Algebraic winding numbers

Daniel Perrucci[♭*] Marie-Françoise Roy[♯]

♭ Departamento de Matemática, FCEN, Universidad de Buenos Aires and IMAS UBA-CONICET, Buenos Aires, Argentina,
♯ IRMAR (UMR CNRS 6625), Université de Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France.

May 15, 2023

Abstract

In this paper we study in detail the properties of the algebraic winding number proposed in [3] with respect to complex root counting in rectangles. We also propose a new algebraic winding number which computes the number of complex roots of a polynomial in a rectangle under no assumptions, including roots on edges or vertices with appropriate counting. We extend both winding numbers to rational functions, obtaining then an algebraic version of the argument principle for rectangles.

Keywords: Root counting, Cauchy index, Winding number, Argument principle.

MSC2020: 12D10, 13J30, 14Q20.

1 Introduction

1.1 Real rational functions: Sign and Cauchy index

Let $\mathbb{N} = \{0, 1, 2, \dots, \}$. Let $\mathbf{R}$ be a real closed field and $x \in \mathbf{R}$. We consider the sign of x as usual as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Given $P, Q \in \mathbf{R}[X] \setminus \{0\}$ they can be written uniquely as

$$P = (X - x)^{\operatorname{mult}_x(P)}\, P_x, \qquad Q = (X - x)^{\operatorname{mult}_x(Q)}\, Q_x,$$

[*]Partially supported by the grants UBACYT 20020190100116BA, PIP 11220200101015CO CONICET and PICT 2018-02315.

with $\text{mult}_x(P), \text{mult}_x(Q) \in \mathbb{N}$, $P_x, Q_x \in \mathbf{R}[X]$ and $P_x(x) \neq 0, Q_x(x) \neq 0$. We consider the valuation defined by x on $\mathbf{R}(X)$ by

$$\text{val}_x(P/Q) = \begin{cases} \text{mult}_x(P) - \text{mult}_x(Q) & \text{if } P \neq 0, \\ +\infty & \text{if } P = 0. \end{cases}$$

Let $P, Q \in \mathbf{R}[X]$. We define the Sign of (P, Q) at x, which is the sign of the rational function P/Q at x whenever this makes sense, and 0 otherwise. The reason why we consider pairs of polynomials instead of rational functions is because in the following sections, the case $Q = 0$ will also be of use.

Definition 1 *For $P, Q \in \mathbf{R}[X]$ and $x \in \mathbf{R}$,*

$$\text{Sign}(P, Q, x) := \begin{cases} \text{sign}\left(P_x(x)Q_x(x)\right) & \textit{if } P \neq 0, Q \neq 0 \textit{ and } \text{val}_x(P/Q) = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

It can be easily checked that whenever $P(x)$ and $Q(x)$ are not simultaneously 0, $\text{Sign}(P, Q, x) = \text{sign}(P(x)Q(x))$.

We also consider the sign variation defined as follows.

Definition 2 *For $P, Q \in \mathbf{R}[X]$ and $x \in \mathbf{R}$,*

$$\text{Var}_x(P, Q) := \frac{1}{2} - \frac{1}{2}\text{Sign}(P, Q, x).$$

Also, for $a, b \in \mathbf{R}$,

$$\text{Var}_a^b(P, Q) := \text{Var}_a(P, Q) - \text{Var}_b(P, Q) = -\frac{1}{2}\text{Sign}(P, Q, a) + \frac{1}{2}\text{Sign}(P, Q, b).$$

Note that $\text{Var}_x(P, Q)$ coincides with the usual notion of sign variation whenever $P(x)$ and $Q(x)$ are not simultaneously 0, since in this case:

$$\text{Var}_x(P, Q) = \begin{cases} 0 & \text{if } P(x)Q(x) > 0, \\ \frac{1}{2} & \text{if } P(x)Q(x) = 0, \\ 1 & \text{if } P(x)Q(x) < 0. \end{cases}$$

We define the Cauchy index of (P, Q) first at a point and then on an interval, following [3]. As before, we need to consider pairs of polynomials instead of rational functions in order to not to exclude the case $Q = 0$.

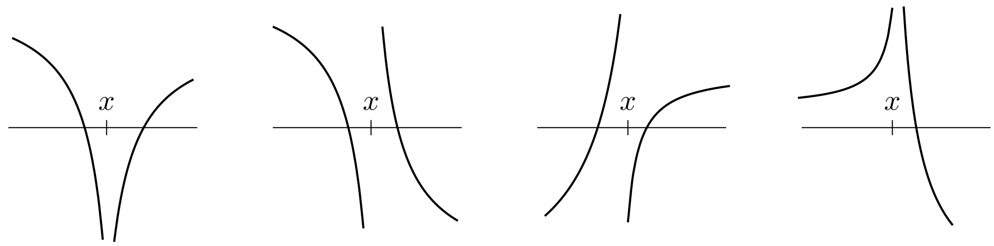
Definition 3 *For $P, Q \in \mathbf{R}[X]$, $x \in \mathbf{R}$ and $\varepsilon \in \{+, -\}$,*

$$\text{Ind}_x^{\varepsilon}(P, Q) := \begin{cases} \frac{1}{2}\text{sign}\left(P_x(x)Q_x(x)\right) & \textit{if } P \neq 0, Q \neq 0, \varepsilon = + \textit{ and } \text{val}_x(P/Q) < 0, \\ \frac{1}{2}(-1)^{\text{val}_x(P/Q)}\text{sign}\left(P_x(x)Q_x(x)\right) & \textit{if } P \neq 0, Q \neq 0, \varepsilon = - \textit{ and } \text{val}_x(P/Q) < 0, \\ 0 & \textit{otherwise;} \end{cases}$$

and

$$\text{Ind}_x(P, Q) := \text{Ind}_x^+(P, Q) - \text{Ind}_x^-(P, Q).$$

It is easy to see that, if $\mathrm{val}_x(P/Q) < 0$, $\mathrm{Ind}_x^+(P,Q)$ is half the sign of P/Q at a sufficiently small interval to the right of x and $\mathrm{Ind}_x^-(P,Q)$ is half the sign of P/Q at a sufficiently small interval to the left of x. We illustrate the definition of Cauchy index at a point considering the graph of the rational function P/Q around x in different cases.



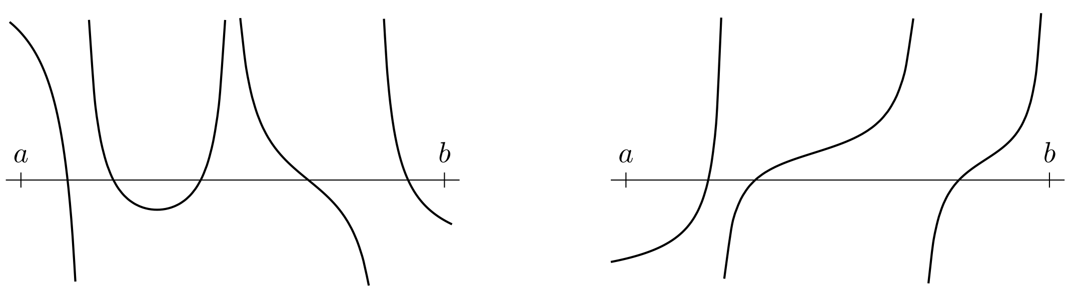
$\mathrm{Ind}_x(P,Q) = 0$ $\quad$ $\mathrm{Ind}_x(P,Q) = 1$ $\quad$ $\mathrm{Ind}_x(P,Q) = -1$ $\quad$ $\mathrm{Ind}_x(P,Q) = 0$

Definition 4 *For $P, Q \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$,*

$$\mathrm{Ind}_a^b(P,Q) := \begin{cases} \mathrm{Ind}_a^+(P,Q) + \sum_{x \in (a,b)} \mathrm{Ind}_x(P,Q) - \mathrm{Ind}_b^-(P,Q) & \text{if } a < b, \\ -\mathrm{Ind}_b^a(P,Q) & \text{if } b < a, \\ 0 & \text{if } a = b. \end{cases}$$

Note that the sum is well-defined since for any P and Q we have $\mathrm{Ind}_x(P,Q) \neq 0$ only for a finite number of $x \in \mathbf{R}$.

In the following picture we consider again the graph of the function P/Q, this time in $[a,b]$.



$\mathrm{Ind}_a^b(P,Q) = 1 + 0 + 1 = 2$ $\quad$ $\mathrm{Ind}_a^b(P,Q) = -1 - 1 - \frac{1}{2} = -\frac{5}{2}$

If P and Q are not simultaneously 0, then any common factor of P and Q can be simplified in both P and Q without changing the value of $\mathrm{Sign}(P,Q,x)$, $\mathrm{Var}_a^b(P,Q)$ or $\mathrm{Ind}_a^b(P,Q)$ for any x, a, b in $\mathbf{R}$.

The algorithmic symbolic computation of the Cauchy index defined as in Definition 4 can be done using Sturm sequences as in [3, Section 3] or subresultant polynomials as in [4]. Classicaly, the Cauchy index is definded on intervals under the assumption that P and Q do not vanish at the endpoints and can be computed by various symbolic methods (see [2, Chapter 9]).

Before going on, we recall the inversion formula (see [3, Theorem 3.9]), which relates the Cauchy Index with the sign variation on an interval.

Theorem 5 *Let $P, Q \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$. Then*

$$\mathrm{Ind}_a^b(P,Q) + \mathrm{Ind}_a^b(Q,P) = \mathrm{Var}_a^b(P,Q).$$

1.2 Complex rational functions: winding number on rectangles

Let $\mathbf{C} = \mathbf{R}[i]$ be the algebraic closure of $\mathbf{R}$.

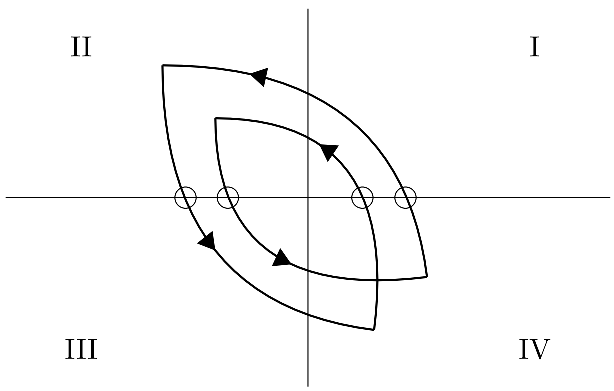
Notation 6 *For $F \in \mathbf{C}[X,Y]$, we denote F_{re} and F_{im} the real and imaginary parts of F, i.e. the unique polynomials in $\mathbf{R}[X,Y]$ such that the identity*

$$F(X,Y) = F_{\mathrm{re}}(X,Y) + iF_{\mathrm{im}}(X,Y)$$

in $\mathbf{C}[X,Y]$ holds.

