NON-SIMULTANEOUS BLOW-UP IN A NUMERICAL APPROXIMATION OF A PARABOLIC SYSTEM

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ABSTRACT. We study the asymptotic behaviour of semidiscrete numerical approximations for a system of two semilinear heat equations $u_t = \Delta u + u^{p_{11}} v^{p_{12}}$, $v_t = \Delta v + u^{p_{21}} v^{p_{22}}$ with homogeneous Dirichlet boundary conditions. We focus in the existence of non-simultaneous blow-up for the discrete solution (U, V). We prove that there are initial data such that U blows up while V does not if and only if $p_{11} > 1$ and $p_{21} < p_{11} - 1$, which is the same condition as the one for the continuous problem. Moreover, we also prove under adequate hypotheses that the methods reproduce the non-simultaneous blow-up.

1. INTRODUCTION.

In this paper we study the behaviour of semidiscrete approximations of the following parabolic system

$$\left\{ \begin{array}{ll} u_t = \Delta u + u^{p_{11}} v^{p_{12}} & \text{in } \Omega \times [0,T), \\ v_t = \Delta v + u^{p_{21}} v^{p_{22}} & \text{in } \Omega \times [0,T), \end{array} \right. \tag{1.1}$$

with homogeneous Dirichlet boundary data, $u = v = 0$ on $\partial\Omega \times [0,T)$, and initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ in Ω. We assume that Ω is a bounded smooth domain, $p_{ij} > 0$ and u_0, v_0 are positive, bounded, compatible with the boundary data and smooth enough to guarantee that u and v are regular. This reaction-diffusion system can be used as a model to describe heat propagation in a two-component combustible mixture. In this case u and v represent the temperatures of the interacting components, thermal conductivity is supposed to be constant and equal for both substances, and a volume energy release given by some powers of u and v is assumed.

Solutions to problem (1.1) exist locally in time, [EL]. The time T is the maximal existence time for the solution, which may be finite or infinite. If T is finite, then (u, v) develops a singularity. More precisely,

$$\limsup_{t \nearrow T} \{ \|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \} = +\infty,$$

and we say that (u, v) *blows up* in finite time T. Blow-up may happen for certain solutions of (1.1), no matter how smooth the initial data are, see [EL].

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This blow-up phenomenon occurs for many parabolic equations and systems and has been widely studied recently, see for example the survey [GV] and the book [SGKM]. For our problem there exist solutions (u, v) that blow up in finite time if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) < p_{12}p_{21}$, see [EL].

A priori there is no reason why both functions u and v should go to infinity simultaneously at time T. In fact, in [QR] the authors prove under adequate hypotheses that there are initial data such that u blows up while v does not if and only if $p_{11} > 1$ and $p_{21} < p_{11} - 1$. They denote this phenomenon as *non-simultaneous blow-up*.

Our interest here is to analyze how the usual numerical methods behave when applied to (1.1). Numerical approximations of blow-up problems have received an increasing interest in recent years, we refer to [ALM1], [ALM2], [BK], [BHR], [C], [DER], [GR], [LR], the survey [BB] and references therein. However, the work cited addresses only the case of a single equation. Here we find that the usual numerical approximations have similar blow-up results to those that hold for the continuous problem (1.1).

We consider a general method for the space discretization with adequate assumptions on the coefficients. More precisely, we take a set of nodes $\{x_1, \dots, x_N\}$, and denote by $(U, V)(t) = ((u_1(t), \dots, u_N(t)), (v_1(t), \dots, v_N(t)))$ the numerical solution. That is, $(u_k(t), v_k(t))$ stands for the values of the numerical approximation at the node x_k at time t. The boundary conditions impose that some $(u_k(t), v_k(t))$ have to be zero. From now on we focus on the other nodes. In what follows h stands for the parameter of the method. We assume that $(U, V) = (U_h, V_h)$ is the solution of the ODE system

$$(1.2) \qquad \left\{ \begin{array}{rcl} MU'(t) & = & -AU(t) + MU^{p_{11}}V^{p_{12}}(t), \\ MV'(t) & = & -AV(t) + MU^{p_{21}}V^{p_{22}}(t), \end{array} \right.$$

with initial data given by $U(0) = U_0$, $V(0) = V_0$. This system can be written explicitly as

$$(1.3) \qquad \left\{ \begin{array}{rcl} m_k u_k'(t) & = & -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) v_k^{p_{12}}(t), \quad 1 \le k \le N, \\ \\ m_k v_k'(t) & = & -\sum_{j=1}^{N} a_{kj} v_j(t) + m_k u_k^{p_{21}}(t) v_k^{p_{22}}(t), \quad 1 \le k \le N, \end{array} \right.$$

with initial data $u_k(0) = u_{0,k}$, $v_k(0) = v_{0,k}$, for $1 \le k \le N$. We denote the maximal existence time for the system by T_h.

