SOME PROPERTIES OF QUASI STATIONARY DISTRIBUTIONS IN THE BIRTH AND DEATH CHAINS: A DYNAMICAL APPROACH

P.A. FERRARI
Instituto de Matemática e Estatistica
Universidade de São Paulo, São Paulo, Brasil.

S. MARTINEZ
Departamento de Ingeniería Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Chile, Santiago, Chile.

P. PICCO
Centre de Physique Théorique, C.N.R.S.
Luminy, Marseille, France.

ABSTRACT. We study the existence of non-trivial quasi-stationary distributions for birth and death chains by using a dynamical approach. We also furnish an elementary proof of the solidarity property.

1. Introduction

Consider an irreducible discrete Markov chain $(X(n))$ on $S^* \cup \{0\}$ where 0 is the only absorbing state and S^* is the set of transient states. Let ν be a probability distribution. Denote by

$$\nu^{(n)}(x) = \mathbb{P}_\nu(X(n) = x \| X(n) \neq 0) \tag{1.1}$$

the conditional probability that at time n the chain is at state x given that it has not been absorbed, starting with the initial distribution ν. A measure μ is called a Yaglom limit if for some probability measure ν we have: $\nu^{(n)}(x) \underset{n\to\infty}{\longrightarrow} \mu(x)$ for all $x \in S^*$.

Now assume that the transition probabilities $p(x,y) = \mathbb{P}(X(n+1) = y | X(n) = x)$ verify the following hypothesis:

$$p(0,0) = 1$$

$$P^* = (p(x,y) : x, y \in S^*) \text{ is irreducible}$$

$$\forall x \in S \text{ the set } \{y \in S : p(y,x) > 0\} \text{ is finite and non-empty}$$

Then it is easy to show that Yaglom limits μ verify the set of equations

$$\forall x \in S^*, \quad \mu(x) = \sum_{y \in S^*} \mu(y)(p(y,x) + p(y,0)\mu(x)) \tag{1.2}$$

or equivalently the row vector $\mu = (\mu(x) : x \in S^*)$ satisfies

$$\mu P^* = \gamma(\mu)\mu \text{ with } \gamma(\mu) = 1 - \sum_{x \in S^*} \mu(x) p(x, 0) \tag{1.3}$$

In general a quasi-stationary distribution (q.s.d.) is a measure μ which verifies (1.3). If μ is also a probability measure we call it a normalized quasi-stationary distribution (n.q.s.d.). Obviously the trivial measure $\mu \equiv 0$ is a q.s.d. It is easy to show that the irreducibility condition we have imposed on the Markov chain implies that for any non-trivial q.s.d., $\mu(x) > 0$ for all $x \in S^*$.

Some of the interesting problems of q.s.d. are concerned with the search for

· necessary and/or sufficient conditions on the transition matrices for the existence of non-trivial q.s.d.,
· domains of attractions of q.s.d.,
· evolution of $\delta_x^{(n)}$, δ_x being the Dirac distribution at point x.

For several kinds of Markov chains it has been proved that $\delta_x^{(n)}$ converges to a n.q.s.d. This was shown for branching process by Yaglom (1947), for finite state spaces by Darroch and Seneta (1965), for continuous time simple random walk on $\mathbb{N}$ by Seneta (1966) and for discrete time random walk on $\mathbb{N}$ by Seneta and Vere-Jones (1966).

For birth and death chains the existence of the limit of the sequence $\delta_x^{(n)}$ does not depend on x, and if the limit exists it is the same for all x. We provide in section 3 an elementary proof of this fact. Good (1968) gave a proof of this result based on some powerful results of Karlin and McGregor (1957); some technical details need additional explanations.

The problem of convergence of $\nu^{(n)}$ for ν other than Dirac distributions was initially considered by Seneta and Vere Jones (1966) for Markov chains with R-positive transition matrix.

