On sum of squares certificates of non-negativity on a strip

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Abstract

In [6], Murray Marshall proved that every $f \in \mathbb{R}[X,Y]$ non-negative on the strip $[0,1] \times \mathbb{R}$ can be written as $f = \sigma_0 + \sigma_1 X(1-X)$ with σ_0, σ_1 sums of squares in $\mathbb{R}[X,Y]$. In this work, we present a few results concerning this representation in particular cases. First, under the assumption $\deg_Y f \le 2$, by characterizing the extreme rays of a suitable cone, we obtain a degree bound for each term. Then, we consider the case of f positive on $[0,1] \times \mathbb{R}$ and *non-vanishing at infinity*, and we show again a degree bound for each term, coming from a constructive method to obtain the sum of squares representation. Finally, we show that this constructive method also works in the case of f having only a finite number of zeros, all of them lying on the boundary of the strip, and such that $\frac{\partial f}{\partial X}$ does not vanish at any of them.

1 Introduction

Let $g_1, \dots, g_s \in \mathbb{R}[X_1, \dots, X_n]$ and consider the basic closed semialgebraic set

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \ge 0, \dots, g_s(x) \ge 0\}.$$

Given $f \in \mathbb{R}[X_1, \dots, X_n]$ such that f is non-negative on S, a classical question is if there is a representation of f which makes evident this fact. Concerning this problem, there are two important algebraic objects associated to $g_1, \dots, g_s$: the preordering

$$T(g_1, \dots, g_s) = \Big\{ \sum_{I \subset \{1,\dots,s\}} \sigma_I \prod_{i \in I} g_i \mid \sigma_I \in \sum \mathbb{R}[X_1, \dots, X_n]^2 \text{ for every } I \subset \{1, \dots, s\} \Big\}$$

and the quadratic module

$$M(g_1, \dots, g_s) = \Big\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_0, \sigma_1, \dots, \sigma_s \in \sum \mathbb{R}[X_1, \dots, X_n]^2 \Big\}.$$

It is clear that $M(g_1, \dots, g_s) \subset T(g_1, \dots, g_s)$, but the equality only holds in some special cases, for instance when $s = 1$. It is also clear that every polynomial $f \in T(g_1, \dots, g_s)$ is non-negative on S, but the converse is not true in general (see [17, Example]).

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Schmüdgen Positivstellensatz ([16]) states that if S is compact, every polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ positive on S belongs to $T(g_1, \dots, g_s)$. On the other hand, Putinar Positivstellensatz ([12]) states that if $M(g_1, \dots, g_s)$ is archimedean, every polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ positive on S belongs to $M(g_1, \dots, g_s)$. Recall that the quadratic module $M(g_1, \dots, g_s)$ is archimedean if there exists $r \in \mathbb{N}$ such that

$$r - X_1^2 - \cdots - X_n^2 \in M(g_1, \dots, g_s).$$

Note that if $M(g_1, \dots, g_s)$ is archimedean, then S is compact, but again, the converse is not true in general (see [3, Example 4.6]).

In the case where $\dim S \geq 3$ or in the case where $n = 2$ and S contains an affine full-dimensional cone, there exist polynomials non-negative on S which do not belong to $T(g_1, \dots, g_s)$ ([14]). On the contrary, M. Marshall proved in [6] the following result for polynomials non-negative on the strip $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$:

Theorem 1 *Let $f \in \mathbb{R}[X, Y]$ with $f \geq 0$ on $[0, 1] \times \mathbb{R}$. Then*

$$f = \sigma_0 + \sigma_1 X(1 - X) \tag{1}$$

with $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X, Y]^2$.

In other words, Theorem 1 states that every polynomial non-negative on the strip $[0, 1] \times \mathbb{R}$ belongs to $M(X(1 - X))$. This result was later extended to other two-dimensional semialgebraic sets in [7] and [15].

In this paper, we present some results concerning effectivity issues around the representation obtained in Theorem 1, in particular cases.

For instance, a natural question is if it is possible to bound the degrees of each term in (1). In Section 2, we prove a degree bound for each term in the case $\deg_Y f \leq 2$. To this end, we first characterize all the extreme rays of a suitable cone containing f and study their representation as in (1). The main result in this section is the following.

Theorem 2 *Let $f \in \mathbb{R}[X, Y]$ with $f \geq 0$ on $[0, 1] \times \mathbb{R}$ and $\deg_Y f \leq 2$. Then f can be written as in (1) with*

$$\deg(\sigma_0), \deg(\sigma_1 X(1 - X)) \leq \deg_X f + 3.$$

In Section 3, we deal again with the question of bounding the degrees of each term in (1) in a different situation. First, in Section 3.1, we consider the case where f is positive on $[0, 1] \times \mathbb{R}$ and does not *vanish at infinity*. To make this concept precise, we introduce the following definition coming from [9]:

Definition 3 *Let $f \in \mathbb{R}[X, Y]$ and $m = \deg_Y f$. The polynomial f is* fully m-ic *on $[0, 1]$ if for every $x \in [0, 1]$, $f(x, Y) \in \mathbb{R}[Y]$ has degree m.*

Given

$$f = \sum_{0 \le i \le m} \sum_{0 \le j \le d} a_{ji} X^j Y^i \in \mathbb{R}[X,Y],$$

define

$$\bar{f} = \sum_{0 \le i \le m} \sum_{0 \le j \le d} a_{ji} X^j Y^i Z^{m-i} \in \mathbb{R}[X,Y,Z].$$

Note that if $f > 0$ on $[0,1] \times \mathbb{R}$ and f is fully m-ic on $[0,1]$ then m is even and $\bar{f} > 0$ on $\{(x,y,z) \mid x \in [0,1],\, y^2 + z^2 = 1\}$.

We note as usual

$$\|f\|_\infty = \max\{|a_{ji}| \mid 0 \le i \le m,\ 0 \le j \le d\}.$$

We prove the following result.

Theorem 4 *Let $f \in \mathbb{R}[X,Y]$ with $f > 0$ on $[0,1] \times \mathbb{R}$, f fully m-ic on $[0,1]$, $d = \deg_X f \ge 2$ and*

$$f^\bullet = \min\{\bar{f}(x,y,z) \mid x \in [0,1],\, y^2+z^2 = 1\} > 0.$$

Then f can be written as in (1) with

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \le \frac{d^3(m+1)\|f\|_\infty}{f^\bullet}.$$

Note that the cases $\deg_X f = 0$ and $\deg_X f = 1$ are not covered by Theorem 4, but these cases are of a simpler nature. If $\deg_X f = 0$, f belongs to $\mathbb{R}[Y]$ and is non-negative on $\mathbb{R}$, then f can simply be written as a sum of squares in $\mathbb{R}[Y]$ with the degree of each term bounded by m (see [5, Proposition 1.2.1] and [4]). If $\deg_X f = 1$, we have

$$f(X,Y) = f(1,Y)X + f(0,Y)(1-X)$$

and, since $f(0,Y)$ and $f(1,Y)$ are non-negative on $\mathbb{R}$, again these polynomials can be written as sums of squares in $\mathbb{R}[Y]$ with the degree of each term bounded by m; then, using the identities

$$X = X^2 + X(1-X) \qquad \text{and} \qquad 1 - X = (1-X)^2 + X(1-X),$$

we take $\sigma_0 = f(1,Y)X^2 + f(0,Y)(1-X)^2$ and $\sigma_1 = f(1,Y) + f(0,Y)$ and the identity $f = \sigma_0 + \sigma_1 X(1-X)$ holds with the degree of each term bounded by $m+2$.

To prove Theorem 4, in Section 3.1 we show a constructive way of producing the representation in Theorem 1 in the case of f positive on $[0,1] \times \mathbb{R}$ and fully m-ic on $[0,1]$, and then we bound the degrees of each term. A similar constructive way of obtaining this representation was already given in [11, Proposition 3] under slightly different hypothesis. The idea behind the construction is to consider the unbounded variable as a parameter and to produce a uniform version of a representation theorem for the segment $[0,1]$ using the effective version of Pólya's Theorem from [10]. This technique was also used in related problems in [9] and [2].

Finally, in Section 3.2, we prove that the constructive method from the previous section also works in the case of f non-negative on the strip and having only a finite number of zeros, all of them lying on the boundary, and such that $\frac{\partial f}{\partial X}$ does not vanish at any of them.

2 The case $\deg_Y f \leq 2$

In this section we consider the problem of finding a degree bound for the representation in Theorem 1 under the assumption $\deg_Y f \leq 2$. Since it will be more convenient to homogenize with respect to the unbounded variable, we introduce the set

$$\mathcal{S} = [0,1] \times (\mathbb{R}^2 \setminus \{(0,0)\}) \subseteq \mathbb{R}^3.$$

It is easy to see that for $\bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2$ non-negative on $\mathcal{S}$ and $x_0 \in [0,1]$, $f_2(x_0) \geq 0$ and $f_0(x_0) \geq 0$ and either $f(x_0, Y, Z) = 0$ or $\deg_Y f(x_0, Y, Z)$ and $\deg_Z f(x_0, Y, Z)$ are even numbers; therefore, if $X - x_0 \mid f_2$ or $X - x_0 \mid f_0$, then $X - x_0 \mid f_1$. Moreover, if $x_0 \in (0,1)$ and $X - x_0 \mid f_2$, then $(X - x_0)^2 \mid f_2$. Similarly, if $x_0 \in (0,1)$ and $X - x_0 \mid f_0$, then $(X - x_0)^2 \mid f_0$.