Definition 7 *(see [3, Definition 4.2]) Let $F \in \mathbf{C}[X,Y]$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ with $x_0 < x_1$ and $y_0 < y_1$, and $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. We consider the following* winding number w *of F on $\partial\Gamma$:*

$$\begin{aligned} \mathrm{w}(F \,|\, \partial\Gamma) \quad := \quad & \tfrac{1}{2}\Big(\mathrm{Ind}_{x_0}^{x_1}(F_{\mathrm{re}}(T,y_0), F_{\mathrm{im}}(T,y_0)) + \mathrm{Ind}_{y_0}^{y_1}(F_{\mathrm{re}}(x_1,T), F_{\mathrm{im}}(x_1,T)) \\ & + \mathrm{Ind}_{x_1}^{x_0}(F_{\mathrm{re}}(T,y_1), F_{\mathrm{im}}(T,y_1)) + \mathrm{Ind}_{y_1}^{y_0}(F_{\mathrm{re}}(x_0,T), F_{\mathrm{im}}(x_0,T))\Big). \end{aligned}$$



$\mathrm{w}(F \,|\, \partial\Gamma) = 2$

The idea behind this definition is, if we go through the curve $F \circ \partial\Gamma$ following the counterclock sense, to count one half of a turn each time this curve crosses the X-axis from quadrant IV to I or from quadrant II to III, and minus one half of a turn each time it crosses the X-axis from quadrant I to IV or from quadrant III to II. Since these crossings coincide with jumps of the rational function $F_{\mathrm{re}}/F_{\mathrm{im}}$ from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ respectively, the Cauchy index is an appropriate algebraic tool to count the number of turns counterclockwise, which is (when F does not vanish on $\partial\Gamma$) the classical definition of the winding number.

We consider a new variable Z together with the inclusion $\mathbf{C}[Z] \subset \mathbf{C}[X,Y]$ through the identity $Z = X + iY$.

Example 8 *Let $F = Z$ and $\Gamma = [0,1] \times [0,1]$, then*

$$\mathrm{w}(F \,|\, \partial\Gamma) = \frac{1}{2}\Big(\mathrm{Ind}_0^1(T,0) + \mathrm{Ind}_0^1(1,T) + \mathrm{Ind}_1^0(T,1) + \mathrm{Ind}_1^0(0,T)\Big) = \frac{1}{4}.$$

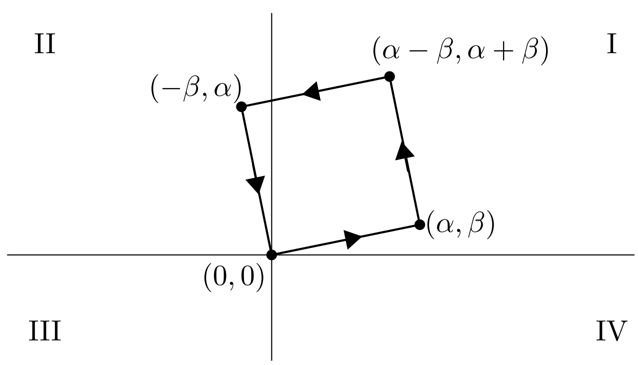
This basic example illustrates why we consider the Cauchy index of a pair of polynomials (P,Q) rather than the Cauchy index of a rational function P/Q: otherwise the Cauchy index $\mathrm{Ind}_0^1(T,0)$ would not be well defined.

Let $\Gamma \subset \mathbf{R}^2$ be a rectangle. For any constant $\gamma \in \mathbf{C}$ we have $\mathrm{w}(\gamma \,|\, \partial\Gamma) = 0$. For a linear polynomial $F(Z) = Z - z_0$ with $z_0 \in \mathbf{C}$ we have (see [3, Proposition 4.4]):

$$\mathrm{w}(F \,|\, \partial\Gamma) = \begin{cases} 1 & \text{if } z_0 \text{ is in the interior of } \Gamma, \\ \frac{1}{2} & \text{if } z_0 \text{ is in one of the edges of } \Gamma, \\ \frac{1}{4} & \text{if } z_0 \text{ is in one of the vertices of } \Gamma, \\ 0 & \text{if } z_0 \text{ is in the exterior of } \Gamma. \end{cases}$$

This may lead to the idea that the winding number w could also work in general in $\mathbf{C}[Z] \setminus \{0\}$ for counting one quarter for roots and poles on vertices. Indeed, this would be truth if only w were additive with respect to multiplication in $\mathbf{C}[Z] \setminus \{0\}$. Unfortunately this is not the general case as shown by the following simple example.

Example 9 *Let $\Gamma = [0,1] \times [0,1] \subset \mathbf{R}^2$, $\alpha, \beta \in \mathbf{R}$ with $0 < \beta < \alpha$ and $\gamma = \alpha + i\beta \in \mathbf{C}$. Then $\mathrm{w}(\gamma \,|\, \partial\Gamma) = 0$, $\mathrm{w}(Z \,|\, \partial\Gamma) = \frac{1}{4}$ but $\mathrm{w}(\gamma Z \,|\, \partial\Gamma) = 0$ instead of $\frac{1}{4}$.*



Note that in the above example, even though the curve does not cross the X-axis, it does cross the Y-axis. This motivates our definition of a new winding number W.

Definition 10 *Let $F \in \mathbf{C}[X,Y]$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ with $x_0 < x_1$ and $y_0 < y_1$, and $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. We define the following* winding number W *of F on $\partial\Gamma$:*

$$\mathrm{W}(F \,|\, \partial\Gamma) \;:=\; \frac{1}{2}\Big(\mathrm{w}(F \,|\, \partial\Gamma) + \mathrm{w}(iF \,|\, \partial\Gamma)\Big).$$

As will be seen later (Proposition 24), the new definition of W is indeed additive with respect to multiplication in $\mathbf{C}[Z] \setminus \{0\}$ and will also be good to deal with the case where F does vanish on edges or vertices of Γ (Theorem 12).

For $F \in \mathbf{C}[X,Y]$, we denote $\overline{F} \in \mathbf{C}[X,Y]$ the conjugate polynomial

$$\overline{F}(X,Y) = F_{\mathrm{re}}(X,Y) - iF_{\mathrm{im}}(X,Y).$$

Now consider $F, G \in \mathbf{C}[X,Y]$ with $G \neq 0$. Then $G_{\mathrm{re}}^2 + G_{\mathrm{im}}^2 \neq 0 \in \mathbf{R}[X,Y]$ and in $\mathbf{C}(X,Y)$ we have the identity

$$\frac{F}{G} = \frac{F\overline{G}}{G\overline{G}} = \underbrace{\frac{(F\overline{G})_{\mathrm{re}}}{G_{\mathrm{re}}^2 + G_{\mathrm{im}}^2}}_{\in \mathbf{R}(X,Y)} + i\,\underbrace{\frac{(F\overline{G})_{\mathrm{im}}}{G_{\mathrm{re}}^2 + G_{\mathrm{im}}^2}}_{\in \mathbf{R}(X,Y)},$$

which defines the real and imaginary part of F/G. Note that the quotient in $\mathbf{R}(X,Y)$ between the real and imaginary parts of $F/G \in \mathbf{C}(X,Y)$ is the same than the quotient in $\mathbf{R}(X,Y)$ of the real and imaginary part of $F\overline{G} \in \mathbf{C}[X,Y]$. This motivates the following definition.

Definition 11 *Let $F/G \in \mathbf{C}(X,Y)$, $x_0, x_1, y_0, y_1 \in \mathbf{R}$ with $x_0 < x_1$ and $y_0 < y_1$, and $\Gamma := [x_0, x_1] \times [y_0, y_1] \subset \mathbf{R}^2$. We define the following* winding numbers *of F/G on $\partial\Gamma$:*

$$\mathrm{w}(F/G \,|\, \partial\Gamma) := \mathrm{w}(F\overline{G} \,|\, \partial\Gamma),$$

$$\mathrm{W}(F/G \,|\, \partial\Gamma) := \mathrm{W}(F\overline{G} \,|\, \partial\Gamma).$$

Definition 11 extends the definitions of w and W made initially for $\mathbf{C}[X,Y]$ to $\mathbf{C}(X,Y)$. From Definitions 7 and 11 we obtain

$$\begin{aligned}\mathrm{w}(F/G \,|\, \partial\Gamma) = \tfrac{1}{2}\Big(&\mathrm{Ind}_{x_0}^{x_1}((F\overline{G})_{\mathrm{re}}(T,y_0), (F\overline{G})_{\mathrm{im}}(T,y_0)) + \mathrm{Ind}_{y_0}^{y_1}((F\overline{G})_{\mathrm{re}}(x_1,T), (F\overline{G})_{\mathrm{im}}(x_1,T)) \\ &+\mathrm{Ind}_{x_1}^{x_0}((F\overline{G})_{\mathrm{re}}(T,y_1), (F\overline{G})_{\mathrm{im}}(T,y_1)) + \mathrm{Ind}_{y_1}^{y_0}((F\overline{G})_{\mathrm{re}}(x_0,T), (F\overline{G})_{\mathrm{im}}(x_0,T))\Big).\end{aligned}$$

It can be easily checked that w and W are well defined for $F/G \in \mathbf{C}(X,Y)$, in the sense that they do not depend on the election of the numerator F and the denominator G in $\mathbf{C}[X,Y]$.

1.3 Goals of this paper

The classical argument principle applied to a rational function $F/G \in \mathbf{C}(Z)$ on a rectangle Γ, states that, as long as F/G has no zeros or poles on $\partial\Gamma$, the winding number of the curve $(F/G) \circ \partial\Gamma$, defined analytically as

$$\frac{1}{2\pi i} \int_{\partial\Gamma} \frac{(F/G)'(z)}{(F/G)(z)} dz$$

counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G inside Γ (see [1, Chapter 4, Section 5.2]).