For random walks it turns of that the Yaglom limit of $\delta_x^{(n)}$ is the minimal n.q.s.d. (this means $\gamma(\mu)$ is minimal). Then the study of the domains of attraction of non-minimal n.q.s.d. concerns the evolution $\nu^{(n)}$ for ν other than Dirac distributions. Recently we proved in [FMP] that the domains of attraction of non-minimal n.q.s.d. are non-trivial. More precisely we show that:

Theorem 1.1. Let μ, μ' be n.q.s.d. with $\gamma(\mu) > \gamma(\mu')$. Assume that ν satisfies:

$$\sup\{|\nu(x) - \mu(x)| \mu'(x)^{-1} : x \in S^*\} < \infty \text{ or } \nu = \eta\mu + (1-\eta)\mu' \text{ for } \eta \in (0, 1]$$

then $\nu^{(n)} \underset{n \to \infty}{\longrightarrow} \mu$. ■

Our main results deal with q.s.d. in birth and death chains. A first study concerning the description of the class of q.s.d.'s for birth and death process was made by Cavender (1978). Roughly, this class was characterized as an ordered one-parameter family and it was proved that any q.s.d. has total mass 0,1 or ∞.

2. Existence of Q.S.D. for Birth and Death Chains

2.1 GENERAL CONDITIONS FOR EXISTENCE

Consider a birth and death chain (X_n) on $\mathbb{N}$ with 0 as its unique absorbing state, so $p(0,0)=1$. Denote $q_x = p(x,x-1)$ and $p_x = p(x,x+1)$, so $p(x,x) = 1 - p_x - q_x$ for all $x \in \mathbb{N}^*$.

For a sequence $\mu = (\mu(x) : x \in \mathbb{N}^*)$ the equations (1.2) take the form,

$$\forall y \in \mathbb{N}^* : (p_y + q_y)\mu(y) = q_{y+1}\mu(y+1) + p_{y-1}\mu(y-1) + q_1\mu(1)\mu(y) \tag{2.1}$$

If $\mu(1) > 0$ we get $\sum_{y=1}^{x} \mu(y) = 1 - \frac{1}{\mu(1)q_1}(q_{x+1}\mu(x+1) - p_x\mu(x))$ so a non-trivial q.s.d. is normalized iff $\mu(x) \underset{x\to\infty}{\longrightarrow} 0$.

Now for $\gamma \neq p_1 + q_1$ define in a recursive way the following sequence $Z_\gamma = (Z_\gamma(x) : x \in \mathbb{N}^*)$,

$$Z_\gamma(1) = \gamma \tag{2.2}$$

$$\forall y \geq 2 : \quad Z_\gamma(y) = f_{\gamma,y}(Z_\gamma(y-1)) \tag{2.3}$$

where

$$f_{\gamma,y}(z) = \gamma + p_y + q_y - p_1 - q_1 - \frac{p_{y-1}q_y}{z} \tag{2.4}$$

Associate to Z_γ the following vector $\mu_{(\gamma)} = (\mu_{(\gamma)}(x) : x \in \mathbb{N}^*)$

$$\mu_{(\gamma)}(1) = \frac{1}{q_1}(p_1 + q_1 - \gamma) \tag{2.5}$$

$$\forall x \geq 2 : \quad \mu_{(\gamma)}(x) = \mu_{(\gamma)}(1) \prod_{y=1}^{x-1} \frac{Z_\gamma(y)}{q_{y+1}} \tag{2.6}$$

In [FMP] it was shown that a vector $\mu = (\mu(x) : x \in \mathbb{N}^*)$ with non-null terms verifies equations (2.1) iff there exists a $\gamma \neq p_1 + q_1$ such that $\mu = \mu_{(\gamma)}$.

In particular this last result implies that there exist non-trivial q.s.d. μ iff for some $\gamma < p_1 + q_1$ the sequence $Z_\gamma = (Z_\gamma(x) : x \in \mathbb{N}^*)$ is strictly positive. Then we search for conditions under which the orbit

$$Z_\gamma(y) = f_{\gamma,y} \circ f_{\gamma,y-1} \circ \cdots \circ f_{\gamma,2}(\gamma)$$

is strictly positive.

Assume for simplicity that $p_x + q_x = 1$ for all $x \in \mathbb{N}^*$ so the evolution functions $f_{\gamma,y}$ take the form,

$$f_{\gamma,y}(z) = \gamma - \frac{p_{y-1} q_y}{z} \tag{2.7}$$

Now make the following hypothesis: there exists a $\bar{q} \in (\frac{1}{2}, \frac{\sqrt{7}-1}{2})$ such that

$$\forall y \in \mathbb{N}^*, \quad \frac{1}{2} < \bar{q} - \frac{1}{2}\left(\bar{q} - \frac{1}{2}\right)^2 < q \le q_y \le q' < \bar{q} + \frac{1}{2}\left(\bar{q} - \frac{1}{2}\right)^2 < 1 \tag{2.8}$$

Denote $p = 1 - q$, $p' = 1 - q'$. Notice that if $\bar{q} = \frac{\sqrt{7}-1}{2}$ then $\bar{q} + \frac{1}{2}(\bar{q} - \frac{1}{2})^2 = 1$. The above condition (2.8) means that the birth and death chain is a perturbation of a random walk of parameter $\bar{q}$.

