Optimization of the first Steklov eigenvalue in domains with holes.

Joint work with
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Discontinuous change in behavior issues in partial differential equations
Crete, June 2006

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The immersion $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact and hence there exist extremals. These are weak solutions to

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega \end{cases}$$

$\lambda =$ Lagrange multiplier

Let $A \subset \Omega$, $|A| = \alpha > 0$ and consider

$$S_A = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 + |u|^2 \, dx}{\int_{\partial\Omega} |u|^2 \, dS} \mid u \in H^1(\Omega), \; u = 0 \text{ a.e. in } A \right\}$$

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If A is closed, the extremals for S_A are weak solutions to

$$\begin{cases} \Delta u = u & \text{in } \Omega \setminus A, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{in } \partial\Omega \setminus A, \\ u = 0 & \text{on } A. \end{cases}$$

$$S_A = \lambda_1 = \lambda_1(A)$$

PROBLEM

Given $0 < \alpha < |\Omega|$, *find a subset* A^* *with Lebesgue measure* $|A^*| = \alpha$ *minimizing* $\lambda_1(A)$ *among all measurable subsets* $A \subset \Omega$ *with* $|A| = \alpha$.

(F. Bonder, Rossi and Wolanski)

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They introduce the problem and prove

1. Existence

2. Symmetry properties of A^*

3. Interior regularity of the boundary of A^*

Related problems

- Regularity Aguilera-Alt-Caffarelli (1986) $\longrightarrow$ *minimize the Dirichlet integral among functions vanishing in a subset with prescribed volume*

- Optimal design for eigenvalue problems Faber-Krahn (1923-25) $\longrightarrow$ *The ball minimizes the first eigenvalue of the Laplace-Dirichlet operator*.

  Henrot (survey) $\longrightarrow$ J. Evol. Equ. 3 (2003)

- Numerical computation Oudet (survey) $\longrightarrow$ ESAIM Control Optim. Calc. Var. (2004)

Our main:

To compute the shape derivative of $\lambda_1(A)$ with respect to the hole A. This allows us to

1. Given a set A, to decide if it is (not) optimal.

2. Numerical methods to compute A^*, S_{A^*}.

SHAPE DERIVATIVE (Hadamard)

$$V : \mathbb{R}^N \to \mathbb{R}^N, \quad supp(V) \subset \Omega, \quad \int_\Omega \text{div } V = 0 \quad V \in C^1.$$

We perturb the hole A in the direction V

$$A_t := (Id + tV)A = \{x + tV(x),\ x \in A\}$$

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Thm: $\lambda_1(A_t)$ is differentiable at $t = 0$ and verifies

$$\frac{d}{dt}\lambda_1(A_t)|_{t=0} = \lambda_1'(A) = -\int_{\partial A} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle V, \nu \rangle \, dS,$$

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Idea of the proof:

1. Differentiability: General theory of Type A operators.

2. Change variables in the weak formulation to keep the domain fixed ($y = x + tV(x)$) and differentiate.

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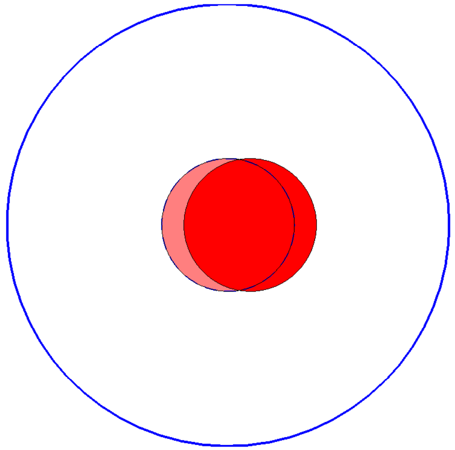
BUT IT IS NOT OPTIMAL

Proof:



Let v the (radial) extremal associated to $B(0, r)$ and let us consider

$$x_t = (x_1 - t, x_2, ..., x_N), \qquad U(t)(x) = v(x_t).$$

The function $U(t)$ vanishes at $A_t := B(0, r)(Id + te_1)$

(it can be used as test function for A_t)

Let us call

$$\Phi(t) := \frac{\int_{B_R} |\nabla U(t)|^2 + |U(t)|^2 \, dx}{\int_{\partial B_R} |U(t)|^2 \, dS}.$$

We have,

$$\Phi(0) = \lambda_1(A).$$

As Φ is an even function,

$$\Phi'(0) = 0.$$

But

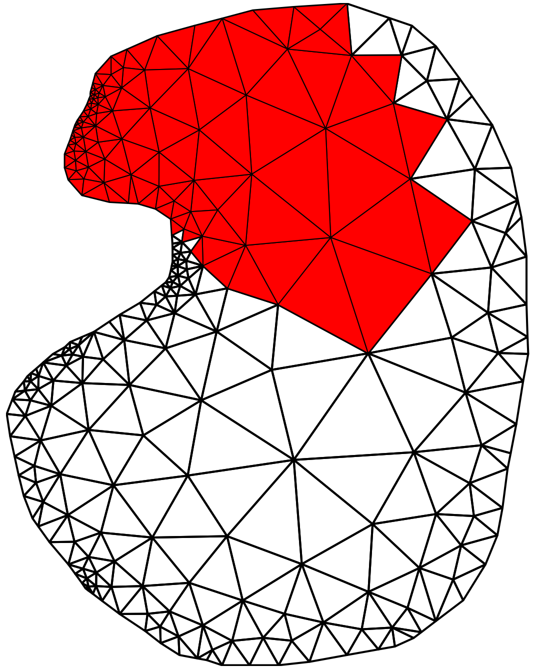
$$\Phi''(0) = \frac{2}{N}\lambda_1\left[1 - \frac{N-1}{R}\lambda_1 - \lambda_1^2\right] < 0.$$

