.6}$$

We have the following result.

Theorem 1.5 *Let $p = m$ and $Y(s)$ given by* (1.5)*, then as s goes to infinity we have that*

$$Y(s) \to W_h,$$

where W_h is a positive stationary solution of (1.6)*.*

Moreover, this W_h converges to the continuous profile, $z(x)$, as h goes to zero.

This result gives that W_h goes to zero outside the set $B(u)$ and we recover the blow-up set by looking at the behaviour of $Y(s)$ for large s.

To end the introduction let us briefly comment on extensions to higher dimensions. For example, if finite differences in a cube $(-L, L)^d \subset \mathbb{R}^d$ are considered to discretize $u_t = \Delta u^m + u^p$, similar results can be obtained applying similar ideas. The only exception being Theorem 1.5 where we use essentially one dimensional arguments.

Organization of the paper: first, in Section 2 we state and prove some auxiliary results that will be used in the rest of the paper and also we prove our convergence result, Theorem 1.1. For expository reasons we divide the proofs in two cases, $p = m$ and $p > m$. In Section 3 we deal with the case $p = m$ and in Section 4 with $p > m$. Finally in Section 5 we present some numerical experiments. We leave for the Appendix some results about existence and uniqueness of the continuous profile $z(x)$.

2 Properties of the numerical scheme

In this section we collect some preliminary results for our numerical method. In particular we prove convergence for regular solutions.

First, we state a symmetry property for the numerical problem (1.3). We call a vector symmetric if verifies that $u_{-k} = u_k$.

Lemma 2.1 *Let $U(0)$ be a symmetric vector then $U(t)$ is also symmetric for all $t \in (0, T_h)$.*

Proof. The result follows from the uniqueness of problem (1.3). Indeed, let $\hat{U}(t)$ be defined as the vector such that

$$\hat{u}_i(t) = u_{-i}(t) \qquad i = -N, \cdots, N.$$

This vector is also a solution of (1.3) and at time $t = 0$ it is equal to $U(0)$. Therefore, by uniqueness, $U(t) = \hat{U}(t)$. □

Remark 2.1 *Since $u_k(0) = \varphi(x_k)$ and φ is symmetric, then $U(0)$ (and therefore $U(t)$) is symmetric. So we can restrict ourselves to the half interval $[0, L]$ reducing the size of the system of ODEs to be solved.*

Now we want to prove a comparison Lemma. To do this we need the following definition,

Definition: We will call $\overline{U}$ a supersolution if it satisfies

$$\begin{cases} \overline{u}_{-N}(t) \geq 1, \\ \\ \overline{u}_k'(t) \geq \dfrac{1}{h^2}(\overline{u}_{k+1}^m(t) - 2\overline{u}_k^m(t) + \overline{u}_{k-1}^m(t)) + \overline{u}_k^p(t), \\ \\ \overline{u}_N(t) \geq 1, \\ \\ \overline{u}_k(0) \geq \varphi(x_k), \qquad -N \leq k \leq N. \end{cases} \tag{2.1}$$

Analogously, we say that $\underline{U}$ is a subsolution if it satisfies (2.1) with the reverse inequalities.

Lemma 2.2 *Let $\overline{U}$ and $\underline{U}$ be a superolution and a subsolution respectively, then*

$$\overline{U}(t) \geq U(t) \geq \underline{U}(t).$$

Proof. By an approximation procedure we restrict ourselves to consider strict inequalities in (2.1). Let us prove that $\overline{U}(t) > U(t)$. We argue by contradiction. Let us assume that there exists a first time t_0 and a node j such that $\overline{u}_j(t_0) = u_j(t_0)$ then $-N+1 < j < N-1$,

$$0 \geq \overline{u}_j'(t_0) - u_j'(t_0) > \frac{1}{h^2}(\overline{u}^m_{j+1}(t_0) - u^m_{j+1}(t_0) + \overline{u}^m_{j-1}(t_0) - u^m_{j-1}(t_0)) \geq 0,$$

a contradiction. This contradiction proves that $\overline{U}(t) > U(t)$.

The inequality $U(t) \geq \underline{U}(t)$ can be handled in a similar way. □

Now we study the monotonicity properties of our numerical scheme in $[0, L]$.

Lemma 2.3 *Let U be a solution of* (1.3) *with $u_k(0) > u_{k+1}(0)$, $k = 0, ..., N$ then*

$$u_k(t) > u_{k+1}(t), \qquad 0 \leq k \leq N-1.$$

Proof. We argue by contradiction, let us assume there exists a first time t_0 and two consecutive nodes where the conclusion of the Lemma fails, let us call them $j, j+1$. So we are assuming that $u_{j+1}(t_0) = u_j(t_0)$. From the equations (1.3) we get

$$0 \geq u_j'(t_0) - u_{j+1}'(t_0) = \frac{1}{h^2}(u^m_{j-1}(t_0) - u^m_{j+2}(t_0)) \geq 0.$$

We conclude that $u_{j-1}(t_0) = u_{j+2}(t_0) = u_j(t_0)$. Using the same argument, we get that all the nodes must be equal to $u_N(t_0) = 1$ at time t_0 but this is impossible since the initial data verifies $U(0) \geq 1$ and hence the solution verifies $u_i(t) > 1$ for all positive t. Indeed, first we use the comparison Lemma to obtain that $u_i(t) \geq 1$ ($\underline{u_i} \equiv 1$ is a subsolution). Next, if all the nodes attains its minimum 1 at the same time t_0 then all of them have derivative less or equal than zero, but at that time t_0 we have $u_j'(t_0) = u_j^p(t_0) = 1 > 0$, and this is a contradiction. □

Remark 2.2 *This ensures that the maximum of $U(t)$ is attained at the node $x_0 = 0$, i.e.,*

$$\max_k u_k(t) = u_0(t).$$

We will be use this fact in the following sections.

Now we prove our convergence result for regular solutions up to $t = T - \tau$.