This paper has two main goals (Theorem 12 and Theorem 13) which can be considered as algebraic versions of the argument principle on rectangles, and work even if the rational function have zeros or poles on $\partial\Gamma$.

Theorem 12 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle. Then $\mathrm{W}(F/G \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G in Γ. Zeros and poles on the edges count for one half. Zeros and poles on the vertices count for one quarter.*

Before stating the next result, we extend the valuation we considered before to $\mathbf{C}(Z)$.

Given $F, G \in \mathbf{C}[Z] \setminus \{0\}$ they can be written uniquely as

$$F = (Z - z)^{\mathrm{mult}_z(F)}\, F_z, \qquad G = (Z - z)^{\mathrm{mult}_z(F)}\, G_z,$$

with $\mathrm{mult}_z(F), \mathrm{mult}_z(G) \in \mathbb{N}$, $F_z, G_z \in \mathbf{C}[Z]$ and $F_z(z) \neq 0, G_z(z) \neq 0$. We consider the valuation defined by z on $\mathbf{C}(Z)$ by

$$\mathrm{val}_z(F/G) = \begin{cases} \mathrm{mult}_z(F) - \mathrm{mult}_z(G) & \text{if } F \neq 0, \\ +\infty & \text{if } F = 0. \end{cases}$$

Theorem 13 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G has even valuation at the vertices of Γ. Then $\mathrm{w}(F/G \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G in Γ. Zeros and poles on the edges count for one half. Zeros and poles on the vertices count for one quarter.*

As a corollary we obtain the following result.

Theorem 14 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G does not have zeros or poles on the vertices of Γ. Then $\mathrm{w}(F/G \,|\, \partial\Gamma)$ counts the number of zeros (with multiplicity) minus the number of poles (with order) of F/G in Γ. Zeros and poles on the edges count for one half.*

The statement of Theorem 14 in the case of polynomials (instead of rational functions) comes from [3, Theorem 5.1]. The proof proposed by [3] had a flaw related to the fact that the auxiliary product formula (1) we discuss in Section 2 (see [3, Theorem 4.5]) does not hold in full generality (see Example 15 below). For a polynomial F which does not vanish on the boundary of Γ, the proof of Theorem 14 has already been fixed (see details in [5, Lemma 20 and Theorem 39]); but to our best knowledge, Theorem 14 for a polynomial F with roots on the edges of Γ, was not proved yet.

The rest of the paper is organized as follows. In Section 2 we characterize all cases where the auxiliary product formula (1) is correct and provide auxiliary product formulas for the remaining cases. In Section 3 we study the additivity of the winding numbers w and W with respect to multipliaction in $\mathbf{C}(Z) \setminus \{0\}$. Finally, in Section 4, we prove Theorem 12 and Theorem 13.

2 Auxiliary product formula for univariate polynomials

Let $F, H \in \mathbf{C}[Z] \setminus \{0\}$ and let $\Gamma \subset \mathbf{R}^2$ be a rectangle. The strategy developed in [3] to relate $\mathrm{w}(FH \,|\, \partial\Gamma)$ with $\mathrm{w}(F \,|\, \partial\Gamma) + \mathrm{w}(H \,|\, \partial\Gamma)$ is to consider the polynomials P, Q, R, S obtained when restricting F_{re}, F_{im}, H_{re}, H_{im} to each of the four sides of Γ, and then relating $\mathrm{Ind}_a^b(PR - QS, PS + QR)$ with $\mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S)$, where $a < b$ are the endpoints of the interval parametrizing the side of Γ under consideration.

For any $P, Q, R, S \in \mathbf{R}[X]$ and $a, b \in \mathbf{R}$ with $a < b$, we consider the following auxiliary product formula coming from [3, Theorem 4.5]:

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) = \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(PS + QR, QS). \tag{1}$$

The auxiliary product formula (1) does not hold with full generality, as can be seen in the following example appearing already in [5].

Example 15 *Let* $a = 0$*,* $b = 1$*,* $P = 1$*,* $Q = X$*,* $R = X - 1$*,* $S = X$*. Then* $PR - QS = -X^2 + X - 1$*,* $PS + QR = X^2$ *and*

$$\mathrm{Ind}_0^1(PR - QS, PS + QR) = -\frac{1}{2}, \quad \mathrm{Ind}_0^1(P, Q) = \frac{1}{2}, \quad \mathrm{Ind}_0^1(R, S) = -\frac{1}{2}, \quad \mathrm{Var}_0^1(PS + QR, QS) = 0,$$

so the auxiliary product formula (1) does not hold. Note that for $F = iZ + 1$ *and* $H = (1 + i)Z - 1$*,* P, Q, R, S *are the restrictions of* F_{re}*,* F_{im}*,* H_{re}*,* H_{im} *to the bottom side of* $\Gamma = [0, 1] \times [0, 1]$*.*

We introduce the following definition.

Definition 16 *Let* $P, Q, R, S \in \mathbf{R}[X]$ *and* $c \in \mathbf{R}$*. We say that* $c \in \mathbf{R}$ *is a* bad number *for* P, Q, R, S*, if* $Q, S \neq 0$ *and* c *satisfies the following two conditions:*

- $\mathrm{val}_c(P/Q) = \mathrm{val}_c(R/S) < 0$,
- $\mathrm{val}_c((PS + QR)/QS) = 0$.

If P, Q, R, S *are clear from the context, we simply say that* c *is a bad number.*

To illustrate this definition, 0 is a bad number and 1 is not a bad number in Example 15. In general, if c is a bad number, then $Q, S \neq 0$ by definition but also necessarily $P, R, PS + QR \neq 0$. Note also that since bad numbers are necessarily roots of Q and S, there is at most a finite number of bad numbers in $\mathbf{R}$.

We now state two results. The first one proves that the only cases where it is necessary to modify the auxiliary product formula is at bad numbers.

Proposition 17 *Let* $P, Q, R, S \in \mathbf{R}[X]$ *with* P *and* Q *not simultaneously* 0 *and* R *and* S *not simultaneously* 0*, and* $a, b \in \mathbf{R}$ *with* $a < b$*. If* a *and* b *are not bad numbers, the auxiliary product formula (1)*

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) = \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \mathrm{Var}_a^b(PS + QR, QS)$$

holds.

The second result studies the situation when a or b are bad numbers.

Proposition 18 *Let* $P, Q, R, S \in \mathbf{R}[X]$ *with* P *and* Q *not simultaneously* 0 *and* R *and* S *not simultaneously* 0*, and* $a, b \in \mathbf{R}$ *with* $a < b$*.*

i) If a *is a bad number and* b *is not a bad number,*

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) \;=\; \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) - \frac{1}{2}\mathrm{Sign}(PS + QR, QS, b).$$

ii) If b *is a bad number and* a *is not a bad number,*

$$\mathrm{Ind}_a^b(PR - QS, PS + QR) \;=\; \mathrm{Ind}_a^b(P, Q) + \mathrm{Ind}_a^b(R, S) + \frac{1}{2}\mathrm{Sign}(PS + QR, QS, a).$$

iii) If a and b are both bad numbers,

$$\mathrm{Ind}_a^b(PR-QS,PS+QR) \ = \mathrm{Ind}_a^b(P,Q)+\mathrm{Ind}_a^b(R,S).$$

It can be checked that in Example 15, the equation of item i) holds instead of equation (1).

The proof of Proposition 17 uses the following two auxiliary lemmas. The proof of these lemmas appears already in [3] as part of the proof of [3, Theroem 4.5] (also as part of the proof of [5, Lemma 20]), but we include these proofs here for completeness.

Lemma 19 *Let $P,Q,R,S \in \mathbf{R}[X]$ and $a,b \in \mathbf{R}$ with $a<b$. If $P=0$ or $Q=0$ but P and Q are not simultaneously 0, the auxiliary product formula (1) holds. Similarly, if $R=0$ or $S=0$ but R and S are not simultaneously 0 the auxiliary product formula (1) holds.*

Proof: If $P=0$ and $Q\neq 0$, using the inversion formula (Theorem (5)) with R and S we have

$$\begin{aligned}\mathrm{Ind}_a^b(PR-QS,PS+QR) &= \mathrm{Ind}_a^b(-S,R)\\ &= \mathrm{Ind}_a^b(R,S)-\mathrm{Var}_a^b(R,S)\\ &= \mathrm{Ind}_a^b(P,Q)+\mathrm{Ind}_a^b(R,S)-\mathrm{Var}_a^b(PS+QR,QS).\end{aligned}$$

On the other hand, if $P\neq 0$ and $Q=0$,

$$\mathrm{Ind}_a^b(PR-QS,PS+QR)=\mathrm{Ind}_a^b(R,S)=\mathrm{Ind}_a^b(P,Q)+\mathrm{Ind}_a^b(R,S)-\mathrm{Var}_a^b(PS+QR,QS).$$

□

Lemma 20 *Let $P,Q,R,S \in \mathbf{R}[X]$ and $a,b\in\mathbf{R}$ with $a<b$. If $Q,S\neq 0$ and $PS+QR=0$, the auxiliary product formula (1) holds.*

Proof: Since $Q\neq 0$, $S\neq 0$ and $PS+QR=0$, we have $P/Q=-R/S\in\mathbf{R}(X)$ and then

$$\mathrm{Ind}_a^b(PR-QS,PS+QR)=0=\mathrm{Ind}_a^b(P,Q)+\mathrm{Ind}_a^b(R,S)-\mathrm{Var}_a^b(PS+QR,QS).$$

□

We are ready for the proof of Proposition 17. Again, part of this proof appears already in the proof of [3, Theorem 4.5] (also in the proof of [5, Lemma 20]). The new part is essentially Case 3.

Proof of Proposition 17 : If at least one of the polynomials P,Q,R,S or $PQ+RS$ is 0, identity (1) holds by Lemmas 19 and 20. So in the rest of the proof we assume that none of these polynomials is 0. We divide P and Q by $\gcd(P,Q)$ and R and S by $\gcd(R,S)$, so without loss of generality we also assume that P and Q are coprime and R and S are coprime.

We divide the interval $[a,b]$ in finitely many subintervals $[a',b']$ and it is enough to prove that identity (1) holds in each of these subintervals. We consider all the roots of P,Q,R,S or $PS+QR$ in $[a,b]$ (possibly none), this includes all bad numbers in $[a,b]$ (again, possibly none). We divide $[a,b]$ in as many subintervals as needed in such a way that each subinterval contains at most one of these roots and additionally:

- if the root is not a bad number, then it is an endpoint of the subinterval,
- if the root is a bad number, then it is an interior point of the subinterval.