The precise assumptions on the matrices involved in the method are: M is a diagonal matrix with positive entries m_k and A is a nonnegative symmetric matrix, with non positive coefficients off the diagonal (that is $a_{ij} \le 0$ if $i \ne j$), $a_{ii} > 0$ and $\sum_{j=1}^{N} a_{ij} \ge 0$. We remark that in general M and A depend on h. We also recall that the boundary nodes (that is, nodes with $u_k(t) = v_k(t) = 0$) do not appear in this system.

As an example, we can consider a linear finite element approximation of problem (1.1) on a regular acute triangulation of Ω (see [Ci]). In this case, let W_h be the subspace of $H_0^1(\Omega) \times H_0^1(\Omega)$ consisting of piecewise linear functions on the triagulation. We impose that the finite element approximation $(u_h, v_h) : [0, T_h) \to W_h$

verifies for each $t \in [0, T_h)$

$$\int_\Omega ((u_h)_t w_1)^I = -\int_\Omega \nabla u_h \nabla w_1 + \int_\Omega ((u_h)^{p_{11}} (v_h)^{p_{12}} w_1)^I,$$
$$\int_\Omega ((v_h)_t w_2)^I = -\int_\Omega \nabla v_h \nabla w_2 + \int_\Omega ((u_h)^{p_{21}} (v_h)^{p_{22}} w_2)^I,$$

for every $(w_1, w_2) \in W_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolate at the nodes of the mesh. These conditions imply that (U, V), the values of (u_h, v_h) at the nodes x_k, must verify a system of the form (1.2). In this case M is the lumped mass matrix and A is the stiffness matrix. The assumptions on the matrices M and A hold as we are considering an acute regular mesh. As initial data we take $(u_k(0), v_k(0)) = (u_0(x_k), v_0(x_k))$. We observe that in this case $(u_h, v_h) = (U, V)^I$.

As another example, if Ω is a cube, $\Omega = (0,1)^n$, we can use a semidiscrete finite differences method to approximate the solution $u(x,t)$ obtaining an ODE system of the form (1.3).

We begin the study of (1.3) by proving that, for times as close to the blow-up time as we want, this method converges uniformly under the assumption of the consistency of the method.

Theorem 1.1. *Let (u, v) be a regular solution of* (1.1) *and (U, V) the numerical approximation given by* (1.3). *We assume that the method is consistent, that is*

$$\begin{aligned} m_k u_t(x_k, t) &= -\textstyle\sum_{j=1}^N a_{kj} u(x_j, t) + m_k u^{p_{11}}(x_k, t) v^{p_{12}}(x_k, t) + \rho_k^1(h), \\ m_k v_t(x_k, t) &= -\textstyle\sum_{j=1}^N a_{kj} v(x_j, t) + m_k u^{p_{21}}(x_k, t) v^{p_{22}}(x_k, t) + \rho_k^2(h), \end{aligned} \tag{1.4}$$

for $t \in [0, T - \tau]$ and there exists a positive function $\rho(h)$ such that

$$\max_i \left\{ \frac{|\rho_k^i(h)|}{m_k} \right\} \le \rho(h), \qquad \max_k |u_0(x_k) - u_k(0)| + |v_0(x_k) - v_k(0)| \le \rho(h),$$

and $\lim_{h\to 0} \rho(h) = 0$. Then there exists a constant C such that

$$\max_{1\le k\le N} \sup_{0\le t\le T-\tau} \left(|u(x_k, t) - u_k(t)| + |v(x_k, t) - v_k(t)| \right) \le C\rho(h).$$

For the scalar problem see [ALM2], where Abia and his collaborators prove consistency in $[0,1] \times [0, T - \tau]$ for every $\tau > 0$ in the case of the finite element method with $\rho(h) = Ch^2$.

In the remainder of the article we will assume that the method converges in the sense described in Theorem 1.1.

As a first step for our analysis of the behaviour of solutions of (1.3) we want to describe when the blow-up phenomenon occurs for the discrete problem. We say that a solution of (1.3) has finite blow-up time if there exists a time T_h such that

$$\lim_{t \nearrow T_h} \{ \|U(t)\|_\infty + \|V(t)\|_\infty \} = \lim_{t \nearrow T_h} \left\{ \max_j u_j(t) + \max_j u_j(t) \right\} = +\infty.$$

We prove that the conditions on the structure of the nonlinearity that characterize the presence of blow-up for the discretized system are precisely those obtained by Escobedo and Levine in [EL] for the continuous problem.

Theorem 1.2. *There exist solutions* (U,V) *of* (1.3) *that blow up in finite time if and only if the exponents* p_{ij} *verify any of the conditions,* $p_{11} > 1$, $p_{22} > 1$ *or* $(p_{11}-1)(p_{22}-1) - p_{12}p_{21} < 0$.

For certain choices of the parameters p_{ij} there are initial data for which one of the components of the semidiscrete system remains bounded while the other blows up. We denote this phenomenon as *non-simultaneous numerical blow-up.* We characterize the range of parameters for which this occurs in the next two theorems.

Theorem 1.3. *Let (U, V) a solution of* (1.3) *such that U blows up at finite time* T_h *and V remains bounded up to that time. Then* $p_{11} > 1$ *and* $p_{21} < p_{11} - 1$.