Proof of Theorem 1.1. In the course of this proof we will denote by C_i a constant independent of h which can be different in different occurrences.

If we rewrite the system (1.3) in terms of $Z = U^m$ we obtain

$$\begin{cases} z_{-N}(t) = 1, \\ (z_k^{1/m})'(t) = \dfrac{1}{h^2}(z_{k+1}(t) - 2z_k(t) + z_{k-1}(t)) + z_k^{p/m}(t), \\ z_N(t) = 1, \\ z_k(0) = \varphi^{1/m}(x_k) \geq 1, \qquad -N+1 \leq k \leq N-1. \end{cases}$$

Let $v_k(t) = u^m(x_k, t)$ where u is the solution of the continuous problem (1.1). We define the error function as

$$e_k = z_k - v_k.$$

Let $t_0 = \max\{t \in [0, T-\tau] : |e_k|(t) \leq 1/2\}$. We perform the following calculations with $t \in [0, t_0]$ and we will prove at the end that $t_0 = T - \tau$ for every h small enough.

The error function satisfies that, for $-N+1 \leq k \leq N-1$,

$$\begin{aligned} \frac{1}{m} z_k^{(1-m)/m} e_k' = & \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) - \frac{1}{m}(z_k^{(1-m)/m} - v_k^{(1-m)/m})\, v_k' \\ + & \; z_k^{p/m} - v_k^{p/m} + C_1 h^2 \\ \leq & \; \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_2 \xi_k^{(1-2m)/m)} |e_k| + C_3 \eta_k^{(p/m)-1} |e_k| + C_1 h^2, \end{aligned}$$

where ξ_k and η_k are intermediate values between z_k and v_k. Taking into account that there exists a constant, C, such that $1 \leq z_k(t) \leq C$ for every $t \in [0, t_0]$ we have

$$e_k' \leq \frac{C_1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_2 |e_k| + C_3 h^2. \tag{2.2}$$

The first and the last nodes verify

$$e_{-N} = e_N = 0. \tag{2.3}$$

We remark that the system (2.2), (2.3), has a comparison principle, that can be proved as in Lemma 2.2. Let us look for a supersolution of the form

$$w_k(t) = g(t),$$

where $g(t)$ is a solution of

$$\begin{cases} g'(t) = C_1 g(t) + C_2 h^2, \\ g(0) = C_3 h^2. \end{cases}$$

That is

$$g(t) = h^2 \left(C_1 e^{C_2 t} + C_3 \right).$$

Therefore,

$$e_k(t) \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

Arguing in the same way with $-e_k$ we obtain

$$|e_k(t)| \le w_k(t) \le C_1 h^2 e^{C_2 T}.$$

From this inequality it is easy to see that $t_0 = T - \tau$ for every h small enough and the Theorem is proved. $\square$

3 Regional blow-up. The case $p = m$

3.1 Blow-up for the numerical scheme

Hereafter we will use $\langle , \rangle$ for the usual inner product in the euclidean space and 1 to denote the vector that has ones in all of its components. We consider the first eigenfunction Ψ which is a solution of (1.4) with $\lambda = \lambda_1(L,h)$, in particular, we choose $\Psi > 0$ such that $\langle M1, \Psi \rangle = 1$. Now, we multiply (1.3) by Ψ to obtain

$$\begin{aligned} \tfrac{d}{dt}\langle MU, \Psi\rangle = & \; -\langle AU^m, \Psi\rangle + \langle MU^m, \Psi\rangle \\ = & \; -\lambda_1(L,h)\langle MU^m, \Psi\rangle + \langle MU^m, \Psi\rangle \\ \ge & \; (1-\lambda_1(L,h))\langle MU, \Psi\rangle^m. \end{aligned}$$

If $\lambda_1(L,h) < 1$ the solution U blows up in finite time and if $\lambda_1(L,h) \ge 1$ we have that $\langle MU, \Psi\rangle$ is non-increasing, and therefore bounded, hence U is global. $\square$

Assume that $\lambda_1(L,h) < 1$, so U has finite time blow-up T_h. From the last inequality we get

$$\frac{\frac{d}{dt}\langle MU, \Psi\rangle}{\langle MU, \Psi\rangle^m} \ge (1-\lambda_1(L,h)) \ge C.$$

Integrating between t and T_h,

$$\int_t^{T_h} \frac{\frac{d}{dt}\langle MU, \Psi\rangle}{\langle MU, \Psi\rangle^m} \ge C(T_h - t).$$

Changing variables we get

$$\int_{\langle MU,\Psi\rangle(t)}^{+\infty} \frac{1}{s^m} \ge C(T_h - t).$$

Therefore we obtain a bound for $T_h - t$ in terms of the size of U. In fact we have that

$$(T_h - t) \le C(\langle MU, \Psi\rangle(t))^{-(m-1)}.$$

This allows us to prove the convergence of the blow-up times. Given $\varepsilon > 0$ let R be such that $CR^{-(m-1)} \leq \varepsilon/2$. As blow-up for u is regional we can choose a time t_0 such that

$$T - t_0 \leq \varepsilon/2 \qquad \text{and} \qquad \int_{-L}^{L} u(s, t_0)\psi(s)\, ds \geq 2R,$$

where ψ is the first eigenfunction of the laplacian in $[-L, L]$. By our uniform convergence result up to $t = t_0 < T$, we have

$$\langle MU, \Psi\rangle(t_0) \to \int_{-L}^{L} u(s, t_0)\psi(s)\, ds, \qquad h \to 0.$$

Therefore for every h small enough, we have that

$$\langle MU, \Psi\rangle(t_0) > R.$$

Hence

$$\begin{aligned} |T - T_h| &\leq |T - t_0| + |T_h - t_0| \\ &\leq \varepsilon/2 + C(\langle MU, \Psi\rangle(t_0))^{-(m-1)} \\ &\leq \varepsilon/2 + CR^{-(m-1)} \\ &\leq \varepsilon, \end{aligned}$$

for every h small enough. This proves that $T_h \to T$ as $h \to 0$.