We introduce the following cone.

Definition 5 *Given $d, e \in \mathbb{N}_0$, we define*

$$\mathcal{C}_{d,e} = \Big\{ \bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2 \in \mathbb{R}[X,Y,Z] \mid$$

$$\bar{f} \geq 0 \text{ on } \mathcal{S},\ \deg f_2 \leq d,\ \deg f_1 \leq \left\lfloor \frac{1}{2}(d+e) \right\rfloor,\ \deg f_0 \leq e \Big\}.$$

We can think of $\mathcal{C}_{d,e}$ as included in $\mathbb{R}^{d+\lfloor\frac{1}{2}(d+e)\rfloor+e+3}$ by identifying each $\bar{f} \in \mathcal{C}_{d,e}$ with its vector of coefficients in some prefixed order. It is easy to see that $\mathcal{C}_{d,e}$ is a closed cone which does not contain lines. Therefore, we can use the following well-known result (see for instance [13, Section 18]).

Theorem 6 *Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a closed cone which does not contain lines, then every element of $\mathcal{C}$ can be written as a sum of elements lying on extreme rays of $\mathcal{C}$.*

For a given $f \in \mathbb{R}[X,Y]$ non-negative on $[0,1] \times \mathbb{R}$, the strategy for proving that Theorem 2 holds for f is to use the classical idea of characterizing the extreme rays of $\mathcal{C}_{d,e}$, then to study the homogenized representation as in Theorem 1 for the elements lying on these rays, and finally to decompose $\bar{f}$ as a sum of them.

Under the additional hypothesis that d and e have the same parity, our characterization of the extreme rays of $\mathcal{C}_{d,e}$ is the following.

Theorem 7 *Let $d, e \in \mathbb{N}_0$ such that $d \equiv e\,(2)$. The extreme rays of $\mathcal{C}_{d,e}$ are the rays generated by the polynomials of the form $r(X)(p(X)Y + q(X)Z)^2$ with*

- *p and q not simultaneously zero and $(p : q) = 1$,*
- *$r \neq 0$, $r \geq 0$ on $[0,1]$ and r with $\deg r$ real roots in $[0,1]$ (counted with multiplicity),*
- *$2\deg p \leq d, 2\deg q \leq e$ and $\deg r = \min\{d - 2\deg p, e - 2\deg q\}$.*

To prove Theorem 7, the idea is to proceed inductively on a sequence of cones ordered *by inclusion*. To do so, we need to show first that given $\bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2 \in \mathcal{C}_{d,e}$ some factors of $f_2(X)$ or $f_0(X)$ are necessarily also factors of $f_1(X)$; in this case, after removing these factors we move to a smaller cone.

The following lemmas are some basic auxiliary results concerning extreme rays of $\mathcal{C}_{d,e}$.

Lemma 8 *Let $d, e \in \mathbb{N}_0$ and let $\bar{f}$ be a generator of an extreme ray of $\mathcal{C}_{d,e}$. Then $\bar{f}$ vanishes at some point of $\mathcal{S}$.*

Proof: Suppose $\bar{f} > 0$ on $\mathcal{S}$ and take

$$c = \min\{\bar{f}(x, y, z) \mid x \in [0, 1],\, y^2 + z^2 = 1\} > 0.$$

Consider cY^2, $c(Y^2 + Z^2) \in \mathcal{C}_{d,e}$. We have

$$0 \le cY^2 \le c(Y^2 + Z^2) \le \bar{f} \ \text{ on } \mathcal{S},$$

but since $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$, $\bar{f}$ is a scalar multiple of both cY^2 and $c(Y^2 + Z^2)$ which is impossible. $\square$

Lemma 9 *Let $d, e \in \mathbb{N}_0$ and let $\bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2$ be a generator of an extreme ray of $\mathcal{C}_{d,e}$. If $f_2 = 0$, $f_1 = 0$ or $f_0 = 0$, then $\bar{f}$ is of the form*

$$r(X)Y^2 \ \text{or}\ r(X)Z^2.$$

Proof: If $f_2 = 0$ then $f_1 = 0$, $\bar{f} = f_0(X)Z^2$ and we take $r(X) = f_0(X)$. Similarly, if $f_0 = 0$ then $f_1 = 0$, $\bar{f} = f_2(X)Y^2$ and we take $r(X) = f_2(X)$. On the other hand, if $f_1 = 0$ and $f_2, f_0 \neq 0$, then

$$0 \le f_2(X)Y^2 \le f_2(X)Y^2 + f_0(X)Z^2 = \bar{f} \ \text{ on } \mathcal{S}$$

which, proceeding similarly to the proof of Lemma 8, is impossible. $\square$

The following lemma shows that the second and third condition in the characterization of the extreme rays in Theorem 7 are indeed consequences of the first condition.

Lemma 10 *Let $d, e \in \mathbb{N}_0$. If $r(X)(p(X)Y + q(X)Z)^2$ with p and q not simultaneously zero and $(p : q) = 1$ generates an extreme ray of $\mathcal{C}_{d,e}$, then*

- *$r \neq 0$, $r \geq 0$ on $[0, 1]$ and r has $\deg r$ real roots in $[0, 1]$ (counted with multiplicity),*
- *$2 \deg p \le d, 2 \deg q \le e$ and $\deg r = \min\{d - 2 \deg p, e - 2 \deg q\}$.*

Proof: Let $\bar{f} = r(X)(p(X)Y + q(X)Z)^2$. Since $\bar{f} \neq 0$, $r \neq 0$, and since $\bar{f} \geq 0$ on $\mathcal{S}$, $r \geq 0$ on $[0, 1]$. If r has a complex non-real root, or a real root which does not belong to the interval $[0, 1]$, it is easy to see that r can be written as $r = r_1 + r_2$ with $r_1, r_2 \in \mathbb{R}[X] - \{0\}$, $\deg r_1, \deg r_2 \leq \deg r$, $\deg r_1 \neq \deg r_2$ and $r_1, r_2 \geq 0$ on $[0,1]$. Then for $i = 1,2$, we take $f_i = r_i(X)(p(X)Y + q(X)Z)^2 \in \mathcal{C}_{d,e}$ and we have

$$0 \leq f_i \leq \bar{f} \text{ on } \mathcal{S},$$

but since $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$, $\bar{f}$ is a scalar multiple of both f_1 and f_2 which is impossible. Since $\bar{f} \in \mathcal{C}_{d,e}$, we have $2\deg p \leq d, 2\deg q \leq e$ and $\deg r \leq \min\{d - 2\deg p, e - 2\deg q\}$. If $\deg r < \min\{d - 2\deg p, e - 2\deg q\}$, we have $X\bar{f} \in \mathcal{C}_{d,e}$ and

$$0 \leq X\bar{f} \leq \bar{f} \text{ on } \mathcal{S}$$

which is again impossible for similar reasons. □

In order to prove Theorem 7, we will do several changes of variables. The following three lemmas summarize the properties we need. We omit their proofs since they are very simple.

Lemma 11 *Let $d, e \in \mathbb{N}_0$ with $d \leq e$, $\bar{f} \in \mathcal{C}_{d,e}$, $\beta \in \mathbb{R}$ and $h \in \mathbb{R}[X,Y,Z]$ defined by*

$$h(X,Y,Z) = \bar{f}(X, Y + \beta Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

Then:

- *h belongs to $\mathcal{C}_{d,e}$.*
- *If $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$, then h generates an extreme ray of $\mathcal{C}_{d,e}$.*
- *If $(x_0, y_0, z_0) \in \mathcal{S}$ with $z_0 \neq 0$ and $\bar{f}(x_0, y_0, z_0) = 0$ and $\beta = y_0/z_0$, then $h_0(x_0) = 0$.*
- *If h can be written as $r(X)(p(X)Y + q(X)Z)^2$ with p and q not simultaneously zero and $(p : q) = 1$, then $\bar{f}$ can be written as*

$$r(X)(p(X)Y + (-\beta p(X) + q(X))Z)^2$$

  with p and $-\beta p + q$ not simultaneously zero and $(p : -\beta p + q) = 1$.

Lemma 12 *Let $d, e \in \mathbb{N}_0$ with $d + 2 \leq e$, $\bar{f} \in \mathcal{C}_{d,e}$, $\ell \in \mathbb{R}[X]$ with $\deg \ell = 1$ and $h \in \mathbb{R}[X,Y,Z]$ defined by*

$$h(X,Y,Z) = \bar{f}(X, Y + \ell(X)Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

Then:

- *h belongs to $\mathcal{C}_{d,e}$.*
- *If $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$, then h generates an extreme ray of $\mathcal{C}_{d,e}$.*
- *If (x_0, y_0, z_0), $(x_1, y_1, z_1) \in \mathcal{S}$ with $x_0 \neq x_1$, $z_0, z_1 \neq 0$, $y_0/z_0 \neq y_1/z_1$ and $f(x_0, y_0, z_0) = f(x_1, y_1, z_1) = 0$ and*

$$\ell(X) = \frac{y_1/z_1 - y_0/z_0}{x_1 - x_0}(X - x_0) + y_0/z_0,$$

  then $h_0(x_0) = h_0(x_1) = 0$.

- *If h can be written as $r(X)(p(X)Y + q(X)Z)^2$ with p and q not simultaneously zero and $(p : q) = 1$, then $\bar{f}$ can be written as*

$$r(X)(p(X)Y + (-\ell(X)p(X) + q(X))Z)^2$$

with p and $-\ell p + q$ not simultaneously zero and $(p : -\ell p + q) = 1$.