This is possible because a and b are not bad numbers. We consider then several cases as follows.

Case 1: There is no root of Q, S or $PS+QR$ in $[a', b']$, then

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = \mathrm{Ind}_{a'}^{b'}(P,Q) = \mathrm{Ind}_{a'}^{b'}(R,S) = 0$$

and

$$\mathrm{Sign}(PS+QR, QS, a') = \mathrm{Sign}(PS+QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2: There is one root of Q, S or $PS+QR$ in $[a', b']$ which is not a bad number, and therefore is an endpoint of $[a', b']$. In this case, by composing with the linear change $X \mapsto a'+b'-X$ (which interchanges a' and b') we can always suppose that the root is a'. We split this case in many cases:

Case 2a: If $Q(a') \neq 0, S(a') \neq 0$ and $(PS+QR)(a') = 0$, then

$$\mathrm{Ind}_{a'}^{b'}(P,Q) = \mathrm{Ind}_{a'}^{b'}(R,S) = \mathrm{Sign}(PS+QR, QS, a') = 0.$$

On the other hand

$$\frac{P(a')}{Q(a')} = -\frac{R(a')}{S(a')},$$

so

$$(PR-QS)(a') = Q(a')S(a')\underbrace{\left(\frac{P(a')}{Q(a')}\frac{R(a')}{S(a')} - 1\right)}_{<0} \neq 0.$$

Write $PS+QR = (X-a')^{\mu}T$ with $\mu = \mathrm{mult}_{a'}(PS+QR) > 0$. Note that $\mathrm{sign}(T(a')) = \mathrm{sign}(T(b')) = \mathrm{sign}((PS+QR)(b'))$. Then

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = -\frac{1}{2}\mathrm{sign}\big(Q(a')S(a')T(a')\big) = -\frac{1}{2}\mathrm{Sign}(PS+QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2b: If $Q(a') = 0$ and $S(a') \neq 0$, since P and Q have no common roots, then $(PS+QR)(a') \neq 0$ and we have that

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = \mathrm{Ind}_{a'}^{b'}(R,S) = \mathrm{Sign}(PS+QR, QS, a') = 0.$$

Write $Q = (X-a')^{\mu}Q_{a'}$ with $\mu = \mathrm{mult}_{a'}(Q) > 0$. Note that $\mathrm{sign}(Q_{a'}(a')) = \mathrm{sign}(Q(b'))$. Then

$$\mathrm{Ind}_{a'}^{b'}(P,Q) = \frac{1}{2}\mathrm{sign}\big(P(a')Q_{a'}(a')\big) = \frac{1}{2}\mathrm{sign}\Big(\big((PS+QR)Q_{a'}S\big)(a')\Big) = \frac{1}{2}\mathrm{Sign}(PS+QR, QS, b')$$

so identity (1) holds in $[a', b']$.

Case 2c: If $Q(a') \neq 0$ and $S(a') = 0$ we proceed in a similar way to the previous case.

Case 2d: If $Q(a') = 0$ and $S(a') = 0$, then $(PS + QR)(a') = 0$, and since P and Q have no common roots and R and S have no common roots, $P(a') \neq 0$, $R(a') \neq 0$.

Write $PS + QR = (X - a')^{\mu_0} T$ with $\mu_0 = \mathrm{mult}_{a'}(PS + QR) > 0$, $Q = (X - a')^{\mu_1} Q_{a'}$ with $\mu_1 = \mathrm{mult}_{a'}(Q) > 0$ and $S = (X - a')^{\mu_2} S_{a'}$ with $\mu_2 = \mathrm{mult}_{a'}(S) > 0$. Note that $\mathrm{val}_{a'}(P/Q) = -\mu_1$, $\mathrm{val}_{a'}(R/S) = -\mu_2$ and $\mathrm{val}_{a'}((PS + QR)/QS) = \mu_0 - \mu_1 - \mu_2$. We denote

$$\begin{aligned} \sigma_1 &:= \mathrm{sign}(P(a')) \in \{-1, 1\}, \\ \sigma_2 &:= \mathrm{sign}(R(a')) \in \{-1, 1\}, \\ \sigma_3 &:= \mathrm{sign}(T(a')) \in \{-1, 1\}, \\ \sigma_4 &:= \mathrm{sign}(Q_{a'}(a')) \in \{-1, 1\}, \\ \sigma_5 &:= \mathrm{sign}(S_{a'}(a')) \in \{-1, 1\}. \end{aligned}$$

Since a' is not a bad number, either $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$ but $\mu_0 \neq \mu_1 + \mu_2$. Note that if $\mu_1 \neq \mu_2$, then again $\mu_0 = \min\{\mu_1, \mu_2\} \neq \mu_1 + \mu_2$. So, in any case we have

$$\mathrm{Sign}(PS + QR, QS, a') = 0.$$

On the other hand, we have

$$\begin{aligned} \mathrm{Ind}_{a'}^{b'}(PR - QS, PS + QR) &= \tfrac{1}{2}\sigma_1\sigma_2\sigma_3, \\ \mathrm{Ind}_{a'}^{b'}(P, Q) &= \tfrac{1}{2}\sigma_1\sigma_4, \\ \mathrm{Ind}_{a'}^{b'}(R, S) &= \tfrac{1}{2}\sigma_2\sigma_5, \\ \tfrac{1}{2}\mathrm{Sign}(PS + QR, QS, b') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5. \end{aligned}$$

We need to prove that

$$\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5 - \sigma_3\sigma_4\sigma_5$$

or, equivalently,

$$\big(\sigma_1\sigma_2 + \sigma_4\sigma_5\big)\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5. \qquad (2)$$

We take into account that $\sigma_1 = \mathrm{sign}(P(b'))$, $\sigma_2 = \mathrm{sign}(R(b'))$, $\sigma_3 = \mathrm{sign}((PS + QR)(b'))$, $\sigma_4 = \mathrm{sign}(Q(b'))$ and $\sigma_5 = \mathrm{sign}(S(b'))$ and one final time we split in cases as follows.

- If $\sigma_1 = \sigma_5$ and $\sigma_2 = \sigma_4$, then $\sigma_3 = 1$ and equation (2) holds.
- If $\sigma_1 = -\sigma_5$ and $\sigma_2 = -\sigma_4$ then $\sigma_3 = -1$ and equation (2) holds.
- In every other case, exactly three elements in the set $\{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$ are equal and the remaining one is different. Then

  $$\sigma_1\sigma_2 + \sigma_4\sigma_5 = \sigma_1\sigma_4 + \sigma_2\sigma_5 = 0$$

  and equation (2) holds.

So, identity (1) holds in $[a', b']$.

Case 3: There is one root c of Q, S or $PS + QR$ in $[a', b']$ which is a bad number, and therefore $c \neq a'$ and $c \neq b'$. Since c is a bad number, c is indeed a root of Q, S and $PS + QR$. Also, since P and Q have no common root and R and S have no common root, $P(c) \neq 0$, $R(c) \neq 0$.

Write $PS+QR = (X-c)^{2\mu}T$, $Q = (X-c)^{\mu}Q_c$, $S = (X-c)^{\mu}S_c$ with $\mu = \mathrm{mult}_c(Q) = \mathrm{mult}_c(S) = -\mathrm{val}_c(P/Q) = -\mathrm{val}_c(R/S) > 0$. We denote

$$\begin{aligned}\sigma_1 &:= \mathrm{sign}(P(c)) \in \{-1,1\},\\ \sigma_2 &:= \mathrm{sign}(R(c)) \in \{-1,1\},\\ \sigma_3 &:= \mathrm{sign}(T(c)) \in \{-1,1\},\\ \sigma_4 &:= \mathrm{sign}(Q_c(c)) \in \{-1,1\},\\ \sigma_5 &:= \mathrm{sign}(S_c(c)) \in \{-1,1\}.\end{aligned}$$

Since $\mathrm{val}_c((PR-QS)/(PS+QR)) = -2\mu$ is even, we have

$$\mathrm{Ind}_{a'}^{b'}(PR-QS, PS+QR) = 0.$$

On the other hand, we have

$$\begin{aligned}\mathrm{Ind}_{a'}^{b'}(P,Q) &= \tfrac{1}{2}(1-(-1)^{\mu})\sigma_1\sigma_4,\\ \mathrm{Ind}_{a'}^{b'}(R,S) &= \tfrac{1}{2}(1-(-1)^{\mu})\sigma_2\sigma_5,\\ \tfrac{1}{2}\mathrm{Sign}(PS+QR, QS, a') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5,\\ \tfrac{1}{2}\mathrm{Sign}(PS+QR, QS, b') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5.\end{aligned}$$

We need to prove that $\sigma_1\sigma_4 + \sigma_2\sigma_5 = 0$. Since $\mu > 0$ and

$$(X-c)^{2\mu}T = PS + QR = (X-c)^{\mu}(PS_c + Q_cR),$$

we conclude that

$$P(c)S_c(c) + Q_c(c)R(c) = 0$$

and therefore

$$\sigma_1\sigma_5 = \mathrm{sign}(P(c)S_c(c)) = -\mathrm{sign}(Q_c(c)R(c)) = -\sigma_2\sigma_4,$$

but then

$$\sigma_1\sigma_4 = \sigma_1\sigma_5^2\sigma_4 = -\sigma_2\sigma_4^2\sigma_5 = -\sigma_2\sigma_5,$$

and identity (1) holds in $[a', b']$.

□

We conclude this section with the proof of Proposition 18.

Proof of Proposition 18: We start with item i). Since there are bad numbers, we have $P, Q, R, S, PS+QR \neq 0$. As in the proof of Proposition 17, we divide P and Q by $\gcd(P,Q)$ and R and S by $\gcd(R,S)$, so without loss of generality, we assume that P and Q are coprime and R and S are coprime.