It can be shown that the hypothesis (2.8) implies the inequality

$$2\sqrt{pq'} < p' + q < 1$$

Call $g_\gamma(z) = \gamma - \frac{p'q}{z}$ and $h_\gamma(z) = \gamma - \frac{pq'}{z}$. It is easy to check that:

$$\forall y \in \mathbb{N}^*,\ z \ge 0: \quad h_\gamma(z) \le f_{\gamma,y}(z) \le g_\gamma(z) \tag{2.9}$$

Take $\gamma \in [2\sqrt{pq'}, 1)$, then $h_\gamma(z)$ has two fixed points (only one if $\gamma = 2\sqrt{pq'}$), a stable one $\xi = \frac{\gamma + \sqrt{\gamma^2 - 4pq'}}{2}$ and an unstable one $\eta = \frac{\gamma - \sqrt{\gamma^2 - 4pq'}}{2}$. Also $g_\gamma(z)$ has two fixed points, a stable one $\tilde{\xi} = \frac{\gamma + \sqrt{\gamma^2 - 4p'q}}{2}$ and an unstable one $\tilde{\eta} = \frac{\gamma - \sqrt{\gamma^2 - 4p'q}}{2}$.

Theorem 2.1. If condition (2.8) holds then there exist n.q.s.d. More precisely, if $\gamma \in [2\sqrt{pq'}, 1)$ then $\mu_{(\gamma)}$ is a non trivial q.s.d. and if $\gamma \in [2\sqrt{pq'}, p' + q]$ then $\mu_{(\gamma)}$ is a n.q.s.d.

Proof. Take $\gamma \in [2\sqrt{pq'}, 1)$. We have $Z_\gamma(1) = \gamma \ge \xi$. Therefore

$$Z_\gamma(y) = f_{\gamma,y} \circ \ldots \circ f_{\gamma,2}(Z_\gamma(1)) \ge h_\gamma^{(y-1)}(Z_\gamma(1)) \ge h_\gamma^{(y-1)}(\xi) = \xi > 0$$

Then $Z_\gamma(y) \ge \xi > 0$. Now $\gamma < 1$ implies $\mu_{(\gamma)}(1) > 0$ and expression (2.6) shows $\mu_{(\gamma)}(x) > 0$ for any $x \ge 2$, so $\mu_{(\gamma)}$ is a non trivial q.s.d.

Now let us prove that:

$$\forall y \in \mathbb{N}^*, \quad Z_\gamma(y) \le \tilde{\xi} + (\gamma - \tilde{\xi})\left(\frac{\tilde{\eta}}{\tilde{\xi}}\right)^{y-1} \tag{2.10}$$

Since $Z_\gamma(1) = \gamma$ the relation (2.10) holds for $y = 1$. Now we have $Z_\gamma(y) = f_{\gamma,y} \circ f_{\gamma,y-1} \circ \cdots \circ f_{\gamma,2}(\gamma) \le g_\gamma^{(y-1)}(\gamma)$ where $g_\gamma^{(x)} = g_\gamma \circ \cdots \circ g_\gamma$ x times. Since $g_\gamma^{(y-1)}(\tilde{\xi}) = \tilde{\xi}$ we get from Taylor formula,

$$g_\gamma^{(y-1)}(\gamma) \le \tilde\xi + (\gamma - \tilde\xi) \sup_{z\in[\tilde\xi,\gamma]} \left\{\frac{\partial}{\partial t} g_\gamma^{(y-1)}(z)\right\}$$

Now

$$\frac{\partial}{\partial z} g_\gamma^{(y-1)}(z) = \prod_{x=0}^{y-2} g'_\gamma(g_\gamma^{(x)}(z))$$

with $g_\gamma^{(0)}(z) = z$ and $g'_\gamma(z) = \frac{p'q}{z^2} = \frac{\tilde\xi\tilde\eta}{z^2}$.

