**Some Bounds for the Number of Components of Real Zero Sets of Sparse Polynomials**

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**Abstract**

We prove that the zero set of a 4-nomial in $n$ variables in the positive orthant has at most 3 connected components. This bound, which does not depend on the degree of the polynomial, not only improves the best previously known bound (which was 10) but is optimal as well. In the general case, we prove that the number of connected components of the zero set of an $m$-nomial in $n$ variables in the positive orthant is lower than or equal to $(n+1)^{m-1}2^{1+(m-1)(m-2)/2}$, improving slightly the known bounds. Finally, we show that for generic exponents, the number of non-compact connected components of the zero set of a 5-nomial in 3 variables in the positive octant is at most 12. This strongly improves the best previously known bound, which was 10384. All the bounds obtained in this paper continue to hold for real exponents.

# 1 Introduction.

Descartes' Rule of Signs provides a bound for the number of positive roots of a given real univariate polynomial which depends on the number of sign changes among its coefficients but not on its degree. One of its consequences is that the number of positive roots of a polynomial with $m$ monomials is bounded above by $m-1$.

Many attempts have been made to generalize Descartes' Rule of Signs (or its corollaries) to multivariate polynomials. Even though this task has not yet been completed, important advances have been made. Let us introduce the notation and terminology we will use throughout this paper.

As usual, $\mathbb{N}$ will denote the set of positive integers. Let $n \in \mathbb{N}$. Given $x \in \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_k > 0 \text{ for } 1 \le k \le n\}$ and $a := (a_1, \dots, a_n) \in \mathbb{R}^n$, $x^a$ will denote $x_1^{a_1} \cdots x_n^{a_n}$.

**Definition 1** *Let $m \in \mathbb{N}$. An $m$-*nomial* in $n$ variables is a function $f : \mathbb{R}^n_+ \to \mathbb{R}$ defined as*

$$f(x) = \sum_{i=1}^{m} c_i x^{a_i},$$

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*where* $c_i \in \mathbb{R}, c_i \neq 0$ *and* $a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in \mathbb{R}^n$ *for* $i = 1, \ldots, m$.

An interesting fact is that Descartes' Rule of Signs continues to hold for real exponents (the adaptation of the proof given in [1, Proposition 1.1.10], for instance, is straightforward).

**Definition 2** *Let* $n, m \in \mathbb{N}$. *Let us consider the functions* $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ *of the form* $F = (f_1, \ldots, f_n)$ *with* $f_i$ *an* $m_i$*-nomial, such that the total number of distinct exponent vectors in* $f_1, \ldots, f_n$ *is less than or equal to* $m$. *We then define* $K(n, m)$ *to be the maximum number of isolated zeroes (in* $\mathbb{R}^n_+$*) an* $F$ *of this type may have. Similarly, we define* $K'(n, m)$ *to be the maximum number of non-degenerate zeroes (in* $\mathbb{R}^n_+$*) an* $F$ *of this type may have.*

A proof of the finiteness of $K(n, m)$ can be found in many sources, for instance [1, Corollary 4.3.8]. The finiteness of $K'(n, m)$ is a consequence of the fact that $K'(n, m)$ is always less than or equal to $K(n, m)$. A bound for $K'(n, m)$ is provided by Khovanski's theorem, which is the most important result in the theory of fewnomials:

**Theorem 1** *Following the notations above,*

$$K'(n, m) \leq (n + 1)^{m-1} 2^{(m-1)(m-2)/2}.$$

For a proof of Khovanski's theorem, see [1, Chapter 4], [2] or [3]. Nevertheless, the statement mentioned above is not exactly equal to any of those in the references. To prove Theorem 1 divide every equation in the system $F(x) = 0$ by $x^a$, where $x^a$ ($a \in \mathbb{R}^n$) is one of the monomials of the system, to make the number of monomials drop and then use [1, Theorem 4.1.1] or [3, Section 3.12, Corollary 6]. Another fact to be highlighted is that here we are allowing fewnomials with real exponents instead of integer exponents as in [1]. Nevertheless, the proof in this reference does not make use of this fact.

Another way to generalize Descartes' Rule of Signs is to increase just the number of variables. In this case, the problem is to find a bound for the number of connected components of the zero set of a single polynomial, which is expected to be a hypersurface. This paper is devoted to the study of this problem, both in particular cases and in the general one. The results presented here are inspired in a paper by Li, Rojas and Wang (see [5]).