Now, we consider $a' \in (a,b)$ such that there is no root of P, Q, R, S or $PS+QR$ in $(a, a']$. Then a' is not a bad number and therefore by Lemma 17, identity (1) holds in $[a', b]$. It is enough then to prove that

$$\mathrm{Ind}_a^{a'}(PR-QS, PS+QR) \;=\; \mathrm{Ind}_a^{a'}(P,Q) + \mathrm{Ind}_a^{a'}(R,S) - \frac{1}{2}\mathrm{Sign}(PS+QR, QS, a').$$

Since $Q(a) = 0$ and $S(a) = 0$, then $(PS + QR)(a) = 0$, and since P and Q have no common roots and R and S have no common roots, $P(a) \neq 0$, $R(a) \neq 0$.

Write $PS + QR = (X - a)^{2\mu} T$, $Q = (X - a)^{\mu} Q_a$ $S = (X - a)^{\mu} S_a$ with $\mu = \mathrm{mult}_a(Q) = \mathrm{mult}_a(S) = -\mathrm{val}_a(P/Q) = -\mathrm{val}_a(R/S) > 0$. We denote

$$\begin{aligned}
\sigma_1 &:= \mathrm{sign}(P(a)) \in \{-1, 1\},\\
\sigma_2 &:= \mathrm{sign}(R(a)) \in \{-1, 1\},\\
\sigma_3 &:= \mathrm{sign}(T(a)) \in \{-1, 1\},\\
\sigma_4 &:= \mathrm{sign}(Q_a(a)) \in \{-1, 1\},\\
\sigma_5 &:= \mathrm{sign}(S_a(a)) \in \{-1, 1\}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{Ind}_a^{a'}(PR - QS, PS + QR) &= \tfrac{1}{2}\sigma_1\sigma_2\sigma_3,\\
\mathrm{Ind}_a^{a'}(P, Q) &= \tfrac{1}{2}\sigma_1\sigma_4,\\
\mathrm{Ind}_a^{a'}(R, S) &= \tfrac{1}{2}\sigma_2\sigma_5,\\
\tfrac{1}{2}\mathrm{Sign}(PS + QR, QS, a') &= \tfrac{1}{2}\sigma_3\sigma_4\sigma_5.
\end{aligned}$$

So we need to prove that

$$\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4 + \sigma_2\sigma_5 - \sigma_3\sigma_4\sigma_5.$$

The rest of the proof is exactly as in Case 2d of the proof of Proposition 17.

The proof of item ii) is similar to the proof of item i). The proof of item iii) follows easily by introducing an intermediate point between a and b which is not a bad number and applying items i) and ii) to the new two subintervals. □

3 Product formulas for complex rational functions

The strategy to study the additivity of w and W with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ is based on the auxiliary product formula (1) proved in Proposition 17 and following the method in [3, Corollary 4.7].

We consider $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ and a rectangle $\Gamma = [x_0, x_1] \times [y_0, y_1]$. Without loss of generality, througout this section we assume that F and G are coprime and H and I are coprime.

We fisrt discuss the relation between $\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma)$ with $\mathrm{w}(F/G \,|\, \partial\Gamma) + \mathrm{w}(H/I \,|\, \partial\Gamma)$.

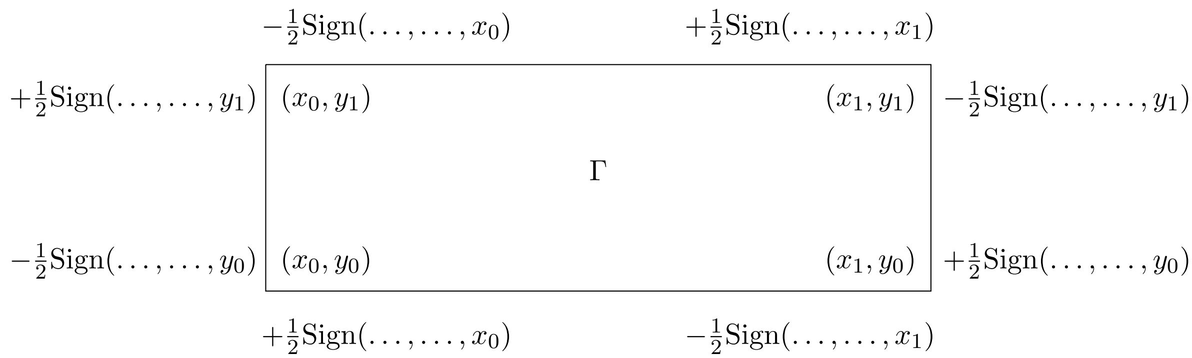
Notation 21 *We take the parametrizations (T, y_0), (x_1, T), (T, y_1) and (x_0, T) of the lines containing the bottom, right, top and left edges of Γ and consider the following polynomials in $\mathbf{R}[T]$:*

$$\begin{array}{llll}
P_1(T) = (F\overline{G})_{\mathrm{re}}(T, y_0), & Q_1(T) = (F\overline{G})_{\mathrm{im}}(T, y_0), & R_1(T) = (H\overline{I})_{\mathrm{re}}(T, y_0), & S_1(T) = (H\overline{I})_{\mathrm{im}}(T, y_0),\\
P_2(T) = (F\overline{G})_{\mathrm{re}}(x_1, T), & Q_2(T) = (F\overline{G})_{\mathrm{im}}(x_1, T), & R_2(T) = (H\overline{I})_{\mathrm{re}}(x_1, T), & S_2(T) = (H\overline{I})_{\mathrm{im}}(x_1, T),\\
P_3(T) = (F\overline{G})_{\mathrm{re}}(T, y_1), & Q_3(T) = (F\overline{G})_{\mathrm{im}}(T, y_1), & R_3(T) = (H\overline{I})_{\mathrm{re}}(T, y_1), & S_3(T) = (H\overline{I})_{\mathrm{im}}(T, y_1),\\
P_4(T) = (F\overline{G})_{\mathrm{re}}(x_0, T), & Q_4(T) = (F\overline{G})_{\mathrm{im}}(x_0, T), & R_4(T) = (H\overline{I})_{\mathrm{re}}(x_0, T), & S_4(T) = (H\overline{I})_{\mathrm{im}}(x_0, T).
\end{array}$$

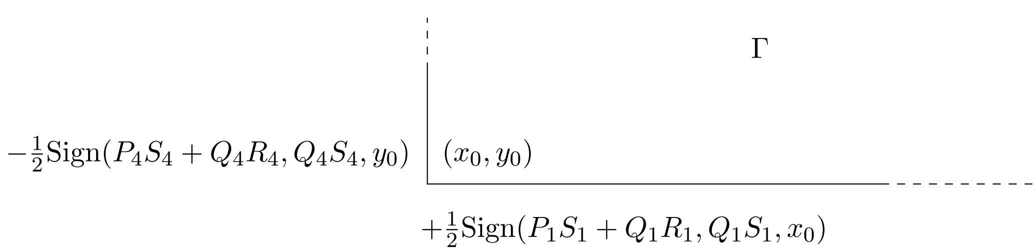
Using Proposition 17 (four times), if x_0, x_1 are not bad numbers for P_1, Q_1, R_1, S_1 or P_3, Q_3, R_3, S_3 and y_0, y_1 are not bad numbers for P_2, Q_2, R_2, S_2 or P_4, Q_4, R_4, S_4,

$$\begin{aligned} & 2\,\mathrm{w}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - 2\,\mathrm{w}(F\overline{G}\,|\,\partial\Gamma) \; - \; 2\,\mathrm{w}(H\overline{I}\,|\,\partial\Gamma) \\ = \;\; & -\,\mathrm{Var}_{x_0}^{x_1}(P_1S_1 + Q_1R_1, Q_1S_1) \; - \; \mathrm{Var}_{y_0}^{y_1}(P_2S_2 + Q_2R_2, Q_2S_2) \\ & -\,\mathrm{Var}_{x_1}^{x_0}(P_3S_3 + Q_3R_3, Q_3S_3) \; - \; \mathrm{Var}_{y_1}^{y_0}(P_4S_4 + Q_4R_4, Q_4S_4). \end{aligned}$$

Therefore, $\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma)$ coincides with $\mathrm{w}(F/G\,|\,\partial\Gamma) + \mathrm{w}(H/I\,|\,\partial\Gamma)$ if and only if the four Var add up to 0.



For instance, zooming around vertex (x_0, y_0) we have:



It would be then enough to prove that at each vertex, the adding Sign cancels with the substracting Sign. Unfortunately, this is not the case in general, as shown by the following example.

Example 22 *(Continuation of Example 9) Let* $F = Z, G = 1,$ $H = \alpha + i\beta$ *with* $0 < \beta < \alpha,$ $I = 1$ *and* $\Gamma = [0,1] \times [0,1]$*. Then*

$$\begin{aligned} P_1(T) = T, \quad Q_1(T) = 0, \quad R_1(T) = \alpha, \quad S_1(T) = \beta, \\ P_4(T) = 0, \quad Q_4(T) = T, \quad R_4(T) = \alpha, \quad S_4(T) = \beta. \end{aligned}$$

Notice that there are no bad numbers for P_1, Q_1, R_1, S_1 *or* P_4, Q_4, R_4, S_4*. We have*

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{Sign}(\beta T, 0, 0) = 0$$

but

$$\mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0) = \mathrm{Sign}(\alpha T, \beta T, 0) = 1.$$

Actually, we have already observed in Example 9 that w *is not additive in this case.*

Nevertheless, next proposition shows that w is indeed additive with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ under mild hypothesis.

Proposition 23 *Let $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G and H/I have even valuation at the vertices of Γ. Then*

$$\mathrm{w}(F/G \cdot H/I \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma) + \mathrm{w}(H/I \,|\, \partial\Gamma).$$

Moreover, next proposition shows that W is additive with respect to product in $\mathbf{C}(Z) \setminus \{0\}$ with full generality.

Proposition 24 *Let $F/G, H/I \in \mathbf{C}(Z) \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle. Then*

$$\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma) = \mathrm{W}(F/G \,|\, \partial\Gamma) + \mathrm{W}(H/I \,|\, \partial\Gamma).$$

Before the proofs of these propositions, we do some preliminary computations and we introduce some notation will use repeteadly in the rest of the section.