3.2 Blow-up rate

From our previous computations we have that

$$\langle MU, \Psi\rangle(t) \leq C(T_h - t)^{-1/(m-1)}.$$

Now we observe that there exists a constant $C = C(h)$ such that

$$\max_k u_k(t) = u_0(t) \leq C\langle MU, \Psi\rangle(t)$$

and we conclude that

$$u_0(t) \leq C(T_h - t)^{-1/(m-1)}.$$

On the other hand, as for all t the solution is symmetric and decreasing in $[0, L]$, we have that

$$u_0'(t) \leq u_0^m(t).$$

Therefore, integrating between t and T_h, we get

$$u_0(t) \geq C(T_h - t)^{-1/(m-1)}.$$

Summing up, we have proved that the blow-up rate is given by

$$C_1(T_h - t)^{-1/(m-1)} \leq \max_k u_k(t) = u_0(t) \leq C_2(T_h - t)^{-1/(m-1)}.$$

3.3 Blow-up set

Theorem 3.1 *If $p = m$ every node is a blow-up point. Moreover in the self-similar variables,*

$$Y(s) \to W, \qquad \text{as} \quad s \to \infty,$$

where $W = (w_{-N}, ..., w_N)$ is a positive solution of the limit problem of (1.6)*. Hence the asymptotic behaviour of $u_k(t)$ is given by*

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k.$$

Proof. In this case, if we write the numerical problem (1.6) in a form similar to (1.2), we have a Lyapunov functional. In fact,

$$\Phi(Y) = \frac{1}{2}\langle A^{1/2}Y^m, A^{1/2}Y^m\rangle + \frac{1}{2m}\langle MY^m, Y^m\rangle - \frac{1}{m^2-1}\langle MY^m, Y\rangle$$

satisfies

$$\frac{d}{ds}\Phi(Y)(s) = -\langle MY', Y^{m-1}Y'\rangle(s) \leq 0.$$

Hence, the orbit $Y(s)$ goes to a stationary state, see [H]. Therefore we study the limit problem of (1.6). By Remark 2.1 we can restrict ourselves to $[0, L]$ arriving to

$$\begin{cases} 0 = \dfrac{2}{h^2}(w_1^m - w_0^m) - \dfrac{1}{m-1}w_0 + w_0^m, \\ \\ 0 = \dfrac{1}{h^2}(w_{k+1}^m - 2w_k^m + w_{k-1}^m) - \dfrac{1}{m-1}w_k + w_k^m, & 0 \leq k \leq N-1, \\ \\ 0 = w_N. \end{cases} \tag{3.1}$$

Moreover, as we begin with a decreasing data $U(0)$ then for a fixed s the vector $Y(s)$ is positive and decreasing, hence we have to look to nonnegative and nondecreasing stationary solutions.

On the other hand, from the blow-up rate we know that $y_0(s) \geq C(h) > 0$, then W is not the trivial solution. We claim that w_k must be positive for all $0 < k < N$. To prove this claim just suppose that there exists some j with $w_j = 0$. Since W is non-negative and non-increasing we have that for $k \geq j$, $w_k = 0$ and therefore $w_k = 0$ for all k, but $w_0 > 0$. This contradiction proves the claim. $\square$

Now our goal is to recover regional blow-up by looking carefully at the behaviour of the stationary solution as the parameter h goes to zero.

When h goes to zero we expect that $Z = W^m$ converges to a solution of the following problem

$$\begin{cases} z_{xx} - \frac{1}{m-1}z^{\frac{1}{m}} + z = 0, & x \in (0, L), \\ z_x(0) = 0, \\ z(L) = 0. \end{cases} \tag{3.2}$$

This is the content of our next Lemma. We remark that since $m > 1$ a non Lipschitz function appears in (3.2).

Lemma 3.1 *Let W be the solution of* (3.1) *and let $z(x)$ be the unique solution of* (3.2). *Then*

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Proof. Multiplying the equation satisfied by Z by

$$\frac{(z_{k+1} - z_k) + (z_k - z_{k-1})}{2}$$

and summing we get

$$0 = \frac{(z_N - z_{N-1})^2}{2h^2} - \frac{(z_1 - z_0)^2}{2h^2} + \sum_{l=1}^{N} \left(z_l - \frac{1}{m-1} z_l^{\frac{1}{m}} \right) \frac{(z_{l+1} - z_{l-1})}{2}.$$

Hence

$$\begin{aligned} 0 = & \; \frac{(z_N - z_{N-1})^2}{2h^2} - \frac{(z_1 - z_0)^2}{2h^2} \\ + & \left(\frac{z_N^2}{2} - \frac{m}{m^2-1} z_N^{(1+m)/m} \right) - \left(\frac{z_0^2}{2} - \frac{m}{m^2-1} z_0^{(1+m)/m} \right) + O(h). \end{aligned}$$

Using the first and the last equations of (3.1) we get that z_{N-1} and z_0 must verify the following polynomial,

$$0 = \frac{z_{N-1}^2}{2h^2} - \frac{m}{m^2-1} z_0^{\frac{1+m}{m}} + \frac{z_0^2}{2} + O(h). \tag{3.3}$$

On the other hand, we know that z_0 is bounded. Therefore, z_{N-1}/h must be bounded. Then we can take a subsequence of h such that $z_{N-1}/h \to \Gamma \geq 0$.

If $\Gamma > 0$ we consider the auxiliary initial value problem

$$\begin{cases} z'' = \frac{1}{m-1} z^{1/m} - z, & x \in [0, L], \\ z(L) = 0, \\ z'(L) = -\Gamma. \end{cases} \tag{3.4}$$

For this problem we define the energy as

$$E_z(x) = \frac{(z'(x))^2}{2} - \frac{m}{m^2-1} z(x)^{(m+1)/m} + \frac{z(x)^2}{2}.$$

Multiplying equation (3.4) by z' and integrating we obtain that it is conservative, i.e. $E_z(x) = E_z(L) = -\Gamma/2$.