Lemma 13 *Let $d, e \in \mathbb{N}_0$ with $d = e$, $\bar{f} \in \mathcal{C}_{d,e}$, $\beta_0, \beta_1 \in \mathbb{R}$ with $\beta_0 \neq \beta_1$ and $h \in \mathbb{R}[X, Y, Z]$ defined by*

$$h(X, Y, Z) = f(X, \beta_0 Y + \beta_1 Z, Y + Z) = h_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

Then:

- *h belongs to $\mathcal{C}_{d,e}$.*
- *If $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$, then h generates an extreme ray of $\mathcal{C}_{d,e}$.*
- *If (x_0, y_0, z_0), $(x_1, y_1, z_1) \in \mathcal{S}$ with $z_0, z_1 \neq 0$, $y_0/z_0 \neq y_1/z_1$ and $f(x_0, y_0, z_0) = f(x_1, y_1, z_1) = 0$ and $\beta_0 = y_0/z_0$, $\beta_1 = y_1/z_1$, then $h_2(x_0) = h_0(x_1) = 0$.*
- *If h can be written as $r(X)(p(X)Y + q(X)Z)^2$ with p and q not simultaneously zero and $(p : q) = 1$, then $\bar{f}$ can be written as*

$$\frac{1}{(\beta_0 - \beta_1)^2} r(X)((p(X) - q(X))Y + (-\beta_1 p(X) + \beta_0 q(X))Z)^2$$

with $p - q$ and $-\beta_1 p + \beta_0 q$ not simultaneously zero and $(p - q : -\beta_1 p + \beta_0 q) = 1$.

We are ready to prove the characterization of the extreme rays of the cone $\mathcal{C}_{d,e}$ given in Theorem 7.

Proof of Theorem 7: We begin by proving that if $\bar{f} = r(X)(p(X)Y + q(X)Z)^2$ with r, p and q as in the statement of Theorem 7, then $\bar{f}$ generates an extreme ray of $\mathcal{C}_{d,e}$. Consider

$$g = g_2(X)Y^2 + g_1(X)YZ + g_0(X)Z^2 \in \mathcal{C}_{d,e}$$

such that $0 \leq g \leq \bar{f}$ on $\mathcal{S}$. We want to show that g is a scalar multiple of $\bar{f}$.

If $p = 0$, since $(p : q) = 1$ we have $q = \lambda \in \mathbb{R} \setminus \{0\}$ and then $\deg r = e$. On the other hand, for every $x \in [0, 1]$, $\bar{f}(x, 1, 0) = 0$. Then, for every $x \in [0, 1]$, $g_2(x) = g(x, 1, 0) = 0$ and this implies $g_2 = g_1 = 0$. Therefore, $g = g_0(X)Z^2$, but since $0 \leq g \leq \bar{f}$ on $\mathcal{S}$, $0 \leq g_0 \leq \lambda^2 r$ on $[0, 1]$. It is easy to see that every root of r is necessarily also a root of g_0 with at least the same multiplicity, then we have $\deg r \leq \deg g_0 \leq e = \deg r$, g_0 is a scalar multiple of r and g is a scalar multiple of $\bar{f}$.

If $p \neq 0$, we consider $G \in \mathbb{R}[X, Y, Z]$ defined by

$$G(X, Y, Z) = p(X)^2 g(X, Y, Z) = g_2(X)(p(X)Y + q(X)Z)^2 + G_1(X)YZ + G_0(X)Z^2.$$

We first see that $G_1 = G_0 = 0$. Take $x_0 \in [0, 1]$ such that $p(x_0) \neq 0$. Since $\bar{f}(x_0, -q(x_0), p(x_0)) = 0$, $G(x_0, -q(x_0), p(x_0)) = 0$ and then

$$-G_1(x_0)q(x_0)p(x_0) + G_0(x_0)p(x_0)^2 = 0. \tag{2}$$

Moreover, since $G \geq 0$ on $\mathcal{S}$,

$$\frac{\partial G}{\partial Y}(x_0, -q(x_0), p(x_0)) = G_1(x_0)p(x_0) = 0. \tag{3}$$

We conclude from (2) and (3) that $G_1(x_0) = G_0(x_0) = 0$. This implies $G_1 = G_0 = 0$ and then $p(X)^2 g(X,Y,Z) = g_2(X)(p(X)Y + q(X)Z)^2$. Since $(p:q) = 1$, $p^2 \mid g_2$ and $g = \tilde{g}_2(X)(p(X)Y + q(X)Z)^2$ for $\tilde{g}_2 = g_2/p^2 \in \mathbb{R}[X]$. Reasoning similarly to the case $p = 0$, we see that $\tilde{g}_2$ is a scalar multiple of r and g is a scalar multiple of $\bar{f}$.

Now we prove that if $\bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2$ generates an extreme ray of $\mathcal{C}_{d,e}$ then $\bar{f}$ can be written as in the statement of Theorem 7. To do so, we use inductive arguments, considering the families of cones ordered *by inclusion*, this is to say,

$$\mathcal{C}_{d_1,e_1} \leq \mathcal{C}_{d_2,e_2} \qquad \text{if} \qquad d_1 \leq d_2 \text{ and } e_1 \leq e_2.$$

Actually, for $(d,e) = (0,0)$, the result is easy to check using Lemma 8, so from now on we assume $(d,e) \neq (0,0)$. Using Lemma 9 and Lemma 10, we can assume $f_2, f_1, f_0 \neq 0$.

First, we prove the result in two particular cases.

A1. There is $x_0 \in [0,1]$ such that $(X - x_0)^2 \mid f_2$ or $(X - x_0)^2 \mid f_0$:

Without loss of generality, suppose $(X - x_0)^2 \mid f_2$, then $X - x_0 \mid f_1$. Consider $h_2 = f_2/(X - x_0)^2$, $h_1 = f_1/(X - x_0) \in \mathbb{R}[X]$ and

$$h = h_2(X)Y^2 + h_1(X)YZ + f_0(X)Z^2 \in \mathbb{R}[X,Y,Z],$$

then

$$h(X, (X - x_0)Y, Z) = \bar{f}(X,Y,Z) \qquad \text{and} \qquad h(X,Y,Z) = \bar{f}\left(X, \frac{Y}{X - x_0}, Z\right)$$

Note that $h \in \mathcal{C}_{d-2,e}$. Indeed, h verifies the degree bounds and $h \geq 0$ on $\{(x,y,z) \in \mathcal{S} \mid x \neq x_0\}$, by continuity, $h \geq 0$ on $\mathcal{S}$. In order to apply the inductive hypothesis, let us prove that h generates an extreme ray of $\mathcal{C}_{d-2,e}$. Given

$$g = g_2(X)Y^2 + g_1(X)YZ + g_0(X)Z^2 \in \mathcal{C}_{d-2,e}$$

such that $0 \leq g \leq h$ on $\mathcal{S}$, we consider

$$\tilde{g} = (X - x_0)^2 g_2(X)Y^2 + (X - x_0)g_1(X)YZ + g_0(X)Z^2 \in \mathbb{R}[X,Y,Z],$$

since $\tilde{g}(X,Y,Z) = g(X, (X - x_0)Y, Z)$, $\tilde{g} \in \mathcal{C}_{d,e}$ and $0 \leq \tilde{g} \leq \bar{f}$ on $\mathcal{S}$. Therefore, $\tilde{g}$ is a scalar multiple of $\bar{f}$ and g is a scalar multiple of h.

By the inductive hypothesis, h is of the form

$$h(X,Y,Z) = \tilde{r}(X)(\tilde{p}(X)Y + \tilde{q}(X)Z)^2$$

with $\tilde{p}$ and $\tilde{q}$ not simultaneously zero and $(\tilde{p} : \tilde{q}) = 1$. Then,

$$\bar{f}(X, Y, Z) = \tilde{r}(X)((X - x_0)\tilde{p}(X)Y + \tilde{q}(X)Z)^2.$$

If $X - x_0 \not| \tilde{q}$, we take $r = \tilde{r}$, $p = (X - x_0)\tilde{p}$ and $q = \tilde{q}$, and if $X - x_0 \,|\, \tilde{q}$, we take $r = (X - x_0)^2\tilde{r}$, $p = \tilde{p}$ and $q = \tilde{q}/(X - x_0) \in \mathbb{R}[X]$. In both cases we have $(p : q) = 1$ and we conclude using Lemma 10.

A2. There is $x_0 \in [0, 1]$ such that $X - x_0 \,|\, f_2, f_0$:

It is clear that $X - x_0 \,|\, f_1$. If $x_0 \in (0, 1)$ it is easy to see that $(X - x_0)^2 \,|\, f_2$ and then we are in case A1, so we can suppose $x_0 \in \{0, 1\}$. Without loss of generality assume $x_0 = 0$. Consider $h = \bar{f}/X \in \mathbb{R}[X, Y, Z]$. Proceeding as in case A1, it is easy to see that h generates an extreme ray of $\mathcal{C}_{d-1,e-1}$, and using the inductive hypothesis we have h is of the form

$$h(X, Y, Z) = \tilde{r}(X)(\tilde{p}(X)Y + \tilde{q}(X)Z)^2$$

with $\tilde{p}$ and $\tilde{q}$ not simultaneously zero and $(\tilde{p} : \tilde{q}) = 1$. Then we take $r = X\tilde{r}$, $p = \tilde{p}$ and $q = \tilde{q}$ and we conclude using Lemma 10.

We consider now an auxiliary list of cases in which we prove the result by reducing to cases A1 and A2.