We consider a new variable $\overline{Z}$, together with the inclusion $\mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y]$ through the identities $Z = X + iY, \overline{Z} = X - iY$. For $G(Z) = \sum_j \gamma_j Z^j \in \mathbf{C}[Z]$, let $\overline{G}(\overline{Z}) = \sum_j \overline{\gamma_j} \overline{Z}^j \in \mathbf{C}[\overline{Z}]$. In this way, note that

$$\overline{G}(\overline{Z}) = \overline{G}(X, Y) \in \mathbf{C}[X, Y].$$

Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$. We define $z_0 = x_0 + iy_0 \in \mathbf{C}$ and

$$\begin{aligned} e &= \mathrm{val}_{z_0}(F/G), \qquad p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0, \\ f &= \mathrm{val}_{z_0}(H/I), \qquad r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0. \end{aligned}$$

If $e \geq 0$, since F and G are coprime, $F(Z)\overline{G}(\overline{Z}) = (Z - z_0)^e F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ with

$$F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z}) = p + iq + (Z - z_0)A(Z, \overline{Z}) + (\overline{Z} - \overline{z_0})B(Z, \overline{Z}) \in \mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y] \tag{3}$$

for some $A, B \in \mathbf{C}[Z, \overline{Z}]$. Therefore,

$$\begin{aligned} P_1(T) &= (T - x_0)^e \left(p + (T - x_0)A_1(T)\right), \\ Q_1(T) &= (T - x_0)^e \left(q + (T - x_0)B_1(T)\right), \end{aligned} \tag{4}$$

for some $A_1, B_1 \in \mathbf{R}[T]$. If e is even,

$$\begin{aligned} P_4(T) &= (-1)^{\frac{e}{2}}(T - y_0)^e \left(p + (T - y_0)A_{4,\mathrm{e}}(T)\right), \\ Q_4(T) &= (-1)^{\frac{e}{2}}(T - y_0)^e \left(q + (T - y_0)B_{4,\mathrm{e}}(T)\right), \end{aligned} \tag{5}$$

for some $A_{4,\mathrm{e}}, B_{4,\mathrm{e}} \in \mathbf{R}[T]$. If e is odd,

$$\begin{aligned} P_4(T) &= (-1)^{\frac{e-1}{2}}(T - y_0)^e \left(-q + (T - y_0)A_{4,\mathrm{o}}(T)\right), \\ Q_4(T) &= (-1)^{\frac{e-1}{2}}(T - y_0)^e \left(p + (T - y_0)B_{4,\mathrm{o}}(T)\right), \end{aligned} \tag{6}$$

for some $A_{4,\mathrm{o}}, B_{4,\mathrm{o}} \in \mathbf{R}[T]$.

Similarly, if $e < 0$, since F and G are coprime, $F(Z)\overline{G}(\overline{Z}) = (\overline{Z} - \overline{z_0})^{-e} F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ with $F_{z_0}(Z)\overline{G}_{z_0}(\overline{Z})$ as in (3). Therefore, we obtain formulas for P_1, Q_1, P_4, Q_4 as in (4), (5) and (6) replacing e by $-e$, and additionally multiplying P_4 and Q_4 by (-1) in (6) (e odd).

If $f \geq 0$, since H and I are coprime, $H(Z)\overline{I}(\overline{Z}) = (Z - z_0)^e H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ with

$$H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z}) = r + is + (Z - z_0)C(Z, \overline{Z}) + (\overline{Z} - \overline{z_0})D(Z, \overline{Z}) \in \mathbf{C}[Z, \overline{Z}] \subset \mathbf{C}[X, Y] \tag{7}$$

for some $C, D \in \mathbf{C}[Z, \overline{Z}]$. Therefore,

$$\begin{aligned} R_1(T) &= (T - x_0)^f \left(r + (T - x_0)C_1(T)\right), \\ S_1(T) &= (T - x_0)^f \left(s + (T - x_0)D_1(T)\right), \end{aligned} \tag{8}$$

for some $C_1, D_1 \in \mathbf{R}[T]$. If f is even,

$$\begin{aligned} R_4(T) &= (-1)^{\frac{f}{2}}(T - y_0)^f \left(r + (T - y_0)C_{4,\mathrm{e}}(T)\right), \\ S_4(T) &= (-1)^{\frac{f}{2}}(T - y_0)^f \left(s + (T - y_0)D_{4,\mathrm{e}}(T)\right), \end{aligned} \tag{9}$$

for some $C_{4,\mathrm{e}}, D_{4,\mathrm{e}} \in \mathbf{R}[T]$. If f is odd,

$$\begin{aligned} R_4(T) &= (-1)^{\frac{f-1}{2}}(T - y_0)^f \left(-s + (T - y_0)C_{4,\mathrm{o}}(T)\right), \\ S_4(T) &= (-1)^{\frac{f-1}{2}}(T - y_0)^f \left(r + (T - y_0)D_{4,\mathrm{o}}(T)\right), \end{aligned} \tag{10}$$

for some $C_{4,\mathrm{o}}, D_{4,\mathrm{o}} \in \mathbf{R}[T]$.

Finally, if $f < 0$, since H and I are coprime, $H(Z)\overline{I}(\overline{Z}) = (\overline{Z} - \overline{z_0})^{-f} H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ with $H_{z_0}(Z)\overline{I}_{z_0}(\overline{Z})$ as in (7). Therefore, we obtain formulas for R_1, S_1, R_4, S_4 as in (8), (9) and (10) replacing the f by $-f$, and additionally multiplying R_4 and S_4 by (-1) in (10) (f odd).

Now, we prove an auxiliary lemma, which is a particular case of Proposition 23 where one of the rational functions is a constant.

Lemma 25 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$, $\gamma \in \mathbf{C} \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle such that F/G has even valuation at the vertices of Γ. Then*

$$\mathrm{w}(F/G \,|\, \partial\Gamma) = \mathrm{w}(\gamma F/G \,|\, \partial\Gamma).$$

Remark 26 *Note that if the valuation of F/G at the vertices of Γ is even, Lemma 25 implies that*

$$\mathrm{W}(F/G \,|\, \partial\Gamma) = \mathrm{w}(F/G \,|\, \partial\Gamma).$$

Proof of Lemma 25: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and $\gamma = \alpha + i\beta$. The statement is clear if $\beta = 0$, so we suppose $\beta \neq 0$. We take $H = \alpha + i\beta$ and $I = 1$, and using Notation 21 we have

$$R_1(T) = R_2(T) = R_3(T) = R_4(T) = \alpha,$$

$$S_1(T) = S_2(T) = S_3(T) = S_4(T) = \beta.$$

Since S_1, S_2, S_3, S_4 are constant, there are no bad numbers for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3 or P_4, Q_4, R_4, S_4. Using Proposition 17 (four times) as explained at the beginning of the section, and the fact that $\mathrm{w}(\alpha + i\beta \,|\, \partial\Gamma) = 0$, we have

$$\begin{aligned} & 2\,\mathrm{w}((\alpha+i\beta)F\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(\alpha+i\beta \,|\, \partial\Gamma) \\ = \;& -\operatorname{Var}_{x_0}^{x_1}(\beta P_1 + \alpha Q_1, \beta Q_1) \;-\; \operatorname{Var}_{y_0}^{y_1}(\beta P_2 + \alpha Q_2, \beta Q_2) \\ & -\operatorname{Var}_{x_1}^{x_0}(\beta P_3 + \alpha Q_3, \beta Q_3) \;-\; \operatorname{Var}_{y_1}^{y_0}(\beta P_4 + \alpha Q_4, \beta Q_4). \end{aligned}$$

Therefore, it is enough to prove that the four Var add up to 0. Concentrating at vertex (x_0, y_0), we will prove that

$$\operatorname{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \operatorname{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0).$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$, we have that $\beta p + \alpha q$, βq are not simultaneously 0. If $e \geq 0$, using (4) and (5) we obtain

$$\operatorname{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \operatorname{sign}((\beta p + \alpha q)\beta q) = \operatorname{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0).$$

If $e < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. $\square$

Proof of Proposition 23: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and take $\gamma = \alpha + i\beta \in \mathbf{C}$. Multiplying F/G by γ and considering for instance the parametrization of the bottom side of Γ, we define

$$P_{\gamma,1}(T) = (\gamma F\overline{G})_{\mathrm{re}}(T, y_0), \quad Q_{\gamma,1}(T) = (\gamma F\overline{G})_{\mathrm{im}}(T, y_0).$$

Using Notation 21 we have

$$\begin{aligned} P_{\gamma,1}(T) &= \alpha P_1(T) - \beta Q_1(T), \\ Q_{\gamma,1}(T) &= \alpha Q_1(T) + \beta P_1(T). \end{aligned}$$

Let $c \in \mathbf{R}$. If $\mathrm{val}_c(P_1/Q_1) < 0$, then for $\beta \neq 0$ we have $\mathrm{val}_c(P_{\gamma,1}/Q_{\gamma,1}) \geq 0$. If $\mathrm{val}_c(P_1/Q_1) \geq 0$, then for generic $\alpha, \beta \in \mathbf{R}$ we still have $\mathrm{val}_c(P_{\gamma,1}/Q_{\gamma,1}) \geq 0$.

This implies that, multiplying F/G and H/I by suitable constants if necessary, we can assume without loss of generality that x_0, x_1 are not bad numbers for P_1, Q_1, R_1, S_1 or P_3, Q_3, S_3, R_3 and y_0, y_1 are not bad numbers for P_2, Q_2, R_2, S_2 or P_4, Q_4, S_4, R_4. In addition we may assume that $F_z(z)\overline{G_z(z)}$ and $H_z(z)\overline{I_z(z)}$ are not real for each of the four vertices z of Γ. By Lemma 25, this does not change $w(F/G \,|\, \partial\Gamma)$, $w(H/I \,|\, \partial\Gamma)$ or $w(F/G \cdot H/I \,|\, \partial\Gamma)$.

Again, using Proposition 17 (four times) we have

$$\begin{aligned} & 2\,\mathrm{w}(F\overline{G}H\overline{I} \,|\, \partial\Gamma) - 2\,\mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(H\overline{I}\, \partial\Gamma) \\ = \;& -\operatorname{Var}_{x_0}^{x_1}(P_1 S_1 + Q_1 R_1, Q_1 S_1) \;-\; \operatorname{Var}_{y_0}^{y_1}(P_2 S_2 + Q_2 R_2, Q_2 S_2) \\ & -\operatorname{Var}_{x_1}^{x_0}(P_3 S_3 + Q_3 R_3, Q_3 S_3) \;-\; \operatorname{Var}_{y_1}^{y_0}(P_4 S_4 + Q_4 R_4, Q_4 S_4). \end{aligned}$$

Therefore, it is enough to prove that the four Var add up to 0. Concentrating at vertex (x_0, y_0), we will prove that

$$\operatorname{Sign}(P_1 S_1 + Q_1 R_1, Q_1 S_1, x_0) = \operatorname{Sign}(P_4 S_4 + Q_4 R_4, Q_4 S_4, y_0).$$

Since $p+iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r+is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $q \neq 0$ and $s \neq 0$, we have that $qs \neq 0$. If $e \geq 0$ and $f \geq 0$, using (4), (5), (8) and (9) we obtain

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{sign}((ps+qr)qs) = \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. □

We focus now on Proposition 24. As before, we prove first an auxiliary lemma, which is a particular case of it where one of the rational functions is a constant.