Problem (3.4) has a unique solution and by classical theory (see [JR]) in the interval where $z(x)$ is positive

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Notice that $z(x)$ is positive in $[L-\delta, L)$. Since the vector Z is decreasing, we have that $z(x)$ is non-increasing in all interval $[0, L]$. Therefore $z(x)$ is positive in $[0, L)$. So, for all $x \in [0, L)$, $Z \to z(x)$.

Then, as the problem is conservative and (3.3) holds, we get that $z(0)$ is the only positive root of

$$\frac{\Gamma^2}{2} - \frac{m}{m^2-1} z(0)^{(m+1)/m} + \frac{z(0)^2}{2} = 0,$$

and

$$z'(0) = 0.$$

So, the constant Γ must be the only constant such that z is a solution of

$$\begin{cases} z'' = \frac{1}{m-1} z^{1/m} - z, & x \in [0, L], \\ z'(0) = 0, & \\ z(L) = 0, & \\ z(x) > 0, & x \in [0, L). \end{cases}$$

This problem has a solution if and only if $\frac{\pi}{2} < L \leq \frac{m\pi}{m-1}$, see the Appendix for the details. So, if $L > \frac{m\pi}{m-1}$ the parameter Γ must be zero.

Now, we assume that $\Gamma = 0$. Then $E_z(L) = 0$ and by (3.3) we obtain that $z_0 \to A$, where A is the only positive root of

$$0 = \frac{1}{2} A^2 - \frac{m}{m^2-1} A^{\frac{1+m}{m}}$$

and that

$$z'(0) = 0.$$

Hence, we consider the problem

$$\begin{cases} z'' = \frac{1}{m-1} z^{1/m} - z & x \in [0, L], \\ z(0) = A, & \\ z'(0) = 0, & \\ z'(x) \leq 0 & x \in [0, L]. \end{cases}$$

This problem has a unique explicit solution given by

$$z(x) = \begin{cases} \left[\frac{2m}{m^2-1} \cos^2\left(\frac{(m-1)x}{2m} \right) \right]^{\frac{m}{m-1}} & x < m\pi/(m-1), \\ 0 & x \geq m\pi/(m-1). \end{cases}$$

Then if $L < \frac{m\pi}{m-1}$ the parameter Γ must be positive.

On the other hand, by classical theory we have that in the interval where $z(x)$ is positive, $[0, m\pi/(m-1))$,

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Since Z is decreasing, we have that for h small enough,

$$z_k \leq z\left(\frac{m\pi}{m-1} - \delta\right) \leq \varepsilon, \qquad \forall\, x_k > \frac{m\pi}{m-1} - \delta.$$

Then, $Z = W^m \to z(x)$ in all the interval $[0, L]$. $\square$

Remark 3.2 *As a consequence of Lemma 3.1 if $L > \frac{m\pi}{m-1}$ we have that*

$$W \to 0, \qquad \text{in } \left[\frac{m\pi}{m-1}, L\right],$$

and we recover the regional blow-up in the sense that the constants that appear in the blow-up rate for the nodes that lie in $[\frac{m\pi}{m-1}, L]$ go to zero as h goes to zero, i.e.,

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k$$

with $w_k \to 0$ as $h \to 0$ for every k such that $x_k \in [\frac{m\pi}{m-1}, L]$.

4 Single point blow-up. Case $p > m$.

4.1 Blow-up for the numerical scheme.

In this section we follow ideas from [GR]. We recall that we are considering a symmetric and decreasing in $[0, L]$ initial data φ. By Lemmas 2.1 and 2.3 $U(t)$ is a symmetric vector and it is decreasing in $[0, L]$. Then, its maximum will be $u_0(t)$.

For the continuous problem (1.1) there exists an energy functional given by

$$\Phi(u)(t) = \frac{1}{2}\int_{-L}^{L} (u^m)_x^2 - \frac{m}{p+m}\int_{-L}^{L} |u|^{p+m},$$

which is nonincreasing along the orbits. So let us define the discrete analogous of Φ, let $\Phi_h(U)$ be the following functional

$$\Phi_h(U)(t) = \frac{1}{2}\langle A^{1/2}U^m, A^{1/2}U^m\rangle - \frac{m}{p+m}\langle MU^p, U^m\rangle,$$

which is also nonincreasing along the orbits.

Lemma 4.1 *Let U be a solution such that $\Phi_h(U)(t_0) < 0$ for some t_0, then $U(t)$ is unbounded.*

Proof. Let us assume that U is bounded, as Φ_h is a Lyapunov functional $U(t)$ must converge to a steady state, W, as t goes to infinity. As $\Phi_h(U)(t_0) < 0$ and Φ_h is nonincreasing, we get that $\Phi_h(W) < 0$. But, multiplying the equation satisfied by W by W^m we get

$$0 = -\langle A^{1/2}W^m, A^{1/2}W^m\rangle + \langle MW^p, W^m\rangle = -2\Phi_h(W) + \frac{p+3m}{p+m}\langle MW^p, W^m\rangle.$$

Hence $\Phi_h(W) \geq 0$, a contradiction. □

Next, we prove that every solution with $\Phi_h(U)(t_0) < 0$ blows up.

Lemma 4.2 *Let U be a solution such that $\Phi_h(U)(t_0) < 0$ for some t_0 then $U(t)$ blows up in finite time T_h. Moreover, there exists a constant C independent of h such that*

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+m}}}. \tag{4.1}$$

Proof. From Lemma 4.1 we can assume that $u_0(t) = \max_j u_j(t)$ becomes unbounded. Therefore, using that $p > m$, we get

$$u_0'(t) = \frac{1}{h^2}(u_1^m(t) - 2u_0^m(t) + u_{-1}^m(t)) + u_0^p(t) \geq \delta u_0^p(t),$$

for every t large enough. As $p > m > 1$ we obtain that u_0 (and then U) blows up in finite time.