B1. There are $x_0 \in \{0, 1\}$ and $(y_0, z_0) \in \{(1, 0), (0, 1)\}$ such that $\bar{f}(x_0, y_0, z_0) = 0$ and $\bar{f}(x, y, z) \neq 0$ for every $(x, y, z) \in \mathcal{S}$ with $x \neq x_0$:

Without loss of generality, suppose $\bar{f}(0, 1, 0) = 0$, then $f_2(0) = 0$ and $X \,|\, f_1$. If $X^2 \,|\, f_2$ we are in case A1 and if $X \,|\, f_0$ we are in case A2. Moreover, if there is $x \in (0, 1]$ with $f_2(x) = 0$, then $\bar{f}(x, 1, 0) = 0$ which contradicts the hypothesis. Similarly, if there is $x \in (0, 1]$ with $f_0(x) = 0$, then $\bar{f}(x, 0, 1) = 0$ which also contradicts the hypothesis. So from now on we assume $X^2 \nmid f_2$, $f_2 > 0$ on $(0, 1]$ and $f_0 > 0$ on $[0, 1]$.

Consider $g_2 = f_2/X, g_1 = f_1/X \in \mathbb{R}[X]$ and note that $g_2 > 0$ in $[0, 1]$. Since $\bar{f}(x, y, z) > 0$ for $(x, y, z) \in \mathcal{S}$ with $x \in (0, 1]$,

$$f_1(x)^2 - 4f_2(x)f_0(x) = x^2g_1^2(x) - 4xg_2(x)f_0(x) < 0,$$

for $x \in (0, 1]$, and then

$$xg_1^2(x) - 4g_2(x)f_0(x) < 0$$

for $x \in (0, 1]$, but since $g_2(0) > 0$ and $f_0(0) > 0$, this last inequality can be extended to $x \in [0, 1]$. We take $\varepsilon > 0$ such that

$$\frac{xg_1^2(x)}{4g_2(x)} - f_0(x) \leq -\varepsilon$$

for $x \in [0, 1]$. Therefore,

$$f_1(x)^2 - 4f_2(x)(f_0(x) - \varepsilon) = x^2g_1^2(x) - 4xg_2(x)(f_0(x) - \varepsilon) \leq 0$$

for $x \in [0, 1]$. Let $h = f_2(X)Y^2 + f_1(X)YZ + (f_0(X) - \varepsilon)Z^2 \in \mathbb{R}[X, Y, Z]$. It follows easily that $h \in \mathcal{C}_{d,e}$ and $0 \leq h \leq \bar{f}$ on $\mathcal{S}$, but then h is a scalar multiple of $\bar{f}$ which is impossible.

B2. There is $(y_0, z_0) \in \{(1,0),(0,1)\}$ such that $\bar{f}(0, y_0, z_0) = \bar{f}(1, y_0, z_0) = 0$ and $\bar{f}(x, y, z) \neq 0$ for every $(x, y, z) \in \mathcal{S}$ with $x \in (0,1)$:

Without loss of generality, suppose $\bar{f}(0,1,0) = \bar{f}(1,1,0) = 0$, then $f_2(0) = f_2(1) = 0$ and therefore $X \mid f_1$ and $X-1 \mid f_1$. If $X^2 \mid f_2$ or $(X-1)^2 \mid f_2$ we are in case A1 and if $X \mid f_0$ or $X-1 \mid f_0$ we are in case A2. Moreover, if there is $x \in (0,1)$ with $f_2(x) = 0$, then $\bar{f}(x,1,0) = 0$ which contradicts the hypothesis. Similarly, if there is $x \in (0,1)$ with $f_0(x) = 0$, then $\bar{f}(x,0,1) = 0$ which also contradicts the hypothesis. So from now on we assume $X^2 \nmid f_2$, $(X-1)^2 \nmid f_2$, $f_2 > 0$ on $(0,1)$ and $f_0 > 0$ on $[0,1]$.

Consider $g_2 = f_2/(X(X-1)), g_1 = f_1/(X(X-1)) \in \mathbb{R}[X]$ and note that $g_2 < 0$ in $[0,1]$. Since $\bar{f}(x,y,z) > 0$ for $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$,

$$f_1(x)^2 - 4f_2(x)f_0(x) = x^2(x-1)^2 g_1^2(x) - 4x(x-1)g_2(x)f_0(x) < 0,$$

for $x \in (0,1)$, and then

$$x(x-1)g_1^2(x) - 4g_2(x)f_0(x) > 0$$

for $x \in (0,1)$, but since $g_2(0) < 0, g_2(1) < 0, f_0(0) > 0$ and $f_0(1) > 0$, this last inequality can be extended to $x \in [0,1]$. We take $\varepsilon > 0$ such that

$$\frac{x(x-1)g_1^2(x)}{4g_2(x)} - f_0(x) \leq -\varepsilon$$

for $x \in [0,1]$. The proof is finished using the same arguments as in case B1.

B3. There are $(y_0, z_0), (y_1, z_1) \in \{(1,0),(0,1)\}$, $(y_0, z_0) \neq (y_1, z_1)$ such that $\bar{f}(0, y_0, z_0) = \bar{f}(1, y_1, z_1) = 0$ and $\bar{f}(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$:

Without loss of generality, suppose $f(0,1,0) = f(1,0,1) = 0$, then $f_2(0) = f_0(1) = 0$ and therfore $X \mid f_1$ and $X-1 \mid f_1$. If $X^2 \mid f_2$ or $(X-1)^2 \mid f_0$ we are in case A1 and if $X \mid f_0$ or $X-1 \mid f_2$ we are in case A2. Moreover, if there is $x \in (0,1)$ with $f_2(x) = 0$, then $\bar{f}(x,1,0) = 0$ which contradicts the hypothesis. Similarly, if there is $x \in (0,1)$ with $f_0(x) = 0$, then $\bar{f}(x,0,1) = 0$ which also contradicts the hypothesis. So from now on we assume $X^2 \nmid f_2$, $(X-1)^2 \nmid f_0$, $f_2 > 0$ on $(0,1]$ and $f_0 > 0$ on $[0,1)$.

Consider $g_2 = f_2/X$, $g_1 = f_1/(X(X-1))$, $g_0 = f_0/(X-1) \in \mathbb{R}[X]$ and note that $g_2 > 0$ in $[0,1]$ and $g_0 < 0$ in $[0,1]$. Since $\bar{f}(x,y,z) > 0$ for $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$,

$$f_1(x)^2 - 4f_2(x)f_0(x) = x^2(x-1)^2 g_1^2(x) - 4x(x-1)g_2(x)g_0(x) < 0$$

for $x \in (0,1)$, and then

$$x(x-1)g_1^2(x) - 4g_2(x)g_0(x) > 0$$

for $x \in (0,1)$, but since $g_2(0) > 0, g_2(1) > 0, g_0(0) < 0$ and $g_0(1) < 0$, this last inequality can be extended to $x \in [0,1]$. We take $\varepsilon > 0$ such that

$$\frac{x(x-1)g_1^2(x)}{4g_2(x)} - g_0(x) \geq \varepsilon$$

for $x \in [0,1]$. Therefore,

$$f_1(x)^2 - 4f_2(x)(x-1)(g_0(x)+\varepsilon) = x^2(x-1)^2 g_1^2(x) - 4x(x-1)g_2(x)(g_0(x)+\varepsilon) \leq 0$$

for $x \in [0,1]$. Let $h = f_2(X)Y^2 + f_1(X)YZ + (X-1)(g_0(X)+\varepsilon)Z^2 \in \mathbb{R}[X,Y,Z]$. It follows easily that $h \in \mathcal{C}_{d,e}$ and $0 \leq h \leq \bar{f}$ on $\mathcal{S}$, but then h is a scalar multiple of $\bar{f}$ which is impossible.

We prove now the general case. Without loss of generality we suppose $d \leq e$. By Lemma 8, $\bar{f}$ vanishes at some point of $\mathcal{S}$. To prove the result we are going to consider three final cases.

C1. There is $(x_0, y_0, z_0) \in \mathcal{S}$ with $x_0 \in (0,1)$ such that $\bar{f}(x_0, y_0, z_0) = 0$:

If $z_0 = 0$, $X - x_0 \mid f_2$, then $(X-x_0)^2 \mid f_2$ and we are in case A1. If $z_0 \neq 0$ we take $\beta = y_0/z_0$ and consider

$$h(X,Y,Z) = \bar{f}(X, Y+\beta Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 11, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_0(x_0) = 0$. Then $(X-x_0)^2 \mid h_0$ and by case A1 applied to h and Lemma 11 the result follows.

C2. There are $x_0 \in \{0,1\}$ and $(y_0, z_0) \in \mathcal{S}$ such that $\bar{f}(x_0, y_0, z_0) = 0$ and $\bar{f}(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \neq x_0$:

Without loss of generality, suppose $x_0 = 0$. If $z_0 = 0$, we can assume $y_0 = 1$ and we are in case B1. If $z_0 \neq 0$, we take $\beta = y_0/z_0$ and consider

$$h(X,Y,Z) = \bar{f}(X, Y+\beta Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 11, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_0(0) = 0$ and $h(0,0,1) = 0$. In addition, $h(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \neq 0$. By case B1 applied to h and Lemma 11 the result follows.

C3. There are $(y_0, z_0), (y_1, z_1) \in \mathcal{S}$ such that $\bar{f}(0, y_0, z_0) = \bar{f}(1, y_1, z_1) = 0$ and $\bar{f}(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$:

If $z_0 = z_1 = 0$, we can assume $y_0 = y_1 = 1$ and we are in case B2.