**Definition 3** *Given a subset* $X$ *of* $\mathbb{R}^n_+$, *we will denote by* $\mathrm{Tot}(X)$, $\mathrm{Comp}(X)$ *and* $\mathrm{Non}(X)$ *the number of connected components, compact connected components and non-compact connected components of* $X$ *respectively.*

*Given* $n, m \in \mathbb{N}$, $P(n, m)$, $P_{comp}(n, m)$ *and* $P_{non}(n, m)$ *are defined in the following way. First, we define the set*

$$\Omega(n, m) := \{f : \mathbb{R}^n_+ \to \mathbb{R} \mid f \text{ is a } k\text{-nomial with } 1 \leq k \leq m\}.$$

*We then define*

$$P(n,m) := \max\{\, \mathrm{Tot}(f^{-1}(0)) \mid f \in \Omega(n,m)\},$$

$$P_{comp}(n,m) := \max\{\, \mathrm{Comp}(f^{-1}(0)) \mid f \in \Omega(n,m)\},$$

$$P_{non}(n,m) := \max\{\, \mathrm{Non}(f^{-1}(0)) \mid f \in \Omega(n,m)\}.$$

It is clear from the definitions that, for all $n, m \in \mathbb{N}$,

$$P_{comp}(n,m) \le P(n,m), \quad P_{non}(n,m) \le P(n,m),$$

$$P(n,m) \le P_{comp}(n,m) + P_{non}(n,m)$$

and that $P$, $P_{comp}$ and $P_{non}$ are increasing functions of their second parameter. For fixed $n, m \in \mathbb{N}$, the finiteness of $P(n,m)$ (and thus that of $P_{comp}(n,m)$ and $P_{non}(n,m)$) is a consequence of the fact that it is bounded from above by $n(n+1)^m\, 2^{n-1} 2^{m(m-1)/2}$ (see [5], Corollary 2). Strongly based on this paper, we will derive a slightly better bound:

**Theorem 2** *Using the previous notation,* $P(n,m) \le (n+1)^{m-1} 2^{1+(m-1)(m-2)/2}$.

Our approach is different from that in [5] in the way we bound the number of non-compact connected components. We state our result in the following theorem, which will also be useful in the last section, when dealing with 5-nomials:

**Theorem 3** *Let us consider $m, n \ge 2$. If $Z := f^{-1}(0) \subset \mathbb{R}^n_+$ with $f$ an $m$-nomial in $n$ variables such that the dimension of the Newton polytope (see Definition 4) of $f$ is $n$, then*

- $\mathrm{Non}(Z) \le 2n\, P(n-1, m-1)$,
- $\mathrm{Tot}(Z) \le \sum_{i=0}^{n-1} 2^i \frac{n!}{(n-i)!} P_{comp}(n-i, m-i)$.

Let us remark that, due to the fact that $\mathbb{R}^n_+$ is not a closed set, a bounded connected component of the zero set of an $m$-nomial may be non-compact. This is the case, for example, when $f$ is the 3-nomial in 2 variables defined by $f(x_1, x_2) = x_1^2 + x_2^2 - 1$.

The next proposition, which will be proved in the next section, shows that, for a fixed number of monomials, a big number of variables will not increase the number of connected components:

**Proposition 1** *Given $m \in \mathbb{N}$, for all $n \in \mathbb{N}$,*

$$P(n,m) \le \begin{cases} m-1 & \textit{if } m \le 2, \\ P(m-2, m) & \textit{if } m \ge 3. \end{cases}$$

One of the goals of this paper is to find a sharp bound for $P(n, 4)$ and this proposition shows that it is enough to find such a bound for $P(2, 4)$. Our result is stated in the following theorem:

**Theorem 4** *Under the previous notation, we have:*

1. $P_{comp}(2, 4) = 1$.
2. $P_{non}(2, 4) = 3$.
3. $P(2, 4) = 3$ *(and thus* $P(n, 4) = 3$*).*
4. *If* $f$ *is a* $4$*-nomial in* $2$ *variables and* $\dim \mathrm{Newt}(f) = 2$*, then* $\mathrm{Tot}(f^{-1}(0)) \leq 2$.

This theorem improves the best previously known bound for $P(n, 4)$, which was 10 ([5, Theorems 2 and 3, and Example 2]). We will state the results used to prove this last bound and sketch a brief proof of it in the next section. Let us remark that in [5, Theorem 3] the equality of the second item is proved in the smooth case.

The techniques we use to prove the previous theorems also allow us to prove the following theorem concerning 5-nomials.

**Theorem 5** *Let* $f$ *be a* $5$*-nomial in* $3$ *variables such that* $\dim \mathrm{Newt}(f) = 3$*. Let* $Z := f^{-1}(0) \subset \mathbb{R}^3_+$*. Then,* $\mathrm{Non}(Z) \leq 12$*.*

This theorem significantly improves the best previously known bound of 10384 (the proof of this bound will be sketched briefly in the next section too).

