Stochastic Differential Equations with explosions

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Joint work with

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The Problem

$$dx = b(x)\,dt + \sigma(x)\,dW$$
$$x(0) = z \in \mathbb{R}_{>0}$$

- W is a one dimensional Wiener process (dW is "white noise")
- b, σ are smooth and positive.

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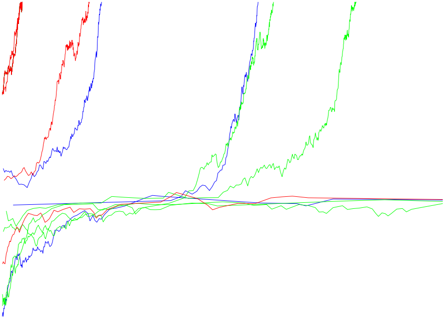
If b is not globally Lipschitz, solutions to this problem may explode in finite time.

Explosions



There exist a stopping time T such that $x(\omega, t)$ is defined in $[0, T(\omega))$, but

$$x(\omega, t) \nearrow +\infty \qquad \text{as } t \nearrow T(\omega).$$

Explosions in evolution problems

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   Example

   $$\dot{u}(t) = Au(t) + b(u(t)), \quad A \in \mathbb{R}^{N\times N} = \text{Discrete Laplacian}$$

   There exist solutions with blow-up (Rossi, G. 2000).

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   - No closed criteria to decide if blow-up will occur.
   - No explicit formula for the blow-up time.
   - The phenomenon is very well understood (blow-up times, blow-up sets, blow-up rates, numerical computation of solutions, etc.) Galaktionov-Vázquez, 2002 (survey).

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No general criteria in higher dimensions

Explosions in evolution problems

5. SPDE: $u_t = u_{xx} + u^p \dot{W}$, $x \in [0,1]$
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Blow-up if $p > 3/2$. Global solutions if $p < 3/2$ (C. Mueller, 2000)

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5. What happens when perturbing the problem? (regularity of the explosion time)
6. How to compute it numerically?

Almost all of these questions are open in the stochastic case

Regularity of the explosion time

Consider the stochastic differential equation

$$\begin{aligned} dx &= b(x)\,dt + \sigma(x) \circ dW \\ x(0) &= x_0. \end{aligned} \tag{1}$$

Theorem 1. *Assume b/σ is nondecreasing, $x(t)$ is a solution to (1) with initial datum x_0 and $x_n(t)$ is a solution to (1) with initial datum x_0^n. Let T and T_n be the explosion times for $x(t)$ and $x_n(t)$ respectively. If $x_0^n \to x_0$ a.s. (in probability) then $T_n \to T$ a.s. (in probability)*

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Theorem 2. *For additive or multiplicative noise, under adequate hypotheses, if $b_n \to b$ and $\sigma_n \to \sigma$ then $T_n \to T$ a.s.*

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Uniform upper explosion estimate: There exists a nondecreasing continuous function G, independent of n, such that

$$\|u_n(t)\| \leq G\left(\frac{1}{T_n - t}\right). \tag{H3}$$

Theorem. *If (H1)–(H3) hold, then*

$$\lim_{n\to\infty} T_n = T.$$

Proof (idea).

- Consider the error $e_n(t) := \|u_n(t) - u(t)\|$.
- Estimate the first time t_n at which $e_n(t_n) = 1$.
- Prove that these times verify $t_n \to T$ and $T_n - t_n \to 0$.

Hence, $\quad |T_n - T| \leq |T - t_n| + |t_n - T| \qquad \blacksquare$

Application to stochastic differential equations

Pathwise solutions of the SDE (Doss-Sussmann)

$$dx = b(x)\,dt + \sigma(x) \circ dW. \tag{1}$$

$$\dot\phi(t,z) = \sigma(\phi(t,z)), \qquad \phi(0,z) = z.$$

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$$H(z,t) := \frac{b(\phi(t,z))\,\sigma(z)}{\sigma(\phi(t,z))} = \frac{b(\phi(t,z))}{\partial_z \phi(t,z)}.$$

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Then $x(t,\omega) := \phi(W(t,\omega), z(t))$ solves (1).

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If σ is globally Lipschitz, $\phi(t,z)$ is globally defined and the explosion times of x and z coincide

Continuity respect to the initial data

$$S := \sup_{n\geq 1}\{T_n; T\}, \qquad \mathbb{P}(S<\infty) = 1.$$

$$A_{K,M} := \{\omega \in \Omega \ : \ S(\omega) \leq K \text{ and } |W(t,\omega)| \leq M, \text{for } t \in [0, K+1]\}.$$

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(H3)

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Let

$$g(\xi) := \left(\int_\xi^{+\infty} \frac{du}{H(u, -M)} \right)^{-1}.$$

Since g is increasing, its inverse $G := g^{-1}$ is also increasing and then we have

Continuity respect to the initial data

$$z_n(t) \leq G\left(\frac{1}{T_n - t}\right).$$

Hence we have a uniform bound for the blow-up rate and the result follows. ■