Lemma 27 *Let $F/G \in \mathbf{C}(Z) \setminus \{0\}$, $\gamma \in \mathbf{C} \setminus \{0\}$ and $\Gamma \subset \mathbf{R}^2$ a rectangle. Then*

$$\mathrm{W}(F/G \,|\, \partial\Gamma) = \mathrm{W}(\gamma F/G \,|\, \partial\Gamma).$$

Proof: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$ and $\gamma = \alpha + i\beta$. The statement is clear if $\beta = 0$, so we suppose $\beta \neq 0$. We take $H = \alpha + i\beta$ and $I = 1$, and using Notation 21 we have

$$R_1(T) = R_2(T) = R_3(T) = R_4(T) = \alpha,$$

$$S_1(T) = S_2(T) = S_3(T) = S_4(T) = \beta.$$

Since S_1, S_2, S_3, S_4 are constant, there are no bad numbers for $P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2;$ P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, R_1, S_1; -Q_2, P_2, R_2, S_2; -Q_3, P_3, R_3, S_3$ or $-Q_4, P_4, R_4, S_4$. Using Proposition 17 (eight times) and the fact that $\mathrm{W}(\alpha + i\beta \,|\, \partial\Gamma) = \mathrm{w}(\alpha + i\beta \,|\, \partial\Gamma) = 0$, we have

$$\begin{aligned}
& 4\,\mathrm{W}((\alpha+i\beta)F\overline{G} \,|\, \partial\Gamma) - 4\,\mathrm{W}(F\overline{G} \,|\, \partial\Gamma) - 4\,\mathrm{W}(\alpha+i\beta \,|\, \partial\Gamma) \\
= \quad & 2\,\mathrm{w}((\alpha+i\beta)F\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(F\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(\alpha+i\beta \,|\, \partial\Gamma) \\
& + 2\,\mathrm{w}((\alpha+i\beta)iF\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(iF\overline{G} \,|\, \partial\Gamma) - 2\,\mathrm{w}(\alpha+i\beta \,|\, \partial\Gamma) \\
= \quad & -\,\mathrm{Var}_{x_0}^{x_1}(\beta P_1 + \alpha Q_1, \beta Q_1) \;-\; \mathrm{Var}_{y_0}^{y_1}(\beta P_2 + \alpha Q_2, \beta Q_2) \\
& -\,\mathrm{Var}_{x_1}^{x_0}(\beta P_3 + \alpha Q_3, \beta Q_3) \;-\; \mathrm{Var}_{y_1}^{y_0}(\beta P_4 + \alpha Q_4, \beta Q_4) \\
& -\,\mathrm{Var}_{x_0}^{x_1}(-\beta Q_1 + \alpha P_1, \beta P_1) \;-\; \mathrm{Var}_{y_0}^{y_1}(-\beta Q_2 + \alpha P_2, \beta P_2) \\
& -\,\mathrm{Var}_{x_1}^{x_0}(-\beta Q_3 + \alpha P_3, \beta P_3) \;-\; \mathrm{Var}_{y_1}^{y_0}(-\beta Q_4 + \alpha P_4, \beta P_4).
\end{aligned}$$

Therefore, it is enough to prove that the eight Var add up to 0. Zooming around vertex (x_0, y_0) we have

Γ

$-\frac{1}{2}\mathrm{Sign}(-\beta Q_4 + \alpha P_4, \beta P_4, y_0)$

$-\frac{1}{2}\mathrm{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0)$ $\quad (x_0, y_0)$

$+\frac{1}{2}\mathrm{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0)$

$+\frac{1}{2}\mathrm{Sign}(-\beta Q_1 + \alpha P_1, \beta P_1, x_0)$

We will prove that

$$\begin{aligned}\operatorname{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) + \operatorname{Sign}(-\beta Q_1 + \alpha P_1, \beta P_1, x_0) &= \\ = \operatorname{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0) + \operatorname{Sign}(-\beta Q_4 + \alpha P_4, \beta P_4, y_0).\end{aligned} \tag{11}$$

Since $p+iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ we have that $\beta p+\alpha q$ and βq are not simultaneously 0 and $-\beta q+\alpha p$, βp are not simultaneously 0. Suppose $e \geq 0$.

If e is even, using (4) and (5) we have

$$\operatorname{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \operatorname{sign}((\beta p + \alpha q)\beta q) = \operatorname{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0),$$

$$\operatorname{Sign}(-\beta Q_1 + \alpha P_1, \beta P_1, x_0) = \operatorname{sign}((-\beta q + \alpha p)\beta p) = \operatorname{Sign}(-\beta Q_4 + \alpha P_4, \beta P_4, y_0).$$

If e is odd, using (4) and (6) we have

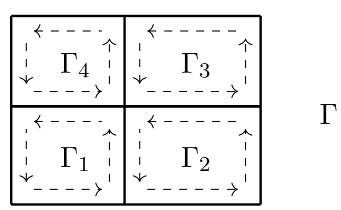
$$\operatorname{Sign}(\beta P_1 + \alpha Q_1, \beta Q_1, x_0) = \operatorname{sign}((\beta p + \alpha q), \beta q) = \operatorname{Sign}(-\beta Q_4 + \alpha P_4, \beta P_4, y_0),$$

$$\operatorname{Sign}(-\beta Q_1 + \alpha P_1, \beta P_1, x_0) = \operatorname{sign}((-\beta q + \alpha p)\beta p) = \operatorname{Sign}(\beta P_4 + \alpha Q_4, \beta Q_4, y_0).$$

Then, in both cases we have that identity (11) holds. If $e < 0$ the same identity holds with a similar proof.

Finally, the analysis for the three remaining vertices (x_1, y_0), (x_1, y_1) and (x_0, y_1) is identical. □

Proof of Proposition 24: Let $\Gamma = [x_0, x_1] \times [y_0, y_1]$. From the definition of w and W, it follows that for any function in $\mathbf{C}[X, Y]$, if we subdivide Γ in four subrectangles $\Gamma_1, \ldots, \Gamma_4$ as in the picture below, then the winding number w or W on $\partial\Gamma$ equals the sum of the respective winding number on $\partial\Gamma_i$ for $1 \leq i \leq 4$.



If F, G, H and I vanish at more than one vertex of Γ, we choose a point $z = (x, y) \in \Gamma$ such that (x, y), (x_0, y), (x_1, y), (x, y_0) and (x, y_1) are not roots of any of them. Using z to subdivide Γ in four rectangles, F, G, H and I vanish at most at one vertex of each of these four rectangles. So without loss of generality we replace Γ by Γ_1 and we make the assumption that F, G, H and I do not vanish at $(x_0, y_1), (x_1, y_1), (x_1, y_0)$.

As in the proof of Proposition 23, multiplying F/G and H/I by suitable constants if necessary, we can suppose without loss of generality that x_0, x_1 and y_0, y_1 are not bad numbers for all the finitely many 4-uples of polynomials we will use along the proof. In addition we may assume that and $F_z(z)\overline{G_z(z)}$ and $H_z(z)\overline{I_z(z)}$ are not real or purely imaginary for each of the four vertices z of Γ. By Lemma 27, this does not change $\mathrm{W}(F/G \,|\, \partial\Gamma)$, $\mathrm{W}(H/I \,|\, \partial\Gamma)$ or $\mathrm{W}(F/G \cdot H/I \,|\, \partial\Gamma)$.

The proof is done in several cases according to the parity of the valuations e and f of F/G and H/I at $z_0 = x_0 + iy_0$.

If e and f are both even, the statement follows from Proposition 23 since, by Remark 26,

$$\mathrm{W}(F/G\,|\,\partial\Gamma) = \mathrm{w}(F/G\,|\,\partial\Gamma),\ \mathrm{W}(H/I\,|\,\partial\Gamma) = \mathrm{w}(H/I\,|\,\partial\Gamma),\ \text{and } \mathrm{W}(F/G{\cdot}H/I\,|\,\partial\Gamma) = \mathrm{w}(F/G{\cdot}H/I\,|\,\partial\Gamma).$$

If e is odd and f is even, suppose first $e \geq 0$ and $f \geq 0$. By Remark 26, $\mathrm{W}(H\overline{I}\,|\,\partial\Gamma) = \mathrm{w}(H\overline{I}\,|\,\partial\Gamma)$. Then, using Proposition 17 (eight times) for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, R_1, S_1$; $-Q_2, P_2, R_2, S_2$; $-Q_3, P_3, R_3, S_3$; and $-Q_4, P_4, R_4, S_4$, we have

$$\begin{aligned}
& 4\,\mathrm{W}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - 4\,\mathrm{W}(F\overline{G}\,|\,\partial\Gamma) - 4\,\mathrm{W}(H\overline{I}\,\partial\Gamma) \\
= {} & 2\,\mathrm{w}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - 2\,\mathrm{w}(F\overline{G}\,|\,\partial\Gamma) - 2\,\mathrm{w}(H\overline{I}\,\partial\Gamma) \\
& + 2\,\mathrm{w}(iF\overline{G}H\overline{I}\,|\,\partial\Gamma) - 2\,\mathrm{w}(iF\overline{G}\,|\,\partial\Gamma) - 2\,\mathrm{w}(H\overline{I}\,\partial\Gamma) \\
= {} & -\mathrm{Var}_{x_0}^{x_1}(P_1S_1 + Q_1R_1, Q_1S_1) - \mathrm{Var}_{y_0}^{y_1}(P_2S_2 + Q_2R_2, Q_2S_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(P_3S_3 + Q_3R_3, Q_3S_3) - \mathrm{Var}_{y_1}^{y_0}(P_4S_4 + Q_4R_4, Q_4S_4) \\
& -\mathrm{Var}_{x_0}^{x_1}(-Q_1S_1 + P_1R_1, P_1S_1) - \mathrm{Var}_{y_0}^{y_1}(-Q_2S_2 + P_2R_2, P_2S_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(-Q_3S_3 + P_3R_3, P_3S_3) - \mathrm{Var}_{y_1}^{y_0}(-Q_4S_4 + P_4R_4, P_4S_4).
\end{aligned}$$