On the other hand, we have that

$$\begin{aligned}
\tfrac{d}{dt}\langle MU(t), \quad & U^m(t)\rangle = -(m+1)\langle A^{1/2}U^m, A^{1/2}U^m\rangle(t) + (m+1)\langle MU^p, U^m\rangle(t) \\
= \quad & -2(m+1)\Phi_h(U(t)) + \frac{(m+1)(p-m)}{(m+p)}\langle MU^p, U^m\rangle(t) \\
\geq \quad & 2(m+1)|\Phi_h(U(t))| + \frac{(m+1)(p-m)}{(m+p)}\langle MU, U^m\rangle^{(m+p)/(m+1)}(t) \\
\geq \quad & C\left(|\Phi_h(U(t))| + C\langle MU, U^m\rangle(t)\right)^{(m+p)/(m+1)}.
\end{aligned}$$

Now, integrating between t_0 and T_h and taking into account that Φ_h is non-increasing, we obtain the desired result. □

We are ready to prove the following proposition, which completes the proof of Theorem 1.2.

Proposition 4.1 *Let φ be such that u blows up in finite time then U, the solution of* (1.3), *also blows up for every h small enough.*

First we observe that if u blows up in finite time T then

$$\lim_{t \to T} \Phi(u)(t) = -\infty,$$

see [CDE].

From our convergence result, it is easy to check that $\Phi_h(u_h(\cdot, t_0)) \to \Phi(u(\cdot, t_0))$ as $h \to 0$ and therefore we conclude that if u blows up in finite time then $\Phi_h(u_h(\cdot, t_0)) < 0$ for some t_0 and every h small enough, and hence u_h blows up in finite time, T_h.

To prove the convergence of the blow-up times we are going to use (4.1). We have that, given $\varepsilon > 0$, we can choose R large enough to ensure that

$$\left(\frac{C}{R^{\frac{p-1}{p+m}}} \right) \leq \frac{\varepsilon}{2}.$$

As u blows up at time T we can choose t_0 such that

$$T - t_0 \leq \varepsilon/2 \qquad \text{and} \qquad |\Phi(u(\cdot, t_0))| \geq 2R.$$

If h is small enough, by the convergence of $\Phi_h(u_h(\cdot, t_0))$ to $\Phi(u(\cdot, t_0))$ we have,

$$|\Phi_h(u_h(\cdot, t_0))| \geq R,$$

and hence, by (4.1),

$$T_h - t_0 \leq \left(\frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+m}}} \right) \leq \left(\frac{C}{R^{\frac{p-1}{p+m}}} \right) \leq \frac{\varepsilon}{2}.$$

Therefore,

$$|T_h - T| \leq |T_h - t_0| + |T - t_0| < \varepsilon. \qquad \square$$

4.2 Blow-up rate

Now we find the blow-up rate. Let us consider a blowing up solution U. Let us remark that by our symmetry assumptions $u_0(t) = \max_j u_j(t)$. Hence

$$u_0^p(t) \geq u_0'(t) \geq u_0^p \left(1 - \frac{2}{h^2} \frac{u_0^m}{u_0^p} \right)(t).$$

Integrating we obtain

$$(T_h - t) \geq \int_t^{T_h} \frac{u_0'}{u_0^p}(s)\, ds \geq \int_t^{T_h} \left(1 - \frac{2}{h^2} \frac{u_0^m}{u_0^p} \right)(s)\, ds.$$

Therefore,

$$(T_h - t) \geq \frac{1}{(p-1)u_0^{p-1}(t)} \geq \int_t^{T_h} \left(1 - \frac{2}{h^2}\frac{u_0^m}{u_0^p}\right)(s)\,ds.$$

As $u_0(t)$ is blowing up and $p > m$ we have that

$$\lim_{t \nearrow T_h} \frac{u_0^m}{u_0^p}(t) = 0.$$

Hence we can conclude that,

$$\lim_{t \nearrow T_h} (T_h - t)^{1/(p-1)} u_0(t) = C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}.$$

This implies that the blow-up rate is given by

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \leq \|U(t)\|_\infty \leq \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

Note that in this case the blow-up rate is independent of the parameter m.

4.3 Blow-up sets

Now we turn our interest to the blow-up set of the numerical solution. For a fixed h we want to look at the set of nodes, x_k, such that $u_k(t) \to +\infty$ as $t \nearrow T_h$. As before, we introduce the self-similar variables given by

$$\begin{cases} y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ \\ (T_h - t) = e^{-s}. \end{cases}$$

In these new variables $Y = (y_k(s))$, problem (1.3) reads as (1.6). We observe that, from the blow-up rates proved in Theorem 1.3, the vector Y is bounded and

$$\lim_{s \to \infty} y_0(s) = C_p > 0.$$

Theorem 4.1 *If $p > m$, then U blows up at exactly K nodes near $x = 0$, i.e.*

$$B(U) = [-Kh, Kh], \qquad K = \left[\frac{\ln((p-1)/(p-m))}{\ln(m)}\right]$$

Moreover, the asymptotic behaviour of the blowing up nodes is given by

$$u_{-j}(t) = u_j(t) \sim (T_h - t)^{\gamma_j}, \qquad \gamma_j = -\frac{m^j}{p-1} + \sum_{l=0}^{j-1} m^l,$$

if $\dfrac{\ln((p-1)/(p-m))}{\ln(m)} \notin \mathbb{N}$ *or* $j \neq K$, *and by*

$$u_{-K}(t) = u_K(t) \sim -\ln(T_h - t), \qquad \textit{if} \quad \frac{\ln((p-1)/(p-m))}{\ln(m)} \in \mathbb{N}.$$