This paper is organized as follows: Section 2 details some preliminaries. Section 3 concerns 4-nomials and contains the proof of Theorem 4. In Section 4, we deal with the general case of $m$-nomials in $n$ variables and we prove Theorems 2 and 3. Finally, in Section 5, we prove Theorem 5.

# 2 Preliminaries

## 2.1 Previously known bounds for some particular cases

The following result provides us with an optimal bound for the number of non-degenerate roots in the positive quadrant for a fewnomial system having at most 4 different monomials.

**Lemma 1** *Following the notation of Definition 2,* $K'(2, 4) = 5$*.*

*Proof:* See [5, Section 2, Proposition 1]. □

The next theorem enables us to get a bound for the number of connected components in the positive orthant of the zero set of a single fewnomial.

**Theorem 6** *(see [5, Theorem 2]) Following the notation of Definition 3, we have:*

- $P_{comp}(n,m) \leq 2\lfloor K'(n,m)/2 \rfloor \leq K'(n,m)$,
- $P_{non}(n,m) \leq 2P(n-1,m)$.

With these results, we can easily prove that $P(n,4) \leq 10$ in the following way:

$$P(n,4) \leq P(2,4) \leq P_{comp}(2,4) + P_{non}(2,4) \leq 2[K'(2,4)/2] + 2P(1,4) = 10,$$

the last equality being true because of Descartes' Rule of Signs. We will improve this bound in Section 3.

In the same way,

$$P_{non}(3,5) \leq 2P(2,5) \leq 2P_{comp}(2,5) + 2P_{non}(2,5) \leq$$

$$\leq 4[K'(2,5)/2] + 4P(1,5) = 10384.$$

We will improve this bound in Section 5.

## 2.2 Monomial changes of variables and Newton polytopes.

Let us start this section with some notation and definitions.

**Notation 1** *Given a non-singular matrix $B \in \mathbb{R}^{n\times n}$, $B = (b_{ij})_{1\leq i,j\leq n}$, we will denote by $B_1, \ldots, B_n$ the columns of $B$. We will call the* monomial change of variables associated to $B$ *the function*

$$h_B : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+, \quad h_B(x) = (x^{B_1}, \ldots, x^{B_n}).$$

The following formulae hold for all $x \in \mathbb{R}^n_+$, $a \in \mathbb{R}^n$ and non-singular matrices $B, C \in \mathbb{R}^{n\times n}$:

- $h_B(x)^a = x^{Ba}$.
- $h_B \circ h_C = h_{CB}$.

Recall the Newton polytope of a polynomial $f$, denoted by $\mathrm{Newt}(f)$, which is a convenient combinatorial encoding of the monomial term structure of a polynomial.

**Definition 4** *Given an $m$-nomial $f$ in $n$ variables, $f(x) := \sum_{i=1}^{m} c_i x^{a_i}$,* $\mathrm{Newt}(f)$ *denotes the smallest convex set containing the set of exponent vectors $\{a_1, \ldots, a_m\}$. The dimension of* $\mathrm{Newt}(f)$*,* $\dim \mathrm{Newt}(f)$*, is defined as the dimension of the smallest translated linear subspace containing* $\mathrm{Newt}(f)$*.*

Therefore, for any $n$-variate $m$-nomial $f$, $\dim \mathrm{Newt}(f) \leq \min\{m-1, n\}$.

Given an $m$-nomial $f(x) = \sum_{i=1}^{m} c_i x^{a_i}$ and a non-singular matrix $B \in \mathbb{R}^{n \times n}$, we have that

$$f \circ h_B(x) = \sum_{i=1}^{m} c_i h_B(x)^{a_i} = \sum_{i=1}^{m} c_i x^{B a_i},$$

and thus

1. $f \circ h_B$ is also an $m$-nomial.
2. $\mathrm{Newt}(f \circ h_B) = \{B\, v \in \mathbb{R}^n \mid v \in \mathrm{Newt}(f)\}$, and then, as $B$ is non-singular, $\dim \mathrm{Newt}(f) = \dim \mathrm{Newt}(f \circ h_B)$.