Using the fact that g_γ is increasing and $\tilde\xi$ is a fixed point of g_γ we get easily that for all $0 \le x \le y-2$, and $z \in [\tilde\xi, \gamma]$, we have $g_\gamma^{(x)}(z) \ge \tilde\xi$. Therefore we get

$$\sup_{z\in[\tilde\xi,\gamma]} \left\{\frac{\partial}{\partial t} g^{(y-1)}(z)\right\} \le \left(\frac{\tilde\eta}{\tilde\xi}\right)^{y-1}$$

Then property (2.10) is fulfilled.

Recall that $q_y \ge q$. Use the bound (2.10) to get from (2.6),

$$\mu_{(\gamma)}(x) \le \mu_\gamma(1)(\prod_{y=1}^{x-1}(1 + \frac{\gamma - \tilde\xi}{\tilde\xi}(\frac{\tilde\eta}{\tilde\xi})^{y-1}))(\frac{\tilde\xi}{q})^{x-1}$$

Since $\sum_{y=1}^{\infty}(\frac{\tilde\eta}{\tilde\xi})^{y-1} < \infty$ we deduce that $C = \prod_{y=1}^{\infty}(1 + \frac{\gamma-\tilde\xi}{\tilde\xi}(\frac{\tilde\eta}{\tilde\xi})^{y-1}) < \infty$. So $\mu_{(\gamma)}(x) \le C\mu_\gamma(1)(\frac{\tilde\xi}{q})^{x-1}$.

Now assume $\gamma \in [2\sqrt{pq'}, p'+q]$. Since $pq' > p'q$ we get:

$$\tilde\xi = \frac{1}{2}(\gamma + \sqrt{\gamma^2 - 4pq'}) \le \frac{1}{2}((p'+q) + \sqrt{(p'+q)^2 - 4p'q}) < \frac{1}{2}((p'+q) + (q-p')) = q$$

Then $\frac{\tilde\xi}{q} < 1$ so $\mu_{(\gamma)}(x) \underset{x\to\infty}{\longrightarrow} 0$. Then $\mu_{(\gamma)}$ is a n.q.s.d. ∎

2.2 LINEAR GROWTH CHAINS WITH IMMIGRATION

These processes are birth and death chains with

$$p_y = \frac{py+1}{(p+q)y+1}, \quad q_y = \frac{qy}{(p+q)y+a} \quad \text{for } y \in \mathbb{N}^*, \tag{2.11}$$

(so $p_y + q_y = 1$) and an absorbing barrier at 0, $p(0,0) = 1$.

We assume conditions

$$p > q \quad \text{and} \quad a < p + q \tag{2.12}$$

It can be shown that these inequalities imply that the sequence of functions $(f_{\gamma,y} : y \in \mathbb{N}^*)$ defined in (2.7), is increasing with y. The pointwise limit of this sequence, when $y \to \infty$, is $f_{\gamma,\infty}(z) = \gamma - \frac{pq}{(p+q)^2 z}$. Then we have:

$$f_{\gamma,2} \leq \cdots \leq f_{\gamma,y} \leq f_{\gamma,y+1} \leq \cdots \leq f_{\gamma,\infty} \tag{2.13}$$

Observe that $f_{\gamma,2}$ plays the role of h_γ and $f_{\gamma,\infty}$ that of g_γ in (2.9).

Now, inequality $p_1 q_2 < 1$ is equivalent to

$$2(q-p)^2 + a(3p + a - 5q) > 0 \tag{2.14}$$

This condition is verified if q is big enough, for instance if $q > p + \frac{5}{4}a + \sqrt{a(p + \frac{17}{16}a)}$. We assume (2.14) holds.

Take $\gamma \in (2\sqrt{p_1 q_2}, 1)$ so $\xi = \frac{\gamma + \sqrt{\gamma^2 - 4p_1 q_2}}{2}$ belongs to the interval $(0, \gamma)$ and it is a fixed point of $f_{\gamma,2}$. Then $Z_\gamma(1) = \gamma$ and,

$$Z_\gamma(y) = f_{\gamma,y} \circ \cdots \circ f_{\gamma,2}(\gamma) > f_{\gamma,2}^{(y-1)}(\xi) = \xi > 0$$

Since $\gamma < 1$, from (2.5) and (2.6) we get $\mu_{(\gamma)}(y) > 0$ for any $y \in \mathbb{N}^*$. Hence $\mu_{(\gamma)}$ is a non trivial q.s.d.