If $z_0 \neq 0$ and $z_1 = 0$, we take $\beta = y_0/z_0$ and consider

$$h(X,Y,Z) = \bar{f}(X, Y+\beta Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 11, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_0(0) = 0$ and $h(0,0,1) = 0$. On the other hand, since $\bar{f}(1, y_1, 0) = 0$, $f_2(1) = 0$ and $h(1,1,0) = 0$. In addition, $h(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$. By case B3 applied to h and Lemma 11 the result follows. If $z_0 = 0$ and $z_1 \neq 0$ we proceed similarly to the case $z_0 \neq 0$ and $z_1 = 0$.

The final case is $z_0, z_1 \neq 0$, but we need to split it in three cases.

If $z_0, z_1 \neq 0$ and $y_0/z_0 = y_1/z_1$, we take $\beta = y_0/z_0$ and consider

$$h(X,Y,Z) = \bar{f}(X, Y+\beta Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 11, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_0(0) = h_0(1) = 0$, then $h(0,0,1) = h(1,0,1) = 0$. In addition, $h(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$. By case B2 applied to h and Lemma 11 the result follows.

If $z_0, z_1 \neq 0$ with $y_0/z_0 \neq y_1/z_1$ and $d = e$, we take $\beta_0 = y_0/z_0$ and $\beta_1 = y_1/z_1$ and consider

$$h(X,Y,Z) = \bar{f}(X, \beta_0 Y + \beta_1 Z, Y + Z) = h_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 13, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_2(0) = h_0(1) = 0$, then $h(0,1,0) = h(1,0,1) = 0$. In addition, $h(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$. By case B3 applied to h and Lemma 13 the result follows.

Finally, if $z_0, z_1 \neq 0$ with $y_0/z_0 \neq y_1/z_1$ and $d < e$, since $d \equiv e(2)$, $d + 2 \leq e$. Then, we take

$$\ell(X) = (y_1/z_1 - y_0/z_0)X + y_0/z_0$$

and consider

$$h(X,Y,Z) = \bar{f}(X, Y + \ell(X)Z, Z) = f_2(X)Y^2 + h_1(X)YZ + h_0(X)Z^2.$$

By Lemma 12, h generates an extreme ray of $\mathcal{C}_{d,e}$ and verifies $h_0(0) = h_0(1) = 0$, then, $h(0,0,1) = h(1,0,1) = 0$. In addition, $h(x,y,z) \neq 0$ for every $(x,y,z) \in \mathcal{S}$ with $x \in (0,1)$. By case B2 applied to h and Lemma 12 the result follows.

$\square$

Finally, we deduce Theorem 2.

Proof of Theorem 2: Take $d = e = \deg_X f$, then $\bar{f} = f_2(X)Y^2 + f_1(X)YZ + f_0(X)Z^2 \in \mathcal{C}_{d,e}$ (note that we *homogenize to degree* 2 even in the case $\deg_Y f = 0$). By Theorems 6 and 7,

$$\bar{f} = \sum_{1 \leq i \leq s} r_i(p_i Y + q_i Z)^2$$

for some $r_i, p_i, q_i \in \mathbb{R}[X]$ as in Theorem 7 for $1 \leq i \leq s$. By studying the factorization in $\mathbb{C}[X]$ of each $r_i \in \mathbb{R}[X]$, it is easy to see that the condition $r_i \geq 0$ on $[0,1]$ implies that there exist $t_i, u_i, v_i, w_i \in \sum \mathbb{R}[X]^2$ such that

$$r_i = t_i + u_i X + v_i(1 - X) + w_i X(1 - X)$$

with $\deg t_i, \deg u_i X, \deg v_i(1 - X), \deg w_i X(1 - X) \leq \deg r_i$. Using the identities

$$X = X^2 + X(1 - X) \qquad \text{and} \qquad 1 - X = (1 - X)^2 + X(1 - X),$$

we take

$$\sigma_0 = \sum_{1 \leq i \leq s} (t_i + u_i X^2 + v_i(1 - X)^2)(p_i Y + q_i)^2$$

and

$$\sigma_1 = \sum_{1 \le i \le s} (u_i + v_i + w_i)(p_i Y + q_i)^2$$

and the identity $f = \sigma_0 + \sigma_1 X(1-X)$ holds. Finally,

$$\deg(\sigma_0) \le \max_{1\le i\le s} \deg(t_i + u_i X^2 + v_i(1-X)^2)(p_i Y + q_i)^2 \le \max_{1\le i\le s} \deg r_i (p_i Y + q_i)^2 + 1 \le \deg_X f + 3$$

and

$$\deg(\sigma_1 X(1-X)) \le \max_{1\le i\le s} \deg(u_i + v_i + w_i)(p_i Y + q_i)^2 X(1-X) \le$$
$$\le \max_{1\le i\le s} \deg r_i (p_i Y + q_i)^2 + 1 \le \deg_X f + 3.$$

□

3 A constructive approach

In this section we show, under certain hypothesis, a constructive approach which also provides a degree bound for each term in the representation in Theorem 1. This approach works in the case that f is positive on the strip and fully m-ic on $[0,1]$ (Section 3.1) and in the case that f is non-negative on the strip, fully m-ic on $[0,1]$, and has only a finite number of zeros, all of them lying on the boundary of the strip and such that $\frac{\partial f}{\partial x}$ does not vanish at any of them (Section 3.2). Finally, we will see in Example 20 that this approach does not work in the general case.

Roughly speaking, the main idea is to lift the interval $[0,1]$ to the standard 1-dimensional simplex

$$\Delta_1 = \{(w,x) \in \mathbb{R}^2 \mid w \ge 0,\, x \ge 0,\, w + x = 1\},$$

to consider Y as a parameter and to produce for each evaluation of Y a certificate of non-negativity on Δ_1 using the effective version of Pólya's Theorem from [10] in a suitable manner so that these certificates can be glued together. We introduce a variable W which is used to lift the interval $[0,1]$ to the simplex Δ_1 and, as before, a variable Z which is used to compactify $\mathbb{R}$.

Notation 14 *Given*

$$f = \sum_{0\le i\le m} \sum_{0\le j\le d} a_{ji} X^j Y^i \in \mathbb{R}[X,Y],$$

define

$$F = \sum_{0\le i\le m} \sum_{0\le j\le d} a_{ji} X^j (W+X)^{d-j} Y^i Z^{m-i} \in \mathbb{R}[W,X,Y,Z].$$

For $N \in \mathbb{N}_0$ and $0 \le j \le N+d$, we define the polynomials $b_j \in \mathbb{R}[Y,Z]$ as follows:

$$(W+X)^N F = \sum_{0\le j\le N+d} b_j(Y,Z) W^j X^{N+d-j}. \tag{4}$$

Note that $(W+X)^N F$ is homogeneous on (W,X) and (Y,Z) of degree $N+d$ and m respectively. Therefore, for $0 \le j \le N+d$, $b_j \in \mathbb{R}[Y,Z]$ is a homogeneous polynomial of degree m.

We introduce the notation

$$C = \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 = 1\}.$$

Proposition 15 *Let $f \in \mathbb{R}[X,Y]$ and $N \in \mathbb{N}_0$ such that for $0 \le j \le N+d$, $b_j \ge 0$ on C. Then f can be written as in (1) with*

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \le N+d+m+1.$$

Proof: Substituting $W = 1-X$ and $Z = 1$ in (4) we have

$$f(X,Y) = \sum_{0 \le j \le N+d} b_j(Y,1)(1-X)^j X^{N+d-j}.$$

For $0 \le j \le N+d$, since $b_j(Y,Z) \ge 0$ on C and b_j is homogeneous, we have $b_j(Y,1) \ge 0$ on $\mathbb{R}$ and therefore $b_j(Y,1)$ is a sum of squares in $\mathbb{R}[Y]$ (see [5, Proposition 1.2.1]) with the degree of each term bounded by m.

If $N+d$ is even, we take

$$\sigma_0 = \sum_{0 \le j \le N+d,\ j \text{ even}} b_j(Y,1)(1-X)^j X^{N+d-j}$$

and

$$\sigma_1 = \sum_{1 \le j \le N+d-1,\ j \text{ odd}} b_j(Y,1)(1-X)^{j-1} X^{N+d-j-1}$$

and the identity $f = \sigma_0 + \sigma_1 X(1-X)$ holds. In addition, we have

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \le N+d+m.$$

If $N+d$ is odd, using the identities

$$X = X^2 + X(1-X) \qquad \text{and} \qquad 1-X = (1-X)^2 + X(1-X),$$

we take

$$\sigma_0 = \sum_{0 \le j \le N+d-1,\ j \text{ even}} b_j(Y,1)(1-X)^j X^{N+d-j+1} + \sum_{1 \le j \le N+d,\ j \text{ odd}} b_j(Y,1)(1-X)^{j+1} X^{N+d-j}$$

and

$$\sigma_1 = \sum_{0 \le j \le N+d-1,\ j \text{ even}} b_j(Y,1)(1-X)^j X^{N+d-j-1} + \sum_{1 \le j \le N+d,\ j \text{ odd}} b_j(Y,1)(1-X)^{j-1} X^{N+d-j}$$

and the identity $f = \sigma_0 + \sigma_1 X(1-X)$ holds. In addition, we have

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \le N+d+m+1.$$

□

In Section 3.1 and Section 3.2, under certain hypothesis, we prove the existence and find an upper bound for $N \in \mathbb{N}_0$ satisfying the hypothesis of Proposition 15. Then, to obtain the representation (1) we proceed as follows. If it possible to compute the upper bound, we compute the expansion of the polynomial $(W+X)^N F$ and then we compute the representation of each $b_j(Y,1)$ as a sum of squares in $\mathbb{R}[Y]$ (see [4]). If it is not possible to compute the upper bound, we pick a value of N and we proceed by increasing N one by one, we check symbolically at each step if it is the case that $b_j(Y,1)$ is non-negative on $\mathbb{R}$ for every $0 \le j \le N+d$ (see [1, Chapter 4] and [8]), and once this condition is satisfied we compute the representation of each $b_j(Y,1)$ as a sum of squares in $\mathbb{R}[Y]$.

