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Fluctuations in the asymmetric simple exclusion process

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Summary. We review recent results about the asymmetric simple exclusion process in $\mathbb{Z}$. In that process particles jump to the right at rate p and to the left at rate q, $p+q=1$. At mosts one particle is allowed in each site. We consider the process in equilibrium with invariant measure ν_ρ, the product measure with density ρ. The first result is the computation of the the diffusion coefficient of the current of particles through a fixed point. We find $D = |p-q|\rho(1-\rho)|1-2\rho|$. A law of large numbers and central limit theorems for the rescaled current are also proven. Analogous results hold for the current of particles through a position travelling at a deterministic velocity r. As a corollary we get that the equilibrium density fluctuations at time t are a translation of the fluctuations at time 0. We also show that the current fluctuations at time t are given, in the scale $t^{1/2}$, by the initial density of particles in an interval of length $|(p-q)(1-2\rho)|t$. The second result is a representation of the position X_t of a tagged particle of the system that allows us to write $X_t = N_t + B_t$, where N_t is a Poisson process of parameter $(1-\rho)(p-q)$ and B_t is a random variable whose absolute value is stochastically bounded above by B, a random variable with distribution independent of t and with a finite exponential moment. This is an extension of Burke's theorem for queuing theory.

Keywords. Asymmetric simple exclusion. Current fluctuations. Burke's Theorem. Tagged particle.

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1. Introduction.

The nearest neighbor one dimensional asymmetric simple exclusion process is the Markov process $\eta_t \in \{0,1\}^{\mathbb{Z}}$ that corresponds to the following description. At most one particle is allowed at each site. If there is a particle at site x, it jumps at rate p to site $x+1$ if there is no particle in $x+1$. Analogously if site $x-1$ is empty, the particle at site x jumps to $x-1$ at rate q. This process was introduced by Spitzer (1970) and has received a great deal of attention. Its ergodic properties are well understood (Liggett (1976, 1985)). The set of invariant measures is the set of convex combinations of the product measures ν_ρ and blocking measures. These measures concentrate on a denumerable set of configurations and for $p > q$ have asymptotic density 0 and 1 to the left and right of the origin, respectively. The hydrodynamical limit was studied by Andjel and Vares (1987) and extended by Benassi et al (1991) for monotone initial density profiles. Rezakhanlou (1990) proposed a general approach to prove a law of large numbers for the density fields of attractive particle systems that works for general initial density profiles. Landim (1992) uses this law of large numbers to prove local equilibrium.

The current through rt at time t is defined by $J_{rt,t}$ = number of particles to the left of the origin at time zero and to the right of rt at time t minus number of particles to the right of the origin at time zero and to the left of rt at time t. We assume that the distribution of the initial configuration is the stationary measure ν_ρ, the product measure with density ρ. Under this initial distribution,

$$EJ_{rt,t} = ((p-q)\rho(1-\rho) - r\rho)t$$

In Ferrari and Fontes (1992a) we show the following results

1. Law of large numbers:

$$\lim_{t\to\infty} \frac{J_{rt,t}}{t} = ((p-q)\rho(1-\rho) - r\rho) \quad a.s.$$

This result is not surprising but its proof for $r \neq 0$ is not immediate because it is not clear how to prove that ν_ρ is an extremal invariant measure for the process as seen from the moving position rt.

2. Central limit theorem: Let $G(0,D)$ be a centered normal random variable with variance D. Then

$$\lim_{t\to\infty} \frac{J_{rt,t} - EJ_{rt,t}}{\sqrt{t}} = G(0, D_J),$$

in distribution, where $D_J = \lim_{t\to\infty}(VJ_{rt,t}/t)$ and V is the variance. For $r = 0$, this result can be proven using techniques of Kipnis (1986), as one can show that the current is a negative correlated process. We get it as a corollary of item 5 below.

3. Computation of the variance. This computation is the key to obtain the other results. The technial point in this computation is to control the behavior of a perturbation of the system. The result is

$$\lim_{t\to\infty} \frac{VJ_{rt,t}}{t} = \lim_{t\to\infty} \frac{E(J_{rt,t} - EJ_{rt,t})^2}{t} = \rho(1-\rho)|(p-q)(1-2\rho) - r|$$

Notice that for $r = (p-q)(1-2\rho)$, $D_J = 0$. The particular case $p = q$, $r = 0$ can also be proven using Arratia (1983). The general case is more surprising. In the next point a more detailed behavior is shown.