**Remark 1** *Given an $m$-nomial $f$ in $n$ variables, $c \in \mathbb{R}, c \neq 0$ and $b \in \mathbb{R}^n$, the function $c^{-1}x^{-b}f$ is an $m$-nomial whose Newton polytope is a translation of* $\mathrm{Newt}(f)$*. Then* $\dim \mathrm{Newt}(c^{-1}x^{-b}f) = \dim \mathrm{Newt}(f)$*. On the other hand, the zero set of $c^{-1}x^{-b}f$ (included in $\mathbb{R}_+^n$ by definition) is equal to the zero set of $f$ (also included in $\mathbb{R}_+^n$). In particular, by choosing $c$ as one of the coefficients of $f$, we will get an $m$-nomial with a coefficient equal to 1. Moreover, by choosing $b$ as one of the exponents of $f$, we will get an $m$-nomial with a non-zero constant term. So, these particularities can be assumed without loss of generality and not modifying the zero set of the $m$-nomial, or the dimension of its Newton polytope. It can also be proved that $p \in f^{-1}(0) \subset \mathbb{R}_+^n$ is a critical point of $f$ if and only if it is a critical point of $c^{-1}x^{-b}f$.*

**Proposition 2** *Let $f$ be an $m$-nomial in $n$ variables, $Z := f^{-1}(0) \subset \mathbb{R}_+^n$ and $d := \dim \mathrm{Newt}(f)$. Then:*

1. *If $d \leq n-1$, then* $\mathrm{Comp}(Z) = 0$ *and* $\mathrm{Non}(Z) \leq P(d, m)$*.*
2. *If $d = m-1$, then* $\mathrm{Comp}(Z) = 0$ *and* $\mathrm{Non}(Z) \leq 1$*.*

*Proof:* Because of Remark 1, we can suppose $f(x) = c_1 + \sum_{i=2}^{m} c_i x^{a_i}$. To prove the first assertion of this Proposition, let us consider a non-singular matrix $B \in \mathbb{R}^{n \times n}$, whose first $d$ columns are a basis of $\langle a_2, \ldots, a_m \rangle$. Let us consider the $m$-nomial $g = f \circ h_{B^{-1}}$. Due to the fact that $h_{B^{-1}}$ is a homeomorphism, the zero sets of $f$ and $g$ have the same number of compact and non-compact connected components. As $g(x) = c_1 + \sum_{i=2}^{m} c_i x^{B^{-1}a_i}$, then $\dim \mathrm{Newt}(g) = \dim\langle B^{-1}a_2, \ldots, B^{-1}a_m \rangle = \dim\langle a_2, \ldots, a_m \rangle = \dim \mathrm{Newt}(f)$. Since the first $d$ columns of $B$ form a basis of the subspace $\langle a_2, \ldots, a_m \rangle$, then each of the vectors

$B^{-1}a_i, i = 2, \ldots, m$, has its $n-d$ last coordinates equal to zero. So, the $m$-nomial $g = f \circ h_{B^{-1}}$ actually involves only $d$ variables. As a consequence of this, the zero set of $g$ may be described as $Z' \times \mathbb{R}_+^{n-d}$, where $Z'$ is the zero set of an $m$-nomial in $d$ variables. Thus, $\mathrm{Comp}(Z) = 0$ and $\mathrm{Non}(Z) \leq P(d, m)$.

Let us prove now the second assertion. Due to the fact that $\dim \mathrm{Newt}(f) = \dim\langle a_2, \ldots, a_m \rangle = m-1$, the $m-1$ vectors $a_2, \ldots, a_m$ are linearly independent. Let us choose the matrix $B$ having the vectors $a_2, \ldots, a_m$ as columns and proceed in the same way as in the above paragraph. The $m$-nomial $g$ turns out to be a linear function. Then, its zero set is the intersection between an hyperplane and the set $\mathbb{R}_+^n$; which is either empty or non-compact and connected. So, $\mathrm{Comp}(Z) = 0$ and $\mathrm{Non}(Z) \leq 1$. □

We can now give a proof of Proposition 1.

*Proof:* Let $f$ be an $m$-nomial in $n$ variables, $Z := f^{-1}(0) \subset \mathbb{R}_+^n$ and $d := \dim \mathrm{Newt}(f)$.

- If $m = 1$, then $f(x) = c_1 x^{a_1} \neq 0$ for all $x \in \mathbb{R}_+^n$. So, $Z = \emptyset$ and then $P(n, 1) = 0$ for all $n \in \mathbb{N}$.
- If $m = 2$, then $f(x) = c_1 x^{a_1} + c_2 x^{a_2}$. In this case, $Z$ is the subset in $\mathbb{R}_+^n$ satisfying the following equation

  $$x^{a_1 - a_2} = -c_2/c_1,$$

  which is empty if $c_1$ and $c_2$ have the same sign, and non-empty, connected and non-compact if they do not.
- If $m \geq 3$, as $d$ is always less than or equal to $m-1$, then we just need to consider the following cases:
  - If