Proof. First we want to prove that $y_j(s) \to 0$ for every $j \neq 0$, that is, the only node that behaves like $C_p(T_h - t)^{-1/(p-1)}$ is $x = 0$. We argue by contradiction. Assume that $u_1(t)(T_h - t)^{1/(p-1)} \to C_p$. Using that U is symmetric and decreasing in $[0, L]$ we have that

$$\begin{aligned}
(u_0 - u_1)'(t) \geq \; & \frac{1}{h^2}(3u_1^m - 3u_0^m)(t) + (u_0^p - u_1^p)(t) \\
= \; & \left[-\frac{3}{h^2}\left(\frac{u_0^m - u_1^m}{u_0 - u_1}\right) + \frac{u_0^p - u_1^p}{u_0 - u_1} \right] (u_0 - u_1)(t) \\
= \; & \left[-\frac{3}{h^2} m \xi^{m-1} + p\eta^{p-1} \right] (u_0 - u_1)(t),
\end{aligned}$$

where ξ and η lie between u_1 and u_0. As we have assumed that both u_0 and u_1 behave like $C_p(T_h - t)^{-1/(p-1)}$ we can conclude that

$$\xi(T_h - t)^{1/(p-1)} \to C_p \qquad \text{and} \qquad \eta(T_h - t)^{1/(p-1)} \to C_p.$$

As $p > m$ we conclude that, for t near T,

$$\frac{d}{dt} \ln(u_0 - u_1)(t) \geq \left[-\frac{3}{h^2} m \xi^{m-1} + p\eta^{p-1} \right] \geq \frac{p(C_p^{p-1} - \varepsilon)}{(T_h - t)}.$$

Integrating we get

$$(u_0 - u_1)(t) \geq C(T_h - t)^{-\frac{p}{p-1} + p\varepsilon}.$$

Using this fact we have,

$$0 = \lim_{t \to T_h} (T_h - t)^{\frac{1}{p-1}} (u_0 - u_1)(t) \geq C \lim_{t \to T_h} (T_h - t)^{\frac{1}{p-1}} (T_h - t)^{-\frac{p}{p-1} + p\varepsilon} = +\infty,$$

a contradiction that proves that $(T_h - t)^{\frac{1}{p-1}} u_1(t) \to 0$.

Now we return to the variable Y. From the blow-up rate for u_0 we have that

$$\lim_{s \to \infty} y_0(s) = C_p.$$

Now we observe that $y_1(s)$ verifies

$$y_1' = e^{\frac{m-p}{p-1}s} \frac{1}{h^2}(y_0^m - 2y_1^m + y_2^m) - \frac{1}{p-1} y_1 + y_1^p \sim C e^{\frac{m-p}{p-1}s} - \frac{1}{p-1} y_1$$

and by integration we get

$$0 \le y_1(s) \sim \begin{cases} C\, e^{-\frac{p-m}{p-1}s} & p < m+1, \\ C\, s\, e^{-\frac{1}{p-1}s} & p = m+1, \\ C\, e^{-\frac{1}{p-1}s} & p > m+1. \end{cases}$$

Notice that in all cases $y_1(s) \to 0$. In the U variables, this translates into,

$$u_1(t) \sim \begin{cases} C(T_h - t)^{\frac{p-1-m}{p-1}} & p < m+1, \\ -C\ln(T_h - t) & p = m+1, \\ C & p > m+1. \end{cases}$$

Therefore, if $p \le 1+m$ the node $u_1(t)$ blows up with a different rate than u_0, and for $p > m+1$ it is bounded.

Repeating the argument for the other nodes we obtain that $y_k(s) \to 0$, for all $k \neq 0$. Moreover, the same calculations used before show how to obtain the asymptotic behaviour of each node using the behaviour of the previous one and the result follows. $\square$

5 Numerical experiments

In this Section we present some numerical experiments that illustrate our results. For the numerical experiments we use the adaptive ODE solvers provided by MATLAB. In all cases we take $L = 15$ and the initial data $\varphi(x) = L^2 - x^2 + 1$.

First we deal with $p = m = 1.5$ and we find that the numerical blow-up time is given by $T_h = 0.1382$. Our numerical results are shown in figures 1 and 2. Figure 1 shows the evolution of the numerical solution. Figure 2 shows the profile of the solution in self-similar variables near T_h, we also put in the same picture the continuous self-similar profile $z(x)$. From this last picture one can appreciate that the numerical calculations show that the blow-up set is $(-9.4248, 9.4248) \approx (-3\pi, 3\pi)$.

Finally, we consider the case $p = 2$, $m = 1.5$. In this case, $T_h = 0.004432$. In Figure 3 one can see the evolution of the solution. In order to obtain the numerical blow-up rates, in Figure 4 we display $\ln(u_i)$ versus $-\ln(T_h - t)$ for $i = 0, 1, 2$. We can appreciate that the first curve (corresponding to $\max u_k = u_0$) has slope 1, the second (for u_1) has slope 1/2 (it becomes parallel to the dotted line which has slope 1/2) and the last one (for u_2) is flat. These behaviours correspond to the expected rates $u_0(t) \sim (T_h - t)^{-1}$, $u_1(t) \sim (T_h - t)^{-1/2}$ and u_2 bounded.