For a homogeneous polynomial

$$g = \sum_{0 \le j \le d} c_j W^j X^{d-j} \in \mathbb{R}[W,X]$$

we note, as in [10],

$$\|g\| = \max \left\{ \frac{|c_j|}{\binom{d}{j}} \mid 0 \le j \le d \right\}.$$

One of the main tools we use is the effective version of Pólya's Theorem from [10]. In the case of a homogeneous polynomial $g \in \mathbb{R}[W,X]$ which is positive on Δ_1, this theorem states that after multiplying for a suitable power of $W+X$, every coefficient is positive. Since we will need an explicit positive lower bound for these coefficients, we present in Lemma 16 a slight adaptation of [10, Theorem 1]. We omit its proof since it can be developed exactly as the proof of [10, Theorem 1] with only a minor modification at the final step.

Lemma 16 *Let $g \in \mathbb{R}[W,X]$ homogeneous of degree d with $g > 0$ on Δ_1 and let $\lambda = \min_{\Delta_1} g > 0$. For $0 \le \epsilon < 1$, if*

$$N + d \ge \frac{(d-1)d\|g\|}{2(1-\epsilon)\lambda},$$

for $0 \le j \le N+d$ the coefficient of $W^j X^{N+d-j}$ in $(W+X)^N g$ is greater than or equal to $\frac{N!(N+d)^d}{j!(N+d-j)!}\epsilon\lambda$.

3.1 The case of f positive on the strip

In this section, we study the case of f positive on $[0,1] \times \mathbb{R}$ and fully m-ic on $[0,1]$ and we prove Theorem 4.

Proposition 17 *Let $f \in \mathbb{R}[X,Y]$ with $f > 0$ on $[0,1] \times \mathbb{R}$, f fully m-ic on $[0,1]$ and*

$$f^\bullet = \min\{\bar{f}(x,y,z) \mid x \in [0,1],\, y^2 + z^2 = 1\} > 0.$$

Then, if

$$N + d > \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2f^\bullet},$$

for every $0 \le j \le N+d$, $b_j \ge 0$ on C.

Proof: Since for every $(w,x,y,z) \in \Delta_1 \times C$, $F(w,x,y,z) = \bar{f}(x,y,z)$ we have $F \geq f^\bullet$ on $\Delta_1 \times C$. On the other hand, it is easy to see that for $(y,z) \in C$,

$$\|F(W,X,y,z)\| \leq (d+1)(m+1) \max_{\substack{0\leq i\leq m\\ 0\leq j\leq d}} \left\{\|a_{ji}X^j(W+X)^{d-j}y^iz^{m-i}\|\right\} \leq (d+1)(m+1)\|f\|_\infty.$$

Using the bound for Polya's Theorem from [10, Theorem 1], if $N \in \mathbb{N}$ verifies

$$N + d > \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2f^\bullet},$$

all the coefficients of the polynomial

$$(W+X)^N F(W,X,y,z) = \sum_{0\leq j\leq N+d} b_j(y,z)W^jX^{N+d-j} \in \mathbb{R}[W,X]$$

are positive. In other words, for $0 \leq j \leq N+d$, $b_j \geq 0$ on C as we wanted to prove. □

We deduce easily Theorem 4.

Proof of Theorem 4: By Proposition 17 if $N \in \mathbb{N}$ is the smallest integer number such that

$$N + d > \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2f^\bullet},$$

then for every $0 \leq j \leq N+d$, $b_j \geq 0$ on C. By Proposition 15, we have that f can be written as in (1) with

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \leq N + d + m + 1.$$

Since

$$\|f\|_\infty \geq |a_{00}| = |f(0,0)| = f(0,0) = \bar{f}(0,0,1) \geq f^\bullet,$$

we have

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \leq N+d+m+1 \leq \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2f^\bullet} + m + 2 \leq \frac{d^3(m+1)\|f\|_\infty}{f^\bullet}.$$

□

3.2 The case of f with a finite number of zeros on the boundary of the strip

Next, we want to relax the hypothesis $f > 0$ on $[0,1] \times \mathbb{R}$ to $f \geq 0$ on $[0,1] \times \mathbb{R}$ and with a finite numbers of zeros on the boundary of the strip. Consider

$$C_+ = \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 = 1,\ z \geq 0\}.$$

For f non-negative in $[0,1] \times \mathbb{R}$ and fully m-ic on $[0,1]$, it is clear that m is even. Then, since each $b_j(Y,Z) \in \mathbb{R}[Y,Z]$ is homogeneous of degree m, to prove that $b_j \geq 0$ on C it is enough to prove that $b_j \geq 0$ on C_+. The advantage of considering C_+ instead of C is simply that under the present hypothesis there is a bijection between the zeros of f in $[0,1]\times\mathbb{R}$ and the zeros of F in $\Delta_1\times C_+$ given by

$$(x,\alpha)\mapsto(1-x,x,y_\alpha,z_\alpha)\qquad\text{with}\qquad(y_\alpha,z_\alpha)=\left(\frac{\alpha}{\sqrt{\alpha^2+1}},\frac{1}{\sqrt{\alpha^2+1}}\right).$$

The idea is to consider separately, for each zero (x,α) of f, the polynomial $F(W,X,y_\alpha,z_\alpha)\in\mathbb{R}[W,X]$ and to find $N_\alpha\in\mathbb{N}_0$ such that $(W+X)^{N_\alpha}F(W,X,y_\alpha,z_\alpha)$ has non-negative coefficients $b_j(y_\alpha,z_\alpha)$. Then, we show that the same N_α works for $(y,z)\in C_+$ close to (y_α,z_α). Finally, in the rest of C_+ we use compactness arguments.

Proposition 18 *Let $f\in\mathbb{R}[X,Y]$ with $f\geq 0$ on $[0,1]\times\mathbb{R}$, f fully m-ic on $[0,1]$ and suppose that f has a finite number of zeros in $[0,1]\times\mathbb{R}$, all of them lying on $\{0,1\}\times\mathbb{R}$, and $\frac{\partial f}{\partial X}$ does not vanish at any of them. Then, there is $N\in\mathbb{N}_0$ such that for every $0\leq j\leq N+d$, $b_j\geq 0$ on C.*

Proof: For $0\leq h\leq d$, we define the polynomials $c_h\in\mathbb{R}[Y,Z]$ as follows:

$$F=\sum_{0\leq h\leq d}c_h(Y,Z)W^hX^{d-h}.$$

Then, for $0\leq h\leq d$,

$$c_h(Y,Z)=\sum_{0\leq i\leq m}\sum_{0\leq j\leq d-h}a_{ji}\binom{d-j}{h}Y^iZ^{m-i}$$

is a homogeneous polynomial in $\mathbb{R}[Y,Z]$ of degree m, and for $(y,z)\in C_+$ we have

$$|c_h(y,z)|\leq(m+1)\|f\|_\infty\sum_{0\leq j\leq d-h}\binom{d-j}{h}=(m+1)\binom{d+1}{h+1}\|f\|_\infty \tag{5}$$

and

$$\|F(W,X,y,z)\|\leq\max\left\{(m+1)\frac{\binom{d+1}{h+1}}{\binom{d}{h}}\|f\|_\infty\mid 0\leq h\leq d\right\}\leq(m+1)(d+1)\|f\|_\infty. \tag{6}$$

Now, since along the proof we will consider several values of N, we add the index N to the notation of polynomials b_j in the following way:

$$(W+X)^NF=\sum_{0\leq j\leq N+d}b_{j,N}(Y,Z)W^jX^{N+d-j}.$$

So we need to prove that there is $N\in\mathbb{N}_0$ such that for every $0\leq j\leq N+d$, $b_{j,N}\geq 0$ on C_+. It is clear that, for a fixed $(y,z)\in C_+$, if $N\in\mathbb{N}_0$ satisfies that for every $0\leq j\leq N+d$, $b_{j,N}(y,z)\geq 0$, then any $N'\in\mathbb{N}_0$ with $N'\geq N$ also satisfies that for every $0\leq j\leq N'+d$, $b_{j,N'}(y,z)\geq 0$.

For $N\in\mathbb{N}_0$ and $\alpha\in\mathbb{R}$, we have the identities

$$b_{0,N}(y_\alpha,z_\alpha)=c_0(y_\alpha,z_\alpha)=F(0,1,y_\alpha,z_\alpha)=\bar{f}(1,y_\alpha,z_\alpha)=\frac{1}{\sqrt{\alpha^2+1}^m}f(1,\alpha) \tag{7}$$

and

$$b_{N+d,N}(y_\alpha,z_\alpha)=c_d(y_\alpha,z_\alpha)=F(1,0,y_\alpha,z_\alpha)=\bar{f}(0,y_\alpha,z_\alpha)=\frac{1}{\sqrt{\alpha^2+1}^m}f(0,\alpha). \tag{8}$$

From (7) and (8) we deduce that for every $N \in \mathbb{N}_0$, $b_{0,N} \geq 0$ on C_+ and $b_{N+d,N} \geq 0$ on C_+. So we need to prove that there is $N \in \mathbb{N}_0$ such that for every $1 \leq j \leq N+d-1$, $b_{j,N} \geq 0$ on C_+.