Theorem 1.4. *If* $p_{11} > 1$ *and* $p_{21} < p_{11} - 1$*, then for every initial datum* $V_0 \neq 0$ *for* (1.3) *there exists an initial datum* U_0 *such that U blows up in finite time* T_h *and V remains bounded up to that time.*

We want to remark that the structure of the nonlinearity needed for the non-simultaneous blow-up in the discretized system is identical to what Quirós and Rossi proved in [QR] for the continuous system. This fact and the convergence of the discrete problem to the continuous one allow us to prove the following theorem for the non-simultaneous blow-up case.

Theorem 1.5. *Let* (u_0, v_0) *be initial data for* (1.1) *such that u and v are strictly increasing in time. Furthermore assume that u blows up at finite time T and v remains bounded up to that time. Then, there exists* h_0 *such that for every* $h < h_0$, *U blows up at finite time* T_h *but V is bounded. Moreover, we have that the blow up rate is given by*

$$\max_i u_k(t) \sim (T_h - t)^{-1/(p_{11}-1)},$$

and the numerical blow-up time converges to the continuous one, i.e.,

$$\lim_{h\to 0} T_h = T.$$

We want to comment on the hypotheses involved in this theorem. They are just technical but we cannot avoid them so far. The first one is the monotonicity assumption. This assumption is usual in problems with blow-up, see [GV]. It is satisfied if the initial data verify

$$\Delta u_0 + u_0^{p_{11}} v_0^{p_{12}} > 0, \qquad \Delta v_0 + u_0^{p_{21}} v_0^{p_{22}} > 0,$$

see Lemma 4.1 for a discrete version of this fact.

The second hypothesis, the non-simultaneous blow-up, is just to guarantee that $p_{11} > 1$. However, we do not need to use explicitly the fact that u blows up and v remains bounded. It is possible to obtain a similar result if $p_{11} > 1$, $p_{22} > 1$, or $p_{21}, p_{12} > 1$, no matter if the blow-up is simultaneous or not. Since our main concern here is the study of non-simultaneous blow-up we prefer to focus out attention on this case.

We remark that our proofs are based in the use of the maximum principle. Hence similar ideas may be used to study systems where this property holds. The application of energy methods would imply a restriction in the range of the exponents, in order to ensure a variational setting for the system.

The blow-up rate obtained in Theorem 1.5 is the same Quirós and Rossi obtained in [QR].

We conclude that the numerical approximations considered here reflect the behaviour of the continuous problem regarding non-simultaneous blow-up. Indeed, the conditions on the structure of the nonlinearity that appear for the continuous problem and for the discrete one are identical. We remark that we are dealing with a wide class of numerical approximations.

The paper is organized as follows: in §2 we prove the convergence result, Theorem 1.1; in §3 the conditions for the existence of blowing-up solutions, Theorem 1.2; in §4 we prove the non-simultaneous blow-up results, Theorems 1.3, 1.4, 1.5. Finally, in §5 we give some numerical experiments.

2. Convergence of the numerical scheme.

In this section we prove a uniform convergence result for regular solutions of the numerical scheme (1.3). Throughout this section, we consider $0 < \tau < T$ fixed. Approximations of regular solutions in one space dimension for a scalar problem with a source in the equation have been analyzed in [Ci], [ALM2], [GR].

We assume that the scheme is consistent, that is (1.4). We want to show that

$$\max_i \{|u_i(t) - u(x_i,t)|,\ |v_i(t) - v(x_i,t)|\} \to 0$$

as $h \to 0$, uniformly in $[0, T-\tau]$, for every $\tau > 0$. This is a natural requirement since before T the exact solution is regular.

Let us begin with a comparison lemma, that will be used throughout the paper.

Definition 2.1. *We say that* $(\overline{U},\overline{V})$ *is a supersolution of* (1.2) *if*

$$\left\{ \begin{array}{l} M\overline{U}' \geq -A\overline{U} + M\overline{U}^{p_{11}}\overline{V}^{p_{12}}, \\ M\overline{V}' \geq -A\overline{V} + M\overline{V}^{p_{21}}\overline{V}^{p_{22}}. \end{array} \right.$$

We say that $\underline{U}$ *is a subsolution of* (1.2) *if*

$$\left\{ \begin{array}{l} M\underline{U}' \leq -A\underline{U} + M\underline{U}^{p_{11}}\underline{V}^{p_{12}}, \\ M\underline{V}' \leq -A\underline{V} + M\underline{V}^{p_{21}}\underline{V}^{p_{22}}. \end{array} \right.$$

The inequalities are understood coordinate by coordinate.