Therefore, it is enough to prove that the eight Var add up to 0. Concentrating around vertex (x_0, y_0) we will prove that

$$\begin{aligned}
& \mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) + \mathrm{Sign}(-Q_1S_1 + P_1R_1, P_1S_1, x_0) = \\
= {} & \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0) + \mathrm{Sign}(-Q_4S_4 + P_4R_4, P_4S_4, y_0).
\end{aligned}$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $p, q, s \neq 0$, we have that $qs \neq 0$ and $ps \neq 0$. Then, using (4), (6), (8) and (9) we have

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{sign}((ps + qr)qs) = \mathrm{Sign}(-Q_4S_4 + P_4R_4, P_4S_4, y_0)$$

$$\mathrm{Sign}(-Q_1S_1 + P_1R_1, P_1S_1, x_0) = \mathrm{sign}((-qs + pr)ps) = \mathrm{Sign}((P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

For the remaining three vertices, since $(F\overline{G})_{\mathrm{re}}$ $(F\overline{G})_{\mathrm{im}}$, $(H\overline{I})_{\mathrm{re}}$ and $(H\overline{I})_{\mathrm{im}}$ do not vanish at them, a simple evaluation gives

$$\begin{aligned}
\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_1) &= \mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_0), \\
\mathrm{Sign}(-Q_1S_1 + P_1R_1, P_1S_1, x_1) &= \mathrm{Sign}(-Q_2S_2 + P_2R_2, P_2S_2, x_0), \\
\mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_1) &= \mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_1), \\
\mathrm{Sign}(-Q_2S_2 + P_2R_2, P_2S_2, y_1) &= \mathrm{Sign}(-Q_3S_3 + P_3R_3, P_3S_3, x_1), \\
\mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_0) &= \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_1), \\
\mathrm{Sign}(-Q_3S_3 + P_3R_3, P_3S_3, x_0) &= \mathrm{Sign}(-Q_4S_4 + P_4R_4, P_4S_4, y_1).
\end{aligned}$$

If e is even and f is odd we interchange F/G with H/I and proceed exactly as before.

If e and f are both odd, suppose first $e \geq 0$ and $f \geq 0$. By Remark 26, $\mathrm{W}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) = \mathrm{w}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) = \mathrm{w}(-F\overline{G}H\overline{I}\,|\,\partial\Gamma)$. Then using Proposition 17 (eight times) for P_1, Q_1, R_1, S_1; P_2, Q_2, R_2, S_2; P_3, Q_3, R_3, S_3; P_4, Q_4, R_4, S_4; $-Q_1, P_1, -S_1, R_1$; $-Q_2, P_2, -S_2, R_2$; $-Q_3, P_3, -S_3, R_3$; and $-Q_4, P_4, -S_4, R_4$, we have

$$\begin{aligned}
& 4\,\mathrm{W}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - 4\,\mathrm{W}(F\overline{G}\,|\,\partial\Gamma) - 4\,\mathrm{W}(H\overline{I}\,|\,\partial\Gamma) \\
= \; & 2\,\mathrm{w}(F\overline{G}H\overline{I}\,|\,\partial\Gamma) - 2\,\mathrm{w}(F\overline{G}\,|\,\partial\Gamma) - 2\,\mathrm{w}(H\overline{I}\,|\,\partial\Gamma) \\
& + 2\,\mathrm{w}(iF\overline{G}iH\overline{I}\,|\,\partial\Gamma) - 2\,\mathrm{w}(iF\overline{G}\,|\,\partial\Gamma) - 2\,\mathrm{w}(iH\overline{I}\,|\,\partial\Gamma) \\
= \; & -\mathrm{Var}_{x_0}^{x_1}(P_1S_1 + Q_1R_1, Q_1S_1) \; - \; \mathrm{Var}_{y_0}^{y_1}(P_2S_2 + Q_2R_2, Q_2S_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(P_3S_3 + Q_3R_3, Q_3S_3) \; - \; \mathrm{Var}_{y_1}^{y_0}(P_4S_4 + Q_4R_4, Q_4S_4) \\
& -\mathrm{Var}_{x_0}^{x_1}(-Q_1R_1 - P_1S_1, P_1R_1) \; - \; \mathrm{Var}_{y_0}^{y_1}(Q_2R_2 - P_2S_2 -, P_2R_2) \\
& -\mathrm{Var}_{x_1}^{x_0}(-Q_3R_3 - P_3S_3, P_3R_3) \; - \; \mathrm{Var}_{y_1}^{y_0}(-Q_4R_4 - P_4S_4, P_4R_4).
\end{aligned}$$

Therefore, it is enough to prove that the eight Var add up to 0. Concentrating around vertex (x_0, y_0), we will prove that

$$\begin{aligned}
& \mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) + \mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_0) \quad = \\
= \; & \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0) + \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_0).
\end{aligned}$$

Since $p + iq = F_{z_0}(z_0)\overline{G_{z_0}(z_0)} \neq 0$ and $r + is = H_{z_0}(z_0)\overline{I_{z_0}(z_0)} \neq 0$ are such that $p, q, r, s \neq 0$, we have that $qs \neq 0$ and $pr \neq 0$. Then, using (4), (6), (8) and (10) we have

$$\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_0) = \mathrm{sign}((ps + qr)qs) = \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_0),$$

$$\mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_0) = \mathrm{sign}(-(ps + qr)pr) = \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_0).$$

If $e < 0$ or $f < 0$ the same identity holds with a similar proof.

For the remaining three vertices, since $(F\overline{G})_{\mathrm{re}}$ $(F\overline{G})_{\mathrm{im}}$, $(H\overline{I})_{\mathrm{re}}$ and $(H\overline{I})_{\mathrm{im}}$ do not vanish at them, a simple evaluation gives

$$\begin{aligned}
\mathrm{Sign}(P_1S_1 + Q_1R_1, Q_1S_1, x_1) &= \mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_0), \\
\mathrm{Sign}(-Q_1R_1 - P_1S_1, P_1R_1, x_1) &= \mathrm{Sign}(-Q_2R_2 - P_2S_2, P_2R_2, y_0), \\
\mathrm{Sign}(P_2S_2 + Q_2R_2, Q_2S_2, y_1) &= \mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_1), \\
\mathrm{Sign}(-Q_2R_2 - P_2S_2, P_2R_2, y_1) &= \mathrm{Sign}(-Q_3R_3 - P_3S_3, P_3R_3, x_1), \\
\mathrm{Sign}(P_3S_3 + Q_3R_3, Q_3S_3, x_0) &= \mathrm{Sign}(P_4S_4 + Q_4R_4, Q_4S_4, y_1), \\
\mathrm{Sign}(-Q_3R_3 - P_3S_3, P_3R_3, x_0) &= \mathrm{Sign}(-Q_4R_4 - P_4S_4, P_4R_4, y_1).
\end{aligned}$$

□

4 Proofs of the main results

Now we focus on the proof of our main results. First we consider the following lemma.

Lemma 28 *Let $\Gamma \subset \mathbf{R}^2$ be a rectangle. For $\gamma \in \mathbf{C}$, $\mathrm{W}(\gamma\,|\,\partial\Gamma) = 0$. For $F(Z) = Z - z_0$ with $z_0 \in \mathbf{C}$,*

$$\mathrm{W}(F\,|\,\partial\Gamma) = \begin{cases} 1 & \textit{if } z_0 \textit{ is in the interior of } \Gamma, \\ \frac{1}{2} & \textit{if } z_0 \textit{ is in one of the edges of } \Gamma, \\ \frac{1}{4} & \textit{if } z_0 \textit{ is in one of the vertices of } \Gamma, \\ 0 & \textit{if } z_0 \textit{ is in the exterior of } \Gamma. \end{cases}$$

and

$$\mathrm{W}(1/F\,|\,\partial\Gamma) = \begin{cases} -1 & \textit{if } z_0 \textit{ is in the interior of } \Gamma, \\ -\frac{1}{2} & \textit{if } z_0 \textit{ is in one of the edges of } \Gamma, \\ -\frac{1}{4} & \textit{if } z_0 \textit{ is in one of the vertices of } \Gamma, \\ 0 & \textit{if } z_0 \textit{ is in the exterior of } \Gamma. \end{cases}$$

We omit the proof of Lemma 28 since it follows from a straightforward computation.

Proof of Theorem 12: By Lemma 28, W does the right counting for non-zero constants, linear monic polynomials and their conjugates, and by Proposition 24, W is additive with respect to multiplication in $\mathbf{C}(Z) \setminus \{0\}$. This proves the theorem. □

Proof of Theorem 13: Theorem 13 is a corollary of Theorem 12, since under the assumption that F/G has even valuation at the vertices of Γ, using Remark 26 we have $\mathrm{W}(F/G\,|\,\partial\Gamma) = \mathrm{w}(F/G\,|\,\partial\Gamma)$. □

An alternative proof for Theorem 13 follows from first proving that it $F(Z) = (Z - z_0)^2$ with z_0 a corner of Γ, w does the right counting for F and $1/F$, and then using the additivity property for w proven in Proposition 23. This proof avoids completely the definition of W.

References

[1] L.V. Ahlfors, *Complex Analysis. An introduction to the theory of analytic functions of one complex variable.* Second edition. Ed. McGraw-Hill Book Company, 1966.

[2] S. Basu, R. Pollack, M–F. Roy, Algorithms in real algebraic geometry. Second edition. Algorithms and Computation in Mathematics, 10. Springer-Verlag, Berlin, 2006.

[3] M. Eisermann, The fundamental theorem of algebra made effective: an elementary real-algebraic proof via Sturm chains. *Amer. Math. Monthly* 119 (2012), no. 9, 715–752.

[4] D. Perrucci, M.-F. Roy, A new general formula for the Cauchy Index on an interval with Subresultants. *J. Symbolic Comput.* 109 (2022), 465-481.

[5] D. Perrucci, M.-F. Roy, Quantitative fundamental theorem of algebra. *Q. J. Math.* 70 (2019), no. 3, 1009–1037.