We note

$$\Pi_f = \{\alpha \in \mathbb{R} \mid f(x,\alpha) = 0 \text{ for some } x \in \{0,1\}\} \subseteq \mathbb{R}.$$

We will show first that for each $\alpha \in \Pi_f$ there is $N_\alpha \in \mathbb{N}_0$ such that for $1 \leq j \leq N_\alpha + d - 1$, $b_{j,N_\alpha}(y_\alpha, z_\alpha)$ is positive on C_+. We consider three cases:

- $f(0,\alpha) = 0$ and $f(1,\alpha) \neq 0$:

  From (8) we have $b_{N+d,N}(y_\alpha, z_\alpha) = 0$ for every $N \in \mathbb{N}_0$ and also $c_d(y_\alpha, z_\alpha) = 0$. We consider the homogeneous polynomial of degree $d-1$

  $$\widetilde{F}_\alpha(W,X) = \frac{F(W,X,y_\alpha,z_\alpha)}{X} = \sum_{0 \leq h \leq d-1} c_h(y_\alpha,z_\alpha) W^h X^{d-h-1} \in \mathbb{R}[W,X].$$

  From (5) we deduce that for $0 \leq h \leq d-1$,

  $$\frac{|c_h(y_\alpha,z_\alpha)|}{\binom{d-1}{h}} \leq (m+1)\frac{\binom{d+1}{h+1}}{\binom{d-1}{h}}\|f\|_\infty = (m+1)\frac{(d+1)d}{(h+1)(d-h)}\|f\|_\infty \leq (m+1)(d+1)\|f\|_\infty$$

  and we have $\|\tilde{F}_\alpha\| \leq (m+1)(d+1)\|f\|_\infty$.

  On the other hand, it is clear that $\widetilde{F}_\alpha > 0$ on $\Delta_1 - \{(1,0)\}$ and, in addition,

  $$\widetilde{F}_\alpha(1,0) = \frac{\partial F(1,0,y_\alpha,z_\alpha)}{\partial X} = \frac{1}{\sqrt{\alpha^2+1}^m}\frac{\partial f}{\partial X}(0,\alpha) > 0$$

  therefore $\widetilde{F}_\alpha(1,0) > 0$. We note

  $$\lambda_\alpha = \min_{\Delta_1} \widetilde{F}_\alpha > 0.$$

  By Lemma 16 with $\epsilon = 1/2$, if $N_\alpha \in \mathbb{N}_0$ satisfies

  $$N_\alpha + d - 1 \geq \frac{(d-2)(d-1)(d+1)(m+1)\|f\|_\infty}{\lambda_\alpha}$$

  and

  $$(W+X)^{N_\alpha}\widetilde{F}_\alpha = \sum_{0 \leq j \leq N_\alpha+d-1} c_j W^j X^{N_\alpha+d-1-j},$$

  for $0 \leq j \leq N_\alpha + d - 1$ we have

  $$c_j \geq \frac{N_\alpha!(N_\alpha+d-1)^{d-1}}{j!(N_\alpha+d-1-j)!}\frac{\lambda_\alpha}{2}.$$

  But since

  $$\sum_{0 \leq j \leq N_\alpha+d-1} c_j W^j X^{N_\alpha+d-j} = (W+X)^{N_\alpha} X \widetilde{F}_\alpha =$$

$$= (W+X)^{N_\alpha} F(W,X,y_\alpha,z_\alpha) = \sum_{0\le j\le N_\alpha+d-1} b_{j,N_\alpha}(y_\alpha,z_\alpha) W^j X^{N_\alpha+d-j}$$

we conclude that for $0 \le j \le N_\alpha + d - 1$,

$$b_{j,N_\alpha}(y_\alpha,z_\alpha) = c_j \ge \frac{N_\alpha!(N_\alpha+d-1)^{d-1}}{j!(N_\alpha+d-1-j)!}\frac{\lambda_\alpha}{2}.$$

- $f(0,\alpha) \ne 0$ and $f(1,\alpha) = 0$:

  From (7) we have $b_{0,N}(y_\alpha,z_\alpha) = 0$ for every $N \in \mathbb{N}_0$ and also $c_0(y_\alpha,z_\alpha) = 0$. We consider the homogeneous polynomial of degree $d-1$

  $$\widetilde{F}_\alpha(W,X) = \frac{F(W,X,y_\alpha,z_\alpha)}{W} = \sum_{1\le h\le d} c_h(y_\alpha,z_\alpha) W^{h-1} X^{d-h} \in \mathbb{R}[W,X].$$

  Then, proceeding similarly to the previous case we prove $\|\tilde{F}_\alpha\| \le \frac{1}{2}(m+1)d(d+1)\|f\|_\infty$. Moreover, since $\widetilde{F}_\alpha > 0$ on $\Delta_1 - \{(0,1)\}$ and

  $$\widetilde{F}_\alpha(0,1) = \frac{\partial F(0,1,y_\alpha,z_\alpha)}{\partial W} = -\frac{1}{\sqrt{\alpha^2+1}^m}\frac{\partial f}{\partial X}(1,\alpha) > 0$$

  we have that $\widetilde{F}_\alpha(1,0) > 0$ and we note

  $$\lambda_\alpha = \min_{\Delta_1} \widetilde{F}_\alpha > 0.$$

  Finally, using Lemma 16 with $\epsilon = 1/2$, we conclude that if $N_\alpha \in \mathbb{N}_0$ satisfies

  $$N_\alpha + d - 1 \ge \frac{(d-2)(d-1)d(d+1)(m+1)\|f\|_\infty}{2\lambda_\alpha},$$

  for $1 \le j \le N_\alpha + d$,

  $$b_{j,N_\alpha}(y_\alpha,z_\alpha) \ge \frac{N_\alpha!(N_\alpha+d-1)^{d-1}}{(j-1)!(N_\alpha+d-j)!}\frac{\lambda_\alpha}{2}.$$

- $f(0,\alpha) = 0$ and $f(1,\alpha) = 0$:

  From (7) and (8) we have $b_{0,N}(y_\alpha,z_\alpha) = b_{N+d,N}(y_\alpha,z_\alpha) = 0$ for every $N \in \mathbb{N}_0$ and also $c_0(y_\alpha,z_\alpha) = c_d(y_\alpha,z_\alpha) = 0$. We consider the homogeneous polynomial of degree $d-2$

  $$\widetilde{F}_\alpha(W,X) = \frac{F(W,X,y_\alpha,z_\alpha)}{WX} = \sum_{1\le h\le d-1} c_h(y_\alpha,z_\alpha) W^{h-1} X^{d-h-1} \in \mathbb{R}[W,X].$$

  Then, proceeding similarly to the previous cases we prove again $\|\tilde{F}_\alpha\| \le \frac{1}{2}(m+1)d(d+1)\|f\|_\infty$. We note

  $$\lambda_\alpha = \min_{\Delta_1} \widetilde{F}_\alpha > 0.$$

  Finally, using Lemma 16 with $\epsilon = 1/2$, we conclude that if $N_\alpha \in \mathbb{N}_0$ satisfies

  $$N_\alpha + d - 2 \ge \frac{(d-3)(d-2)d(d+1)(m+1)\|f\|_\infty}{2\lambda_\alpha},$$

  for $1 \le j \le N_\alpha + d - 1$,

  $$b_{j,N_\alpha}(y_\alpha,z_\alpha) \ge \frac{N_\alpha!(N_\alpha+d-2)^{d-2}}{(j-1)!(N_\alpha+d-1-j)!}\frac{\lambda_\alpha}{2}.$$

Now, our next goal is to compute a radios $r_\alpha > 0$ around each (y_α, z_α) so that for $(y,z) \in C_+$ with $\|(y,z) - (y_\alpha, z_\alpha)\| \le r_\alpha$, for $1 \le j \le N_\alpha + d - 1$, we have $b_{j,N_\alpha}(y,z) \ge 0$. First, we do some auxiliary computations.