Recall that $f_{\gamma,2} \leq f_{\gamma,\infty}$ is equivalent to $\frac{pq}{(p+q)^2} < p_1 q_2$. Take $\gamma \in (\frac{2\sqrt{pq}}{(p+q)}, 2\sqrt{p_1 q_2})$. Then the point $\tilde{\eta} = \frac{\gamma - \sqrt{\gamma^2 - \frac{4pq}{(p+q)^2}}}{2}$ and $\tilde{\xi} = \frac{\gamma + \sqrt{\gamma^2 - \frac{4pq}{(p+q)^2}}}{2}$ are respectively the unstable and the stable fixed points of $f_{\gamma,\infty}$. Replacing g_γ by $f_{\gamma,\infty}$ we get that condition (2.9) holds with $\tilde{\eta}$, $\tilde{\xi}$ the fixed points of $f_{\gamma,\infty}$. Then,

$$\mu_{(\gamma)}(x) \leq \mu_{(\gamma)}(1) \{ \prod_{y=1}^{x-1} (1 + \frac{\gamma - \tilde{\xi}}{\tilde{\xi}} (\frac{\tilde{\eta}}{\tilde{\xi}})^{y-1} \} \tilde{\xi}^{x-1} \prod_{y=1}^{x-1} \frac{1}{q_{y+1}} \tag{2.15}$$

Denote $C = \prod_{y=1}^{\infty} (1 + \frac{\gamma - \tilde{\xi}}{\tilde{\xi}} (\frac{\tilde{\eta}}{\tilde{\xi}})^{y-1})$ which is finite.

We have $\prod_{y=1}^{x-1} \frac{1}{q_{y+1}} = (\frac{p+q}{q})^{x-1} \prod_{y=1}^{x-1} (1 + \frac{a}{(p+q)(y+1)}) \leq (\frac{p+q}{q})^{x-1} \exp \{ \frac{a}{p+q} \sum_{y=1}^{x-1} \frac{1}{y+1} \}$. Then $\prod_{y=1}^{x-1} \frac{1}{q_{y+1}} \leq (\frac{p+q}{q})^{x-1} (x-1)^{\frac{a}{(p+q)}}$. Hence

$$\mu_{(\gamma)}(x) \leq \mu_{(\gamma)}(1) C (\tilde{\xi} \frac{(p+q)}{q})^{x-1} (x-1)^{\frac{a}{p+q}} \tag{2.16}$$

It can be easily verified that our assumptions imply that $\tilde{\xi} < \frac{q}{p+q}$. Then $\mu_{(\gamma)}(x) \underset{x\to\infty}{\longrightarrow} 0$. Then, for $\gamma \in (2\frac{\sqrt{pq}}{p+q}, 2\sqrt{p_1 q_2})$ the q.s.d. $\mu_{(\gamma)}$ is normalized.

3. Solidarity Property for Birth and Death Chains

Concerning the convergence of point measures to some Yaglom limit, the deepest results have been established in [S2,SV-J] for random walks ($q_x = q, p_x = 1-q$) with continuous and discrete time. Here we shall show a solidarity process which asserts that it suffices to have the convergence for the probability measure concentrated at 1. Our proof is elementary, in fact it does not use any higher technique. We must point out that Good [G] has also shown this result but in his proof some technical steps have been overlooked.

Theorem 3.1. If $\delta_1^{(n)}$ converges to a q.s.d. μ then for any $x \in \mathbb{N}^*$, $\delta_x^{(n)}$ converges to μ.

Proof. For $x, n \in \mathbb{N}^*$ set:

$$\alpha_x(n) = \frac{\mathbb{P}_{x+1}(X(n-1) \neq 0)}{\mathbb{P}_x(X(n) \neq 0)}, \quad \beta_x(n) = \frac{\mathbb{P}_x(X(n-1) \neq 0)}{\mathbb{P}_x(X(n) \neq 0)},$$
$$\xi_x(n) = \frac{\mathbb{P}_{x-1}(X(n-1) \neq 0)}{\mathbb{P}_x(X(n) \neq 0)}$$

Observe that $\xi_1(n) = 0$ for all $n \in \mathbb{N}^*$, all other terms being > 0.