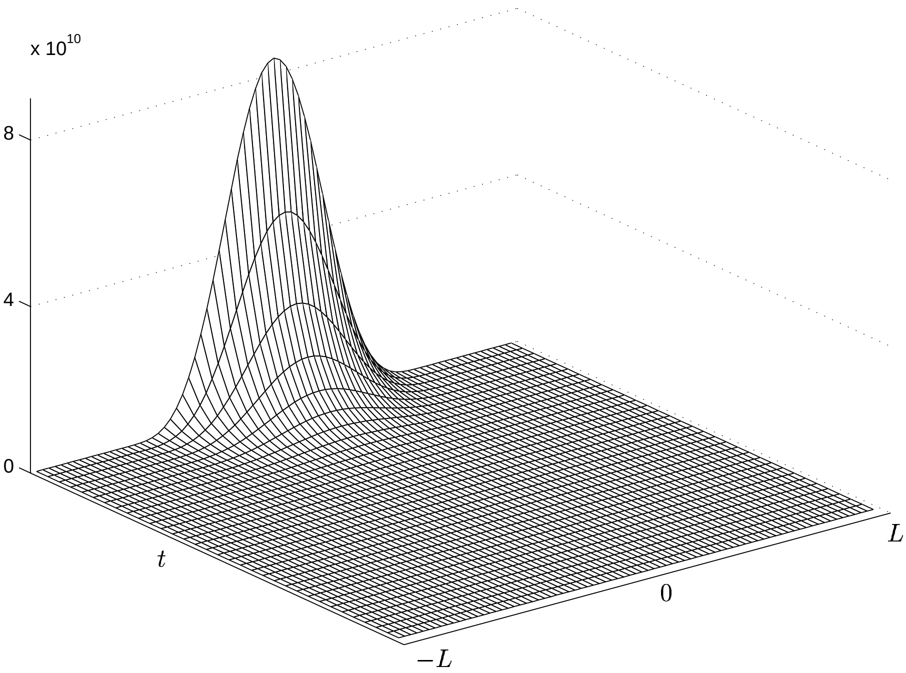

Figure 1. Evolution of the numerical solution ($p = m$).

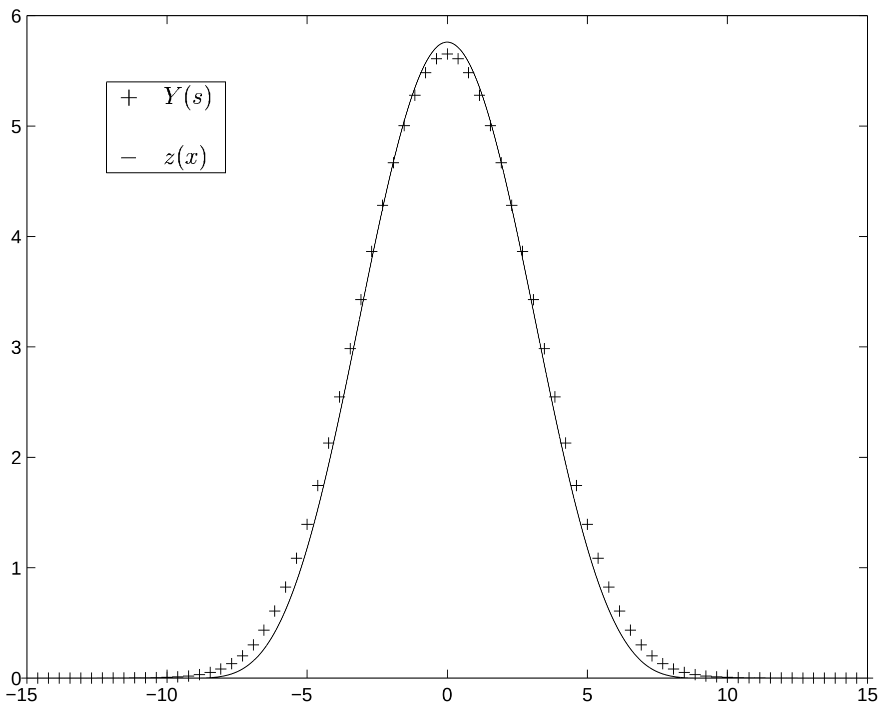

Figure 2. Self-similar profiles ($p = m$).

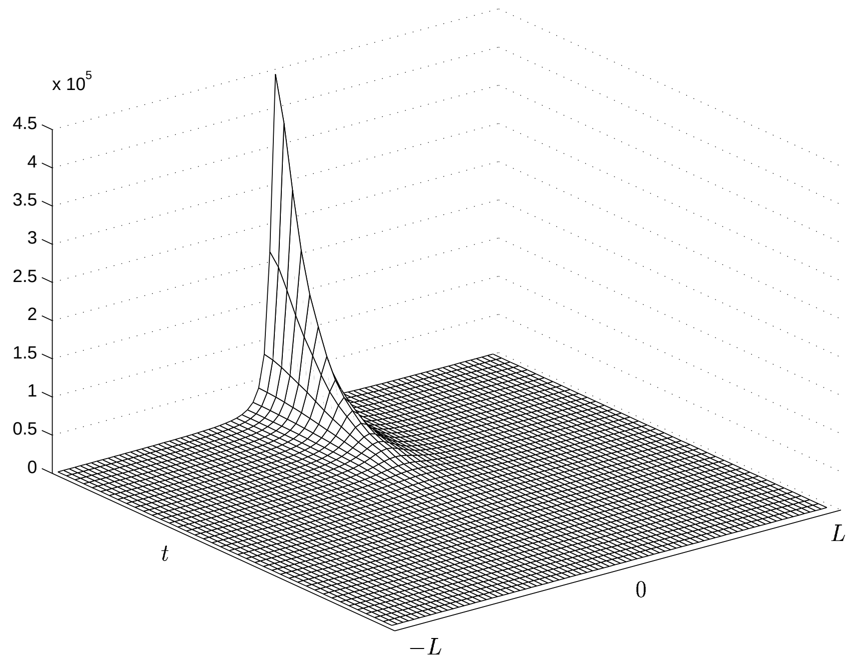

Figure 3. Evolution of the numerical solution ($p > m$).