For $0 \le h \le d$ and $(y,z) \in \mathbb{R}^2$ with $y^2 + z^2 \le 1$ we have

$$\begin{aligned}
\|\nabla c_h(y,z)\| &\le \left|\frac{\partial c_h}{\partial Y}(y,z)\right| + \left|\frac{\partial c_h}{\partial Z}(y,z)\right| \\
&\le \sum_{1\le i\le m}\sum_{0\le j\le d-h} |a_{ji}| \binom{d-j}{h} i + \sum_{0\le i\le m-1}\sum_{0\le j\le d-h} |a_{ji}|\binom{d-j}{h}(m-i) \\
&\le m(m+1)\binom{d+1}{h+1}\|f\|_\infty \\
&\le m(m+1)(d+1)\binom{d}{h}\|f\|_\infty.
\end{aligned}$$

Then, for $(y,z) \in C_+$,

$$|c_h(y,z) - c_h(y_\alpha, z_\alpha)| \le m(m+1)(d+1)\binom{d}{h}\|f\|_\infty \|(y,z) - (y_\alpha, z_\alpha)\|.$$

We introduce now some notation following [10]. For $t \in \mathbb{R}$, $m \in \mathbb{N}_0$ and a variable U,

$$(U)_t^m := U(U-t)(U-2t)\cdots(U-(m-1)t) = \prod_{0\le i\le m-1}(U - it) \in \mathbb{R}[U].$$

Also, for $t \in \mathbb{R}$

$$F_t(W,X,Y,Z) = \sum_{0\le h\le d} c_h(Y,Z)(W)_t^h (X)_t^{d-h}.$$

By [10, (4)], for $N \in \mathbb{N}_0$ and $0 \le j \le N + d$ we have

$$b_{j,N}(y,z) = \frac{N!(N+d)^d}{j!(N+d-j)!} F_{\frac{1}{N+d}}\left(\frac{j}{N+d}, \frac{N+d-j}{N+d}, y, z\right).$$

Then, using the Vandermonde-Chu identity (see [10, (6)]), for $(y,z) \in C_+$ we have

$$\begin{aligned}
&\left|F_{\frac{1}{N+d}}\left(\frac{j}{N+d}, \frac{N+d-j}{N+d}, y, z\right) - F_{\frac{1}{N+d}}\left(\frac{j}{N+d}, \frac{N+d-j}{N+d}, y_\alpha, z_\alpha\right)\right| \\
&\le \sum_{0\le h\le d} |c_h(y,z) - c_h(y_\alpha,z_\alpha)| \left(\frac{j}{N+d}\right)^h_{\frac{1}{N+d}} \left(\frac{N+d-j}{N+d}\right)^{d-h}_{\frac{1}{N+d}} \\
&\le m(m+1)(d+1)\|f\|_\infty \|(y,z)-(y_\alpha,z_\alpha)\| \left(\sum_{0\le h\le d}\binom{d}{h}\left(\frac{j}{N+d}\right)^h_{\frac{1}{N+d}} \left(\frac{N+d-j}{N+d}\right)^{d-h}_{\frac{1}{N+d}}\right) \\
&= m(m+1)(d+1)\|f\|_\infty \|(y,z)-(y_\alpha,z_\alpha)\| (1)^d_{\frac{1}{N+d}} \\
&\le m(m+1)(d+1)\|f\|_\infty \|(y,z)-(y_\alpha,z_\alpha)\|.
\end{aligned}$$

Consider $\alpha \in \Pi_f$. If $f(0,\alpha) = 0$ and $f(1,\alpha) \neq 0$ we take

$$r_\alpha = \frac{\lambda_\alpha (N_\alpha + d - 1)^{d-1}}{2(N_\alpha + d)^d m(m+1)(d+1)\|f\|_\infty}.$$

Then, for $(y,z) \in C_+$ with $\|(y,z) - (y_\alpha, z_\alpha)\| \le r_\alpha$ and $1 \le j \le N_\alpha + d - 1$ we have

$$\begin{aligned} b_{j,N}(y,z) &= b_{j,N}(y_\alpha, z_\alpha) + b_{j,N}(y,z) - b_{j,N}(y_\alpha, z_\alpha) \\ &\ge \frac{N_\alpha!(N_\alpha + d - 1)^{d-1}}{j!(N_\alpha + d - 1 - j)!}\frac{\lambda_\alpha}{2} - \frac{N_\alpha!(N_\alpha + d)^d}{j!(N_\alpha + d - j)!} m(m+1)(d+1)\|f\|_\infty r_\alpha \\ &\ge 0. \end{aligned}$$

If $f(0,\alpha) \neq 0$ and $f(1,\alpha) = 0$ we take again

$$r_\alpha = \frac{\lambda_\alpha (N_\alpha + d - 1)^{d-1}}{2(N_\alpha + d)^d m(m+1)(d+1)\|f\|_\infty}$$

and if $f(0,\alpha) \neq 0$ and $f(1,\alpha) = 0$ we take

$$r_\alpha = \frac{\lambda_\alpha (N_\alpha + d - 2)^{d-2}}{2(N_\alpha + d)^d m(m+1)(d+1)\|f\|_\infty}$$

and in both cases we proceed in a similar way.
Now, consider $K \subseteq C_+$ defined by

$$K = \{(y,z) \in C_+ : \|(y,z) - (y_\alpha, z_\alpha)\| \ge r_\alpha \text{ for all } \alpha \in \Pi_f\}.$$

Since K is compact and $\lambda_K = \min_{\Delta_1 \times K} F > 0$, by [10, Theorem 1] using (6), if

$$N + d > \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2\lambda_K},$$

for $0 \le j \le N + d$, $b_{j,N}(y,z) \ge 0$ for every $(y,z) \in K$.
Finally, if $N \in \mathbb{N}$,

$$N = \max\left\{ \left\lfloor \frac{(d-1)d(d+1)(m+1)\|f\|_\infty}{2\lambda_K} \right\rfloor - d + 1,\ \max\{N_\alpha \,|\, \alpha \in \Pi_f\} \right\},$$

we conclude that for $0 \le j \le N + d$, $b_{j,N} \ge 0$ on C_+. □

From Proposition 15 and Proposition 18 we deduce the following result.

Theorem 19 *Let $f \in \mathbb{R}[X,Y]$ with $f \ge 0$ on $[0,1] \times \mathbb{R}$, f fully m-ic on $[0,1]$ and suppose that f has a finite number of zeros in $[0,1] \times \mathbb{R}$, all of them lying on $\{0,1\} \times \mathbb{R}$, and $\frac{\partial f}{\partial X}$ does not vanish at any of them. Then, for $N \in \mathbb{N}_0$ as in Proposition 18, f can be written as in (1) with*

$$\deg(\sigma_0), \deg(\sigma_1 X(1-X)) \le N + d + m + 1.$$

We conclude with an example of a polynomial $f \in \mathbb{R}[X,Y]$ with $f \geq 0$ on $[0,1] \times \mathbb{R}$, f fully m-ic on $[0,1]$, with only one zero in $[0,1] \times \mathbb{R}$ lying on $\{0,1\} \times \mathbb{R}$ but $\frac{\partial f}{\partial X}$ vanishing at it, and such that f does not admit a value of $N \in \mathbb{N}_0$ as in Proposition 15. Note that in this example, f is itself a sum of squares, so the representation as in (1) is already given; nevertheless, our purpose is to show that there is no hope of applying the method underlying Proposition 15 in full generality.

Example 20 *Let*

$$f(X,Y) = (Y^2 - X)^2 + X^2 = Y^4 - 2XY^2 + 2X^2.$$

Then

$$F(W,X,Y,Z) = (W+X)^2Y^4 - 2X(W+X)Y^2Z^2 + 2X^2Z^4.$$

and for $N \in \mathbb{N}$,

$$(W+X)^N F(W,X,Y,Z) = Y^4W^{N+2} + Y^2\left((N+2)Y^2 - 2Z^2\right)W^{N+1}X + \ldots$$

It is easy to see that it does not exist $N \in \mathbb{N}_0$ such that

$$b_{N+1}(Y,Z) = Y^2\left((N+2)Y^2 - 2Z^2\right)$$

is non-negative on C.

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References

[1] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[2] Paula Escorcielo and Daniel Perrucci. A version of Putinar's positivstellensatz for cylinders. Manuscript. Arxiv: 1811.03586, 2018.

[3] Thomas Jacobi and Alexander Prestel. Distinguished representations of strictly positive polynomials. *J. Reine Angew. Math.*, 532:223–235, 2001.

[4] Victor Magron, Mohab Safey El Din, and Markus Schweighofer. Algorithms for weighted sum of squares decomposition of non-negative univariate polynomials. *J. Symbolic Comput.*, 93:200–220, 2019.

[5] Murray Marshall. *Positive polynomials and sums of squares*, volume 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[6] Murray Marshall. Polynomials non-negative on a strip. *Proc. Amer. Math. Soc.*, 138(5):1559–1567, 2010.

[7] Ha Nguyen and Victoria Powers. Polynomials non-negative on strips and half-strips. *J. Pure Appl. Algebra*, 216(10):2225–2232, 2012.

[8] Daniel Perrucci and Marie-Françoise Roy. A new general formula to compute the cauchy index with subresultants on an interval. Manuscript. Arxiv: 1812.02470, 2019.

[9] Victoria Powers. Positive polynomials and the moment problem for cylinders with compact cross-section. *J. Pure Appl. Algebra*, 188(1-3):217–226, 2004.

[10] Victoria Powers and Bruce Reznick. A new bound for Pólya's theorem with applications to polynomials positive on polyhedra. *J. Pure Appl. Algebra*, 164(1-2):221–229, 2001. Effective methods in algebraic geometry (Bath, 2000).

[11] Victoria Powers and Bruce Reznick. Polynomials positive on unbounded rectangles. In *Positive polynomials in control*, volume 312 of *Lect. Notes Control Inf. Sci.*, pages 151–163. Springer, Berlin, 2005.

[12] Mihai Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.*, 42(3):969–984, 1993.

[13] Ralph Tyrrell Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[14] Claus Scheiderer. Sums of squares of regular functions on real algebraic varieties. *Trans. Amer. Math. Soc.*, 352(3):1039–1069, 2000.

[15] Claus Scheiderer and Sebastian Wenzel. Polynomials nonnegative on the cylinder. In *Ordered algebraic structures and related topics*, volume 697 of *Contemp. Math.*, pages 291–300. Amer. Math. Soc., Providence, RI, 2017.

[16] Konrad Schmüdgen. The K-moment problem for compact semi-algebraic sets. *Math. Ann.*, 289(2):203–206, 1991.

[17] Gilbert Stengle. Complexity estimates for the Schmüdgen Positivstellensatz. *J. Complexity*, 12(2):167–174, 1996.