Lemma 2.1. *Let* $(\overline{U},\overline{V})$ *and* $(\underline{U},\underline{V})$ *be a super and a subsolution of* (1.2) *respectively such that* $(\overline{U},\overline{V})(0) \geq (\underline{U},\underline{V})(0)$. *Then*

$$(\overline{U},\overline{V})(t) \geq (\underline{U},\underline{V})(t).$$

Proof: Let $(W,Z) = (\overline{U} - \underline{U}, \overline{V} - \underline{V})$. We can assume, by an approximation argument, that we have strict inequalities in Definition 2.1 and that $W(0), Z(0) > 0$. We observe that W verifies

$$\begin{aligned} MW' &> -AW + M\left(\overline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\overline{V}^{p_{12}} + \underline{U}^{p_{11}}\overline{V}^{p_{12}} - \underline{U}^{p_{11}}\underline{V}^{p_{12}}\right) \\ &= -AW + M\overline{V}^{p_{12}}\left(\frac{\overline{U}^{p_{11}} - \underline{U}^{p_{11}}}{\overline{U} - \underline{U}}\right)W + M\underline{U}^{p_{11}}\left(\frac{\overline{V}^{p_{12}} - \underline{V}^{p_{12}}}{\overline{V} - \underline{V}}\right)Z. \end{aligned}$$

Now, suppose that the conclusion of the Lemma is false. Thus, let t_0 be the first time such that $\min\{(W,Z)(t)\} = 0$. We can assume that W attains the minimum.

At that time, there must be a node j such that $w_j(t_0) = 0$. But on one hand $w_j'(t_0) \le 0$ and, on the other hand, by our hypotheses on A,

$$\begin{aligned} m_j w_j' &> -\sum_{i=1}^{N} a_{ij} w_i + m_j \overline{v}_j^{p_{12}} \left(\frac{\overline{u}_j^{p_{11}} - \underline{u}_j^{p_{11}}}{\overline{u}_j - \underline{u}_j} \right) w_j + m_j \underline{u}_j^{p_{11}} \left(\frac{\overline{v}_j^{p_{12}} - \underline{v}_j^{p_{12}}}{\overline{v}_j - \underline{v}_j} \right) z_j \\ &\ge -\sum_{i \ne j} a_{ij} w_i + m_j \overline{v}_j^{p_{12}} p_{11} \underline{u}_j^{p_{11}-1} w_j + m_j \underline{u}_j^{p_{11}} p_{12} \underline{v}_j^{p_{12}-1} z_j \ge 0, \end{aligned}$$

a contradiction that completes the proof. $\square$

We are ready to prove the convergence result.

Proof of Theorem 1.1: Let us start by defining the error functions

$$e_k(t) = u(x_k,t) - u_k(t), \qquad \varepsilon_k(t) = v(x_k,t) - v_k(t).$$

By (1.4), these functions verify

$$\begin{aligned} m_k e_k' &= -\sum_{i=1}^{N} a_{ik} e_i + m_k (u^{p_{11}}(x_k,t) v^{p_{12}}(x_k,t) - u_k^{p_{11}} v_k^{p_{12}}) + \rho_k^1(h), \\ m_k \varepsilon_k' &= -\sum_{i=1}^{N} a_{ik} \varepsilon_i + m_k (u^{p_{21}}(x_k,t) v^{p_{22}}(x_k,t) - u_k^{p_{21}} v_k^{p_{22}}) + \rho_k^2(h). \end{aligned}$$

Let $t_0 = \max\{t : \ t < T - \tau, \ \max_k |e_k(t)| \le 1, \max_k |\varepsilon_k(t)| \le 1\}$. We will see by the end of the proof that $t_0 = T - \tau$ for h small enough. In $[0, t_0]$ we have

$$m_k e_k' \le -\sum_{i=1}^{N} a_{ik} e_i + K m_k e_k(t) + K m_k \varepsilon_k(t) + \rho_k^1(h),$$

where

$$\begin{aligned} K = \max \quad &\{(\|v\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p_{12}} p_{11} (\|u\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p_{11}-1}, \\ &(\|u\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p_{11}} p_{12} (\|v\|_{L^\infty(\Omega\times[0,T-\tau])} + 1)^{p_{12}-1}\}. \end{aligned}$$

An analogous inequality holds for ε_k. Hence, in $[0, t_0]$, $E = (e_1, ..., e_N), \Sigma = (\varepsilon_1, \dots, \varepsilon_N)$ satisfies

$$\begin{aligned} ME' &\le -AE + KM(E + \Sigma) + \rho(h) M (1, ..., 1)^t, \\ M\Sigma' &\le -A\Sigma + KM(E + \Sigma) + \rho(h) M (1, ..., 1)^t. \end{aligned}$$

Let us now define the pair of functions $(W, Z) = (w_1, \dots, w_N, z_1, \dots, z_N)$, which will be used as a supersolution.

$$w_k(t) = z_k(t) = e^{(2K+1)t} (\|e(0)\|_{L^\infty(\Omega)} + \rho(h)).$$

It is easy to check that $(W(t), Z(t))$ verifies

$$\begin{aligned} MW' &> -AW + KM(W + Z) + \rho(h) M (1, ..., 1)^t, \\ MZ' &> -AZ + KM(W + Z) + \rho(h) M (1, ..., 1)^t. \end{aligned}$$

Hence (W, Z) is a supersolution. A comparison result analogous to Lemma 2.1 holds for this system, so we get

$$\max_{1 \le j \le N} e_j(t) \le e^{(2K+1)t} (\max_j |e_j(0)| + \rho(h)), \qquad t \in [0, t_0].$$

The corresponding estimate also holds for $\max \varepsilon_j$.