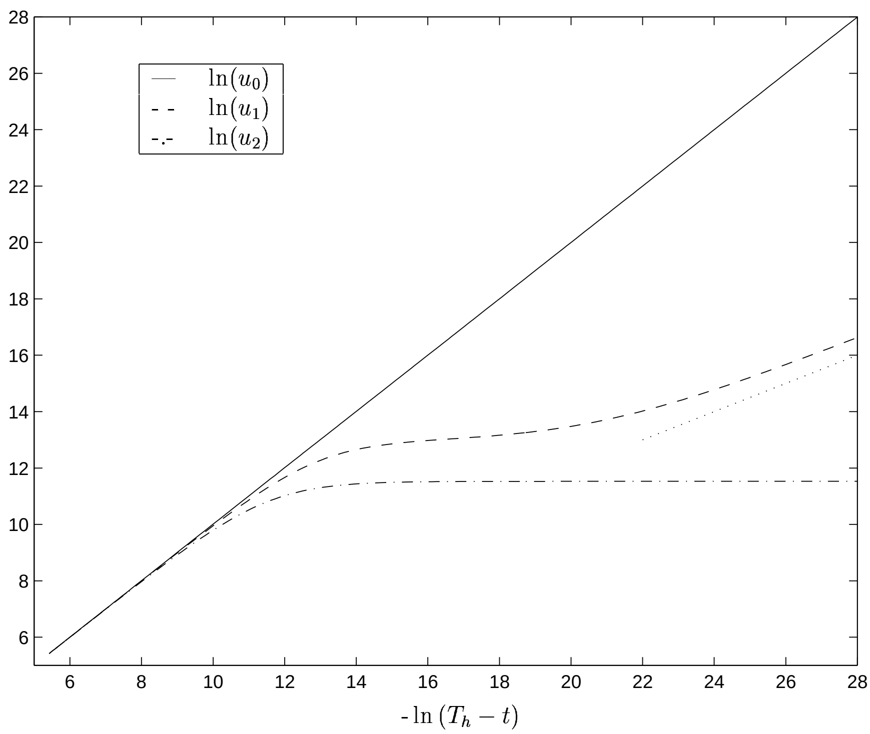

Figure 4. Blow-up rates ($p > m$).

6 Appendix

In this appendix we search for nonnegative monotone solutions of the following boundary value problem

$$\begin{cases} (f^m)'' = \frac{1}{m-1} f - f^m, & x \in [0, L], \\ f'(0) = 0, \qquad f(L) = 0, \end{cases} \tag{6.1}$$

We note that if $f(x)$ is a solution of this problem then

$$u(x,t) = (T-t)^{-1/(m-1)} f(x)$$

is a solution of

$$\begin{cases} u_t = (u^m)_{xx} + u^m, & (x,t) \in (0,L) \times [0,T), \\ (u^m)_x(0,t) = 0, & t \in [0,T), \\ u(L,t) = 0, & t \in [0,T), \\ u(0,t) = T^{-1/(m-1)} f(x). \end{cases}$$

This problem has been studied in [SGKM] and we know that for all positive initial data

(i) $u(x,t)$ blows up if and only if $L > \pi/2$.

(ii) The blow rate is given by $\|u(x,t)\|_\infty \sim (T-t)^{-1/(m-1)}$.

On the other hand, following a standard technique we introduce the self similar variables

$$v(x,\tau) = (T-t)^{\frac{1}{m-1}} u(x,t), \qquad \tau = -\ln(T-t).$$

This function $v(x,\tau)$ satisfies the following problem,

$$\begin{cases} v_\tau = (v^m)_{xx} - \frac{1}{m-1} v + v^m, & (x,\tau) \in (0,L) \times (0,+\infty), \\ (v^m)_x(0,\tau) = 0, & \tau \in (0,+\infty), \\ v(L,\tau) = 0, & \tau \in (0,+\infty), \\ v(x,0) = T^{\frac{1}{m-1}} u(x,0), & x \in (0,L). \end{cases} \tag{6.2}$$

We remark that the stationary solutions of (6.2) are solutions of (6.1).

Now, we prove that $v(x,\tau)$ converges (in terms of ω-limits) to a stationary state. For this we multiply the equation by $(v^m)_\tau$ and integrate with respect to x to obtain

$$\int_0^L (v^m)_\tau v_\tau = -\frac{d}{dt} F(v),$$

where

$$F(v) = \int_0^L \frac{(v^m)_x^2}{2}(s,\tau)\, ds + \frac{m}{m^2-1} \int_0^L v^{m+1}(s,\tau)\, ds - \frac{1}{2} \int_0^L v^{2m}(s,\tau)\, ds.$$

Hence $F(v)$ is a Lyapunov functional for problem (6.2) and we can conclude that the $\omega-$limit set of $v(x,\tau)$ consists of nontrivial stationary solutions of (6.2).

Lemma 6.1 *Let $f(x)$ be a solution of* (6.1)*. Then,*

(i) if $L \leq \pi/2$ no solution exists,

(ii) if $\pi/2 < L < (m\pi)/(m-1)$ there exists a unique positive solution and

(iii) if $L \geq (m\pi)/(m-1)$ the unique monotone solution is given by

$$f(x) = \begin{cases} \left[\frac{2m}{m^2-1}\cos^2\left(\frac{(m-1)x}{2m}\right)\right]^{\frac{1}{m-1}} & x < m\pi/(m-1), \\ 0 & x \geq m\pi/(m-1). \end{cases}$$

Proof. From the previous argument we have that problem (6.1) has a solution if and only if $L > \pi/2$.

In order to prove the uniqueness we consider two different solutions of (6.1), $f_1(x)$ and $f_2(x)$. These profiles must have some intersections. If this is not the case, they are ordered and hence the corresponding solutions in variables (x,t) do not have the same blow-up time. Since the energy

$$E_f(x) = \frac{(f'(x))^2}{2} - \frac{m}{m^2-1} f(x)^{(m+1)/m} + \frac{f(x)^2}{2}$$

is a nonnegative constant function, we obtain that two different profiles have only one intersection point. Moreover, $f_1'(L) < f_2'(L) \leq 0$ if and only if $f_1(0) > f_2(0)$, which is a contradiction with the fact that they have only one intersection point. Finally we remark that, as the energy must be nonnegative, we have

$$f(0) \geq A = \left(\frac{2m}{m^2-1}\right)^{\frac{1}{m-1}},$$

and for each $f(0) \geq A$ there exists a unique $L \in (\pi/2, (m\pi)/(m-1)]$ such that $f(x)$ is a positive solution of (6.1). For $L > (m\pi)/(m-1)$ we extend by zero for all $x \in [(m\pi)/(m-1), L]$. □

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