These quantities are related by the identity

$$\forall x \in \mathbb{N}^*, \quad \xi_{x+1}(n) = \beta_x(n)\beta_{x+1}(n)(\alpha_x(n))^{-1} \tag{3.1}$$

On the other hand from the equation

$$\begin{aligned}\mathbb{P}_x(X(n) \neq 0) = {} & q_x \mathbb{P}_{x-1}(X(n-1) \neq 0) \\ & + (1 - p_x - q_x)\mathbb{P}_x(X(n-1) \neq 0) + p_x \mathbb{P}_{x+1}(X(n-1) \neq 0)\end{aligned} \tag{3.2}$$

we deduce that

$$\forall x, n \in \mathbb{N}^*, \quad q_x \xi_x(n) + (1 - p_x - q_x)\beta_x(n) + p_x \alpha_x(n) = 1 \tag{3.3}$$

Also from definition we get

$$\beta_x(n) = (\mathbb{P}_x(X(n) \neq 0 | X(n-1) \neq 0))^{-1} = (1 - \delta_x^{(n-1)}(1) q_1)^{-1} \tag{3.4}$$

If the limit of a sequence $\eta(n)$ exits denote it by $\eta(\infty)$. So the hypothesis of the theorem is: $\forall z \in \mathbb{N}^*$, $\delta_1^{(\infty)}(z)$ exists.

Since $\delta_1^{(\infty)}(1)$ exists and belongs to [0,1] we deduce from (3.4) that $\beta_1(\infty)$ exists and belongs to $[1, \frac{1}{1-q_1}]$. From $\xi_1(n) = 0$ and (3.3) we get that $\alpha_1(\infty)$ exists and is bigger or equal than $\frac{1}{p_1}(1 - \frac{(1-p_1-q_1)}{1-q_1}) = \frac{1}{1-q_1}$.

Now let us show that,

$$\forall x \in \mathbb{N}^*, \quad \liminf_{n\to\infty} \alpha_x(n) > 0 \tag{3.5}$$

This holds for $x = 1$. Now from (3.2) evaluated at $x + 2$ we deduce the inequality

$$\mathbb{P}_{x+1}(X(n-1) \neq 0) \leq q_{x+2}^{-1} \mathbb{P}_{x+2}(X(n) \neq 0)$$

On the other hand since $\mathbb{P}_y(X(n) \neq 0)$ increases with $y \in \mathbb{N}^*$ and decreases with $n \in \mathbb{N}^*$ we get the following relations

$$\mathbb{P}_x(X(n) \neq 0) \geq p_x \mathbb{P}_{x+1}(X(n-1) \neq 0)$$

$$\alpha_{x+1}(n) = \frac{\mathbb{P}_{x+2}(X(n-1) \neq 0)}{\mathbb{P}_{x+1}(X(n) \neq 0)} \geq \frac{\mathbb{P}_{x+2}(X(n) \neq 0)}{\mathbb{P}_{x+1}(X(n-1) \neq 0)}$$

Hence we obtain:

$$\begin{aligned} \alpha_x(n) = \frac{\mathbb{P}_{x+1}(X(n-1) \neq 0)}{\mathbb{P}_x(X(n) \neq 0)} &\leq (p_x q_{x+2})^{-1} \frac{\mathbb{P}_{x+2}(X(n) \neq 0)}{\mathbb{P}_{x+1}(X(n-1) \neq 0)} \\ &\leq (p_x q_{x+2})^{-1} \alpha_{x+1}(n) \end{aligned}$$

Then $\alpha_{x+1}(n) \geq p_x q_{x+2} \alpha_x(n)$. So $\liminf_{n\to\infty} \alpha_{x+1}(n) > 0$ and relation (3.5) holds.

Now let us prove by recurrence that:

$$\text{the limits } \alpha_x(\infty), \beta_x(\infty), \xi_x(\infty) \text{ and } \delta_x^{(\infty)}(z) \text{ for all } z \in \mathbb{N}^*, \text{ exist} \tag{3.6}$$

We show above that these limits exist for $x = 1$. Assuming that property (3.6) holds for $x \in \{1, ..., y\}$, we shall prove that it is also satisfied for $x = y + 1$. With this purpose in mind, condition on the first step of the chain to get,

$$\begin{aligned} \mathbb{P}_y(X(n) = z) = q_y \mathbb{P}_{y-1}(X(n-1) = z) + (1 - p_y - q_y) \mathbb{P}_y(X(n-1) = z) \\ + p_y \mathbb{P}_{y+1}(X(n-1) = z) \end{aligned} \tag{3.7}$$