Arguing along the same lines with $(-E,-\Sigma)$, we obtain

$$\max_{1\le j\le N}|e_j(t)|\le e^{(2K+1)t}(\max_j|e_j(0)|+\rho(h))\le C\rho(h),\qquad t\in[0,t_0],$$

by our hypotheses on the convergence of the initial data. Using this fact, since $\rho(h)$ goes to zero, we get that $|e_k|,|\varepsilon_k|\le 1$ for every $t\in[0,T-\tau]$ for every h small enough. Therefore $t_0=T-\tau$ for h small enough, and the result follows. □

3. Blow-up for the numerical scheme.

In this section we prove the conditions on the exponents for the existence of blowing up solutions of the numerical scheme, Theorem 1.2.

First, we prove an auxiliary lemma.

Lemma 3.1. *Let $p_{11}\le 1$, $p_{22}\le 1$ and $(p_{11}-1)(p_{22}-1)<p_{12}p_{21}$. Let α_1, α_2 be the solution of*

$$\begin{pmatrix} p_{11}-1 & p_{12}\\ p_{21} & p_{22}-1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}-1\\ -1\end{pmatrix}.$$

If $\alpha_1/\alpha_2\ge 1$ then there exists a constant $C>0$ such that the solution of (1.3) *satisfies*

$$Cu_i(t)\ge v_i^{\alpha_1/\alpha_2}(t).$$

Proof: Let $z_i(t)=Cu_i(t)$ and $w_i(t)=v_i^{\alpha_1/\alpha_2}(t)$, with C a positive constant. Then $Z=(z_i)$ satisfies,

$$MZ'=-AZ+C^{1-p_{11}}MZ^{p_{11}}W^{p_{12}\alpha_2/\alpha_1}.$$

Using the convexity of the function x^{α_1/α_2} $(\alpha_1/\alpha_2\ge 1)$ and the properties of the matrix A we have that

$$-\frac{\alpha_1}{\alpha_2}v_i^{(\alpha_1/\alpha_2)-1}\left(\sum_{j=1}^N a_{ij}v_j\right)\le -\left(\sum_{j=1}^N a_{ij}v_j^{\alpha_1/\alpha_2}\right),$$

hence,

$$MW'\le -AW+\frac{\alpha_1}{\alpha_2C^{p_{21}}}MZ^{p_{21}}W^{(\alpha_2(p_{22}-1)+\alpha_1)/\alpha_1}.$$

We can choose $C>0$ large enough such that

$$z_i(0)=Cu_i(0)>v_i^{\alpha_1/\alpha_2}(0)=w_i(0)$$

for every node. Recall that the nodes such that $u_k(t)=0$ do not appear in the system. Assume that there exists a first time t_0 and a node x_i such that $z_i(t_0)=w_i(t_0)$. Using that $(C^{1-p_{11}}-(\alpha_1/\alpha_2C^{p_{21}}))>0$ (this can be done choosing C larger if necessary), and observing that $p_{11}+p_{12}(\alpha_1/\alpha_2)=(\alpha_2(p_{22}-1)+\alpha_1)/\alpha_1+p_{21}$, at t_0 we have

$$\begin{aligned} 0\ge\ & m_i(z_i-w_i)'(t_0)\\ \ge\ & -\sum_{j=1}^N a_{ij}(z_j-w_j)(t_0)+m_i\left(C^{1-p_{11}}-\frac{\alpha_1}{\alpha_2C^{p_{21}}}\right)z_i^{p_{11}+p_{12}(\alpha_2/\alpha_1)}>0,\end{aligned}$$

a contradiction. □

Now we prove Theorem 1.2, which states a necessary and sufficient condition for the existence of solutions of the discrete problem with blow-up.

Proof of Theorem 1.2: Let us define,

$$T_h = \sup\{t \text{ such that } (U,V)(s) \text{ is defined for } s \in [0,t]\}.$$

If T_h is finite, then by a classical result from ODE theory we have

$$\lim_{t \nearrow T_h} \left\{ \max_j u_j(t) + \max_j v_j(t) \right\} = +\infty.$$

As mentioned in the introduction, this means that (U,V) blows up at time T_h.

First, assume that $p_{11} > 1$. As $v_i(t)$ is bounded from below in $[0,t_0]$, we have that

$$u_k'(t) \geq u_k(t) \left(-\frac{a_{kk}}{m_k} + \delta^{p_{12}} u_k^{p_{11}-1}(t) \right).$$

If the initial data u_0 is such that $u_i(0)$ is large enough (depending on h) for some node x_i then we have that $u_i'(t) \geq c u_i^{p_{11}}(t)$ for $t \in [0,t_0]$. Since $p_{11} > 1$, integrating this inequality we obtain that u_i blows up in finite time (smaller than t_0 if $u_i(0)$ is large enough). Hence (U,V) cannot be global.

The same argument applies when $p_{22} > 1$, using V instead of U.

