Numerical approximation of SDE with explosions.

Joint work with
Juán Dávila, U. de Chile
Julián Fernández Bonder, U. de Buenos Aires
Julio D. Rossi, U. de Buenos Aires
Mariela Sued, U. de Buenos Aires.

1^{o} ERPEM, Buenos Aires, 2004.

The Problem

$$dx = b(x)\,dt + \sigma(x)\,dW$$
$$x(0) = z \in \mathbb{R}_{>0}$$

- W is a one dimensional Wiener process.

- b, σ are smooth and positive.

If b is not globally Lipschitz, solutions to this problem may explode in finite time.

There exist a stopping time T such that $x(\omega, t)$ is defined in $[0, T(\omega))$, but

$$x(\omega, t) \nearrow +\infty \qquad \text{as } t \nearrow T(\omega).$$

Fatigue Cracking

This kind of SDE are used, for example, to model fatigue cracking (fatigue failures in solid materials)

$x(t)$ represents the evolution of the length of the largest crack.

The explosion time corresponds to the time of ultimate damage or fatigue failure in the material.

The *Feller Test for Explosions* provides a precise criteria to determine, in terms of b and σ whether solutions explode with probability zero, positive or one.

We assume

- $0 < C_1 \leq \sigma^2(s) \leq C_2 b(s)$.
- $b(s)$ is nondecreasing for $s > s_0$ and $\int^{\infty} \frac{1}{b(s)}\, ds < \infty$.

Under these conditions, explosions occur with probability one.

Example:

$$dx = (1 + x^2)\, dt + dW$$

How a numerical method should be in order to reproduce this behavior?

-

-

-

How a numerical method should be in order to reproduce this behavior?

- Convergence of the numerical solutions to the continuous one.

-

-

How a numerical method should be in order to reproduce this behavior?

- Convergence of the numerical solutions to the continuous one.

- Explosions in the numerical solutions for small choices of the parameter.

-

How a numerical method should be in order to reproduce this behavior?

- Convergence of the numerical solutions to the continuous one.

- Explosions in the numerical solutions for small choices of the parameter.

- Convergence of the numerical explosion times to the continuous one.

The Euler-Maruyama method (for bounded solutions)

$$X_i \approx x(t_i)$$

$$h = t_{i+1} - t_i, \qquad \Delta W_i = W(t_{i+1}) - W(t_i).$$

$$X_{i+1} = X_i + hb(X_i) + \sigma(X_i)\Delta W_i, \qquad X_0 = x(0) = z,$$

$$X_{i+1} = X_i + hb(X_i) + \sigma(X_i)\Delta W_i$$

- **Not suitable for solutions with explosions!** The numerical solution is defined for every positive time.

- The time step h can not be constant. It must be adapted as the solution increase.

- We propose $h_i = \frac{h}{b(X_i)}$. i.e. $t_{i+1} - t_i = \frac{h}{b(X_i)}$

$$X_{i+1} = X_i + h_i b(X_i) + \sigma(X_i)\Delta W_i = X_i + h + \sigma(X_i)\Delta W_i$$

The numerical solution

$$X(t_i) = X_i,$$

$$X(t) = X_i + (t - t_i)b(X_i) + \sigma(X_i)(W(t) - W(t_i)), \qquad \text{for } t \in [t_i, t_{i+1}).$$

is a well defined process up to time

$$T_h = \sum_{i=1}^{\infty} h_i = \sum_{i=1}^{\infty} \frac{h}{b(X_i)}.$$

We say that the numerical solution explode in finite time T if

$$T_h < \infty$$

Mean Square Convergence (while solutions are bounded)

Theorem 1: Fix a time $S > 0$ and a constant $M > 0$. Consider the stopping times given by

$$R^M := \inf\{t : x(t) = M\} \quad R_h^M := \inf\{t : X(t) = M\}$$

$$\tau_h = R^M \wedge R_h^{2M} \wedge S$$

Then

$$\lim_{h \to 0} \mathbb{E}\left[\sup_{0 \le t \le \tau_h} |x(t) - X(t)|^2\right] = 0.$$

Explosions in the Numerical Scheme

Theorem 2:

1. $X(\cdot)$ explodes in finite time with probability one.
2. For every $h > 0$ we have,

$$\lim_{i\to\infty} \frac{X(t_i)}{hi} = 1 \quad \text{i.e} \quad X(t_i) \sim hi$$

(This is the asymptotic behavior of the numerical solution, since the behavior of

$$t_i = \sum^{i-1} \frac{h}{b(X_j)}$$

can also be computed.)

For example, if $b(s) \sim s^p$ ($p > 1$), the explosion rate is

$$X(t_i)(T_h - t_i)^{1/(p-1)} \longrightarrow \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \text{ as } t_i \nearrow T_h.$$

Convergence of the Numerical Explosion Times

Theorem 3: ?

The Numerical Explosion Times T_h converges to the continuous one T in probability as $h \to 0$, that is, for every $\varepsilon > 0$

$$\mathbb{P}(|T_h - T| > \varepsilon) \to 0 \qquad \text{as } h \to 0.$$

Convergence of the Numerical Explosion Times

Theorem 3: ?

The Numerical Explosion Times T_h converges to the continuous one T in probability as $h \to 0$, that is, for every $\varepsilon > 0$

$$\mathbb{P}(|T_h - T| > \varepsilon) \to 0 \qquad \text{as } h \to 0.$$

THE END