Now from definitions of $\alpha_y(n)$, $\beta_y(n)$, $\xi_y(n)$ we have the following identities for $y \geq 2$:

$$\begin{aligned} \mathbb{P}_y(X(n) \neq 0) &= \alpha_y(n)(\mathbb{P}_{y+1}(X(n-1) \neq 0))^{-1} = \beta_y(n)(\mathbb{P}_y(X(n-1) \neq 0))^{-1} \\ &= \xi_y(n)(\mathbb{P}_y(X(n-1) \neq 0))^{-1} \end{aligned}$$

Develop $\delta_y^{(n)}(z) = \mathbb{P}_y(X(n) = z)(\mathbb{P}_y(X(n) \neq 0))^{-1}$ according to (3.7) and the last equalities to get:

$$\delta_y^{(n)}(z) = \delta_{y-1}^{(n-1)}(z) q_y \xi_y(n) + \delta_y^{(n-1)}(z)(1 - p_y - q_y)\beta_y(n) + \delta_{y+1}^{(n-1)}(z) p_y \alpha_y(n) \qquad (3.8)$$

This last equality holds for any $y \geq 1$ (recall $\xi_1(n) = 0$).

Since $\delta^{(\infty)}(z), \xi_x(\infty), \beta_x(\infty), \alpha_x(\infty)$ exist for any $x \leq y$ and $z \in \mathbb{N}^*$, and, by (3.5), $\alpha_x(\infty) > 0$ we get that $\delta_{y+1}^{(\infty)}(z)$ exists for any $z \in \mathbb{N}^*$. On the other hand equality (3.4) implies that $\beta_{y+1}(\infty)$ exists. Then by (3.1) the limit $\xi_{y+1}(\infty)$ exists and equation (3.3) implies the existence of $\alpha_{y+1}(\infty)$.

From (3.5) and (3.4) we deduce $\alpha_x(\infty) > 0$ and $\beta_x(\infty) > 0$ for any $x \in \mathbb{N}^*$. So (3.1) implies $\xi_x(\infty) > 0$ for $x \geq 2$.

Then if $\delta_y^{(\infty)}(z) = 0$ for some $y, z \in \mathbb{N}^*$ we can deduce from equality (3.8) that $\delta_x^{(\infty)}(z) = 0$ for all $x \in \mathbb{N}^*$. So the q.s.d.'s which are the limits of $\delta_x^{(n)}$ are all trivial or normalized.

Assume that $\delta_1^{(n)}$ converges to a normalized q.s.d. μ. Let us prove by recurrence that $\delta_x^{(n)}$ converges to μ for all x.

Since the limits $\delta_y^{(\infty)}$ exists and $\delta_1^{(\infty)}(z) > 0$, when we evaluate (3.8) at $y = 1$, $n = \infty$ we get the following equation:

$$1 = (1 - p_1 - q_1)\beta_1(\infty) + \left(\frac{\delta_2^{(\infty)}(z)}{\delta_1^{(\infty)}(z)}\right) p_1 \alpha_1(\infty)$$

Comparing this equation with (3.3) evaluated at $x = 1$, $n = \infty$, and by taking into account that $\xi_1(\infty) = 0$ we deduce $\delta_2^{(\infty)}(z) = \delta_1^{(\infty)}(z)$ for any $z \in \mathbb{N}^*$.

Assume we have shown for any $y \in \{1, ..., y_0\}$ that: $\forall z \in \mathbb{N}^*$, $\delta_y^{(\infty)}(z) = \delta_1^{(\infty)}(z)$. Let us show that this last set of equalities also hold for $y_0 + 1$. Evaluate equation (3.8) at $y = y_0$, $n = \infty$ to get

$$1 = q_{y_0} \xi_{y_0}(\infty) + (1 - p_{y_0} - q_{y_0})\beta_{y_0}(\infty) + \left(\frac{\delta_{y_0+1}^{(\infty)}(z)}{\delta_{y_0}^{(\infty)}(z)}\right) p_{y_0} \alpha_{y_0}(\infty)$$

Comparing this equation with (3.3) evaluated at $x = y_0$, $n = \infty$ we deduce that $\delta_{y_0+1}^{(\infty)}(z) = \delta_{y_0}^{(\infty)}(z)$. Hence the recurrence follows and for any $x \in \mathbb{N}^*$, $\delta_x^{(n)}$ converges to $\mu = \delta_1^{(\infty)}$. ■

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