If $p_{11} \leq 1$ and $p_{22} \leq 1$ then we can apply Lemma 3.1 to obtain that, if $v_i(0)$ is large enough, for some node x_i we have

$$\begin{aligned} v_i'(t) &\geq -\frac{a_{ii}}{m_k} v_i(t) + u_i^{p_{21}}(t) v_i^{p_{22}}(t) \\ &\geq -\frac{a_{ii}}{m_k} v_i(t) + C v_i^{p_{22}+p_{21}(\alpha_1/\alpha_2)}(t) \\ &\geq c v_i^{p_{22}+p_{21}(\alpha_1/\alpha_2)}(t). \end{aligned}$$

Our hypotheses on the exponents imply that $p_{22} + p_{21}(\alpha_1/\alpha_2) > 1$ and therefore (U,V) cannot be global. □

4. Non-simultaneous blow-up.

In this section we focus on the numerical non-simultaneous blow-up. We consider positive solutions of (1.3) with h fixed and we denote by C a positive constant that may depend on h and may vary from one line to another.

Proof of Theorem 1.3: First we observe the equations involving U'. We have

$$m_k u_k'(t) = -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) v_k^{p_{12}}(t).$$

Suppose $p_{11} \leq 1$. Since V is bounded we have

$$m_k u_k'(t) \leq -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) C^{p_{12}}.$$

We define $w(t) = \sum_{k=1}^{N} u_k(t)$. As u blows up at time T_h there exists a time t_0 such that for every $t \in [t_0, T_h)$ we have

$$w'(t) \le -\sum_{k=1}^{N}\sum_{j=1}^{N} \frac{a_{kj}}{m_k} u_j(t) + C\sum_{k=1}^{N} u_k^{p_{11}} \le C w(t)^{p_{11}}.$$

Since $p_{11} \le 1$, w is bounded, which implies that U is bounded. Then p_{11} must be strictly greater than one.

We want to get a bound from below for the blow-up rate. For $t \in [t_0, T_h)$ we can integrate the above inequality between t and T_h to obtain

$$\int_t^{T_h} \frac{w'(s)}{w^{p_{11}}(s)}\, ds \le C(T_h - t).$$

Changing variables we get

$$\int_{w(t)}^{+\infty} \frac{1}{s^{p_{11}}}\, ds \le C(T_h - t).$$

Hence

$$w(t) \ge C(T_h - t)^{-1/(p_{11}-1)}.$$

Using that there exists a constant $C = C(h)$ such that

$$\max_j u_j(t) \ge C \sum_{i=1}^{N} u_i(t),$$

we have

$$\max_j u_j(t) \ge C(T_h - t)^{-1/(p_{11}-1)}. \tag{4.1}$$

We consider now the equations involving V'

$$m_k v_k'(t) = -\sum_{j=1}^{N} a_{kj} v_j(t) + m_k u_k^{p_{21}}(t) v_k^{p_{22}}(t).$$

Since there exists $\delta > 0$ such that $v_k(t) > \delta$ for $t \in [t_0, T_h)$ and taking into account the bound from below for the blow-up rate we have that for $t \in [t_0, T_h)$

$$m_k v_k'(t) \ge -\sum_{j=1}^{N} a_{kj} v_j(t) + C(T_h - t)^{-p_{21}/(p_{11}-1)} \delta^{p_{22}}.$$

If $p_{21} \ge p_{11} - 1$ then v_k blows up, a contradiction. Hence we have that $p_{21} < p_{11} - 1$, and this ends the proof. □

Now we prove the converse, Theorem 1.4.

Proof of Theorem 1.4: Let V_0 any initial data for V. There exists T, $\delta > 0$ such that for every node we have that $v_k(t) \ge \delta$ for all $t_0 \le t \le T$. For u_k we have

$$m_k u_k'(t) \ge -\sum_{j=1}^{N} a_{kj} u_j(t) + m_k u_k^{p_{11}}(t) \delta^{p_{22}} \ge C u_k^{p_{11}}(t)$$

if $u_k(0)$ is large enough. As $p_{11} > 1$, $u_k(t)$ cannot be global. Then $U(t)$ blows up in finite time. Actually $U(t)$ exists only until one of its components blows up. Moreover, the blow-up rate is bounded by

$$u_k(t) \leq C(T_h - t)^{-1/(p_{11}-1)}. \tag{4.2}$$

Then V verifies

$$MV'(t) \leq -AV + MC(T_h - t)^{-p_{12}/(p_{11}-1)} V^{p_{22}}.$$

As $p_{12} < p_{11} - 1$, if T_h is small enough, V remains bounded up to T_h. Now we can choose $U(0)$ such that U blows up at time T_h small enough to ensure that V is bounded and the result follows. □

To prove Theorem 1.5 we need a lemma which states that, for solutions that increase in time, if u blows up and v is bounded then (U, V) cannot be global and provides a bound for the blow-up rate independent of h.