4. Current and second class particle. For $p = 1$ and $r = (1-2\rho)$ we show

$$VJ_{(1-2\rho)t,t} = \rho(1-\rho)E|R_t^0 - (1-2\rho)t|$$

where R_t^0 is the position of a second class particle initially located at the origin. For $p = 1$, a second class particle interacts with the other particles in the following way: it jumps to empty sites to the right at rate 1 and interchange positions with ("first class") particles to its left at rate 1. A perturbation of the system behaves as a second class particle. Spohn (1991) gives heuristic arguments suggesting that VR_t^0 behave as $t^{4/3}$. This would imply that the variance of the current through $(1-2\rho)t$ behaves as $t^{2/3}$.

5. Dependence on the initial configuration.

$$\lim_{t\to\infty} \frac{E(J_{rt,t} - N_{th(r,\rho)} - \rho^2 t)^2}{t} = 0$$

where $h(r,\rho) = r - (1-2\rho)(p-q)$, $N_r(\eta) = \sum_{x=0}^{r} \eta(x)$ for $r > 0$ and $N_r(\eta) = -\sum_{x=r}^{0} \eta(x)$ for $r \le 0$. $N_{th}(\eta)$ depends only on the initial configuration η. A result like this was conjectured by Spohn (1991) and proved by Gärtner and Presutti (1989) and Ferrari (1992a,b) for the position of a tagged particle.

6. Equilibrium fluctuations translate rigidly in time. Let ξ_t^ε be the fluctuations fields defined by

$$\xi_t^\varepsilon(\Phi) = \varepsilon^{1/2} \sum_x \Phi(\varepsilon x)[\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)]$$

for smooth integrable functions Φ. Call $\bar r = (p-q)(1-2\rho)$, then

$$\lim_{\varepsilon\to 0} E(\xi_t^\varepsilon - \tau_{\varepsilon^{-1}\bar r t}\xi_0^\varepsilon)^2 = 0$$

where the translation τ is defined by $\tau_y \xi_t^\varepsilon(\Phi) = \xi_t^\varepsilon(\tau_y\Phi)$ and $\tau_y\Phi(x) = \Phi(x+y)$. In other words, the limiting equilibrium fluctuations fields satisfy the linear equation

$$\frac{\partial}{\partial t}\xi_t(r) = \bar r \frac{\partial}{\partial r}\xi_t(r)$$

with initial condition ξ_0, the Gaussian field with zero mean and covariance given by

$$E(\xi(\Psi)\xi(\Phi)) = \rho(1-\rho)\int dr\,\Psi(r)\Phi(r)$$

as conjectured by Spohn (1991).

Our other result has to do with the position of a tagged particle. Let X_t be the position of a tagged particle that at time zero was put at the origin. The joint process (η_t, X_t) is Markov and the process $\tau_{X_t}\eta_t$ has as extremal invariant measure $\nu'_\rho = \nu_\rho(.|\eta(0) = 1)$ (Ferrari (1986)). Under this distribution,

$$EX_t = (1-\rho)(p-q)t$$

Kipnis (1986) proved the following law of large numbers

$$\lim_{t\to\infty} \frac{X_t}{t} = (1-\rho)(p-q)$$

and central limit theorem:

$$\lim_{t\to\infty} \frac{X_t - (1-\rho)(p-q)t}{\sqrt{t}} = G(0, D_X),$$

in distribution. The variance D_X is given by

$$D_X = \lim_{t\to\infty} \frac{VX_t}{t} = \lim_{t\to\infty} \frac{E(X_t - (1-\rho)(p-q)t)^2}{t} = (1-\rho)(p-q).$$

The limit was computed by De Masi and Ferrari (1985). Ferrari and Fontes (1992b) show the following stronger result:

7. Assume that the initial distribution of ξ_t is given by ν'_ρ. Then

$$X_t = N_t + B_t$$

where N_t is a Poisson point process of parameter $(1-\rho)(p-q)$, and $|B_t| \le B$ stochastically, where B is a random variable on $\mathbb{N}$ with a finite exponential moment. That is there exists a positive θ such that

$$E\exp(\theta B) < \infty$$

In words, it is possible to represent the position of a tagged particle almost as a Poisson process. This can be understood as an extension of Burke's theorem in queueing theory. In the language of particle systems, Burke's theorem is just the statement above for $p = 1$ and $B_t \equiv 0$. This was observed by Kesten and quoted by Spitzer (1970). See also Kipnis (1986) and Ferrari (1992b).

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