Hence, for small t

$$\lambda_1(A_t) \leq \Phi(t) < \Phi(0) = \lambda_1(A)$$

NUMERICAL APROXIMATION

Finite Element Method



$\mathcal{V}_h \subset H^1(\Omega)$ piecewise linear continuous

$$\mathcal{T}_h := \{T_j^h \ : \ 1 \leq j \leq J_h, \ \bigcup_j T_j^h = \Omega\}$$

Let us consider the class $\mathcal{O}_\alpha^h$ of "numerical holes of measure α".

$$\mathcal{O}_\alpha^h := \left\{ A_h : A_h = \cup_k T_{j_k}^h, \ |A_h| \geq \alpha, \ |A_h - T_{j_k}^h| < \alpha \text{ for some } k \right\}$$

We call

$$\lambda_{1,h}(A_h) := \inf \left\{ \int_\Omega |\nabla v|^2 + v^2 \, dx \ : \ v \in \mathcal{V}_h, \ \|v\|_{L^2(\partial\Omega)} = 1 \text{ and } v|_{A_h} \equiv 0 \right\}.$$

The numerical optimal hole is

$$\lambda_{1,h}(A_h^*) = \min_{\mathcal{O}^h} \lambda_{1,h}(A_h).$$

Thm:

$$\lim_{h \to 0} \lambda_{1,h}(A_h^*) = \lambda_1(A^*).$$

Moreover, for any sequencer $h_j \to 0$, there exists $h_{j_k} \to 0$ and $u \in H^1(\Omega)$ such that

$$u_{h_{j_k}} \to u \qquad \text{strongly in} H^1(\Omega).$$

The function u verifies $A^* := \{u = 0\}$ is optimal for $\lambda_1(A)$ and u is an eigenfunction associated to $\lambda_1(A^*)$. Finally,

$$|A^*_{h_{j_k}} \triangle A^*| \to 0, \quad \text{as } k \to \infty.$$

Algorithm to find $\lambda_{1,h}(A_h^*)$

1. Choose an initial hole $A_h^0 \in \mathcal{O}_\alpha^h$.

2. Compute $\lambda_{1,h}(A_h^0)$ and the extremal u_h^0.

3. Compute $\frac{\partial u_h^0}{\partial \nu}$ on ∂A_h^0.

4. Remove the triangles with larger normal derivative from the hole and add (to the hole) triangles in regions of the boundary where the normal derivative is small to obtain a new hole $A_h^1 \in \mathcal{O}_\alpha^h$.

5. Go to 2.

Numerical Experiments

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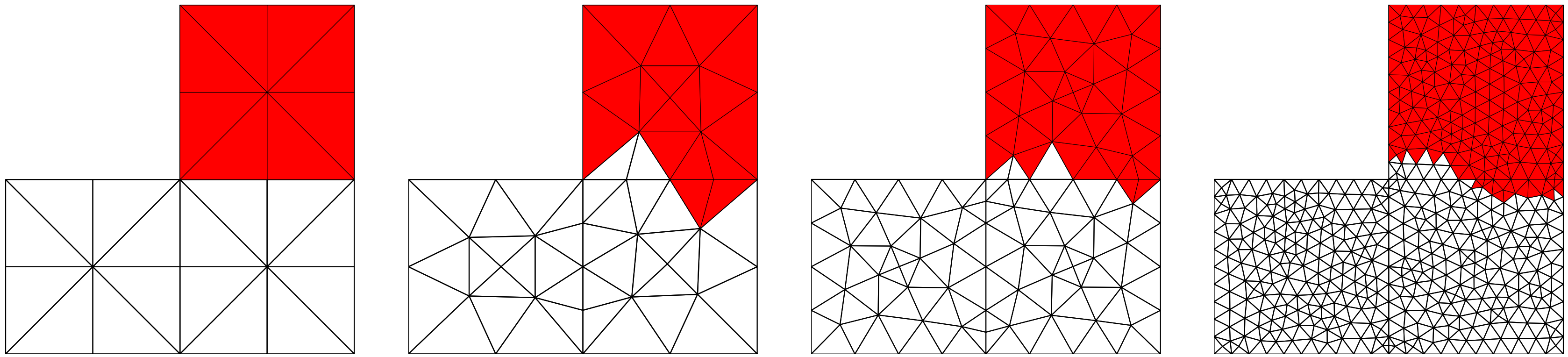
Behavior of the numerical optimal hole as $h \to 0$.



h=0.80 h=0.50 h=0.25 h=0.1

Some comments on the method.

- Convergence of the algorithm to the discrete optimal hole for fixed h is not proved.

- An adequate method should be used to compute the eigenfunction for a fixed hole A_h.

- Changes in the topology of the hole are allowed by the method.

- The method, as described, seems better suited to find the location of the hole rather than the fine resolution of its boundary. Adaptivity should be used to achieve this goal.