Lemma 4.1. *Let (U, V) be a solution of (1.2) with $p_{11} > 1$ and such that $u_k'(0) \geq \delta u_k^{p_{11}}(0)$ and $v_k'(0) \geq 0$. Then $u_k'(t) \geq \delta u_k^{p_{11}}(t)$ and $v_k'(t) \geq 0$ for every $t < T_h$.*

Proof. First, we claim that both $u_k'(t)$ and $v_k'(t)$ are nonnegative. In order to do that, let us define $w_k(t) = u_k'(t)$ and $z_k(t) = v_k'(t)$. Therefore, by a simple computation, (W, Z) verifies

$$MW' = -AW + D_1 W + D_2 Z,$$

$$MZ' = -AZ + D_3 W + D_4 Z,$$

where D_i are time dependent matrices with nonnegative coefficients. By the minimum principle, since $W(0), Z(0) \geq 0$, the claim follows.

Now, let us check that $u_k'(t) \geq \delta u_k^{p_{11}}(t)$. Let $w_k(t) = u_k'(t) - \delta u_k^{p_{11}}(t)$. We want to use the minimum principle to show that $w_k(t)$ is positive. To this end, we observe that w_k verifies

$$m_k w_k' + \sum_{j=1}^N a_{kj} w_j = m_k(u_k'' - \delta p_{11} u_k^{p_{11}-1} u_k') + \sum_{j=1}^N a_{kj}(u_j' - \delta u_j^{p_{11}})$$

$$= -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + m_k(p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} + p_{12} v_k^{p_{12}-1} v_k' u_k^{p_{11}}) - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$\geq -\delta m_k p_{11} u_k^{p_{11}-1} u_k' + m_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$= -\delta p_{11} u_k^{p_{11}-1} \left(\sum_{j=1}^N a_{kj} u_j + m_k u_k^{p_{11}} \right) + m_k p_{11} u_k^{p_{11}-1} u_k' v_k^{p_{12}} - \delta \sum_{j=1}^N a_{kj} u_j^{p_{11}}$$

$$= m_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j\neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1} u_j) + a_{kk}(1-p_{11}) u_k^{p_{11}} \right)$$

$$= m_k p_{11} u_k^{p_{11}-1} v_k^{p_{12}} w_k - \delta \left(\sum_{j\neq k} a_{kj}(u_j^{p_{11}} - p_{11} u_k^{p_{11}-1}(u_j - u_k) - u_k^{p_{11}}) \right.$$

$$\left. + \sum_{j=1}^N a_{kj}(1-p_{11}) u_k^{p_{11}} \right).$$

As $f(u) = u^{p_{11}}$ is convex ($p_{11} > 1$) and from our hypotheses on the matrix A it follows that $W = (w_1, \ldots, w_N)$ verifies

$$MW' \geq -AW + Mp_{11} U^{p_{11}-1} V^{p_{12}} W.$$

Since $W(0) > 0$ and the minimum principle holds for this equation, the result follows. □

Proof of Theorem 1.5: In [QR] the authors prove that if u blows up and v remains bounded then $p_{11} > 1$ and $p_{21} < p_{11} - 1$, so we are going to begin the proof assuming $p_{11} > 1$. The blow-up rate follows from (4.1) and (4.2). From the previous lemma we can obtain an upper bound for the blow-up rate of U independent of h. From our hypotheses on the initial data we have that there exists δ independent of h such that the hypothesis of the previous lemma holds. Integrating we obtain that if $u_k(t)$ is blowing up then

$$\int_t^{T_h} \frac{u_k'}{u_k^{p_{11}}}(s)\, ds \geq \delta(T_h - t),$$

so that

$$u_k(t) \leq C(T_h - t)^{-\frac{1}{p_{11}-1}}, \tag{4.3}$$

where C depends only on p_{11} and δ. Moreover,

$$(T_h - t) \leq \frac{1}{\delta} \int_{\max_k u_k(t)}^{+\infty} \frac{1}{x^{p_{11}}}\, dx. \tag{4.4}$$

This bound says that if $U(t)$ is large then $T_h - t$ is small.

For the remainder of the proof the idea is as follows, as u is blowing up and v is bounded, from the convergence of the method we get that $U(t_0)$ is large and $V(t_0)$ is bounded for t_0 close to T. Using the bound (4.4) we obtain that $T_h - t_0$ is small and $V(t)$ verifies

$$(4.5) \qquad \begin{aligned} &MV'(t) \le -AV + MC(T_h - t)^{-p_{12}/(p_{11}-1)} V^{p_{22}}, \\ &V(t_0) \le \|v\|_\infty + 1. \end{aligned}$$

Hence $V(t)$ is bounded up to T_h. The proof is similar to the one of Theorem 1.4.

Let us write down the details of the proof. Given $\varepsilon > 0$ we can choose K large enough to guarantee

$$\frac{1}{\delta} \int_K^\infty \frac{1}{x^{p_{11}}}\, dx < \varepsilon,$$

and $t_0 < T$ such that

$$\|u(\cdot, t_0)\|_\infty > 2K.$$

As $U(t)$ converges to u in $[0, t_0]$ we can choose h_0 such that for every $h < h_0$

$$\|U(t_0)\|_\infty > K.$$

Hence, from (4.4) we get

$$T_h - t_0 < \varepsilon,$$

using this fact and that $p_{12} < p_{11} - 1$, we can assert that the solution of

$$(4.6) \qquad \begin{aligned} &\overline{v}_k(t) = C(T_h - t)^{-p_{12}/(p_{11}-1)} \overline{v}_k^{p_{22}}(t), \\ &\overline{v}_k(t_0) = \|v\|_\infty + 1, \end{aligned}$$

is bounded up to T_h.

As $\|V(t_0)\|_\infty \le \|v\|_\infty + 1$ for every h small enough and (4.5) holds we can use $\overline{V}(t)$ as a supersolution for $V(t)$ to conclude that $V(t)$ is bounded up to T_h. Therefore we have that U blows up and V remains bounded for every h small enough. Hence we get non-simultaneous numerical blow-up for every h small enough.

Now we prove the convergence of the blow-up times. Given $\varepsilon > 0$ we can repeat the arguments used above to obtain that for some $t_0 < T$ such that $T - t_0 < \varepsilon$,

$$|T_h - t_0| \le \frac{1}{\delta} \int_K^{+\infty} \frac{1}{x^{p_{11}}}\, dx < \varepsilon,$$

therefore

$$|T_h - T| \le |T_h - t_0| + |T - t_0| < 2\varepsilon.$$

This ends the proof. □

5. Numerical experiments.

In this Section we present some numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one performs numerical computations. For simplicity we restrict ourselves to an interval, $\Omega = (0, 1)$ and perform the semidiscrete method in a uniform mesh using finite elements with mass lumping (this coincides with the central differences scheme). As we mentioned in the introduction this method satisfies our general assumptions. To integrate in time we use an adaptive ODE solver.

We begin by considering non-simultaneous blow-up. We perform the calculations with the following parameters, $p_{11} = 2$, $p_{12} = 1$, $p_{21} = 1/2$, $p_{22} = 1/2$, with initial data $u_0(x) = 200x(1-x)$, $v_0(x) = 5x(1-x)$. In this case the numerical blow-up time is $T_h = 0.01902$. In Figure 1 we show the behavior of $\max U(t)$ and of $\max V(t)$. For a better understanding we show the evolution until $t_0 = 0.0187$. We remark that though V does not blow-up, V' does. This numerical experiment suggests that the monotonicity assumption in Theorem 1.5 is not essential.

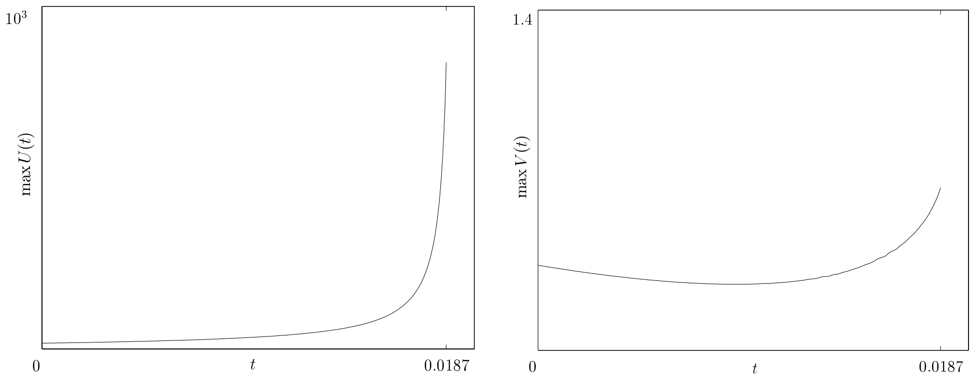

Figure 1. $\max U(t)$ blowing up and $\max V(t)$ bounded.

Now, in Figure 2 we show simultaneous numerical blow-up. For these pictures we have only changed the value of p_{21} from $1/2$ to 2. This forces simultaneous blow-up.

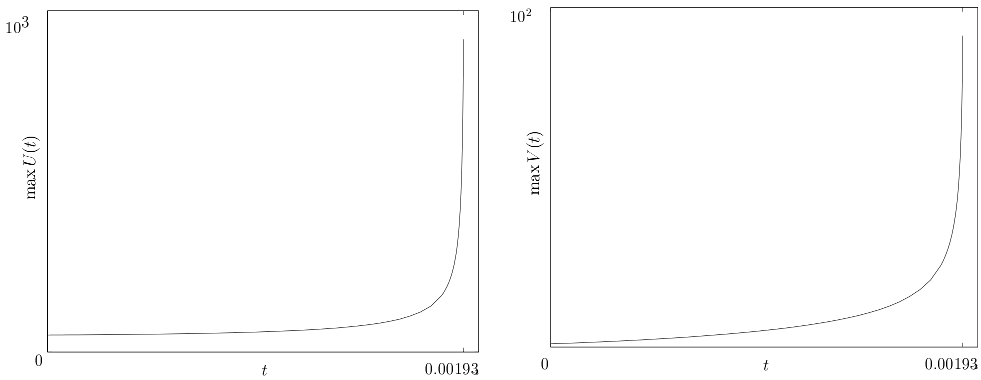

Figure 2. $\max U(t)$ and $\max V(t)$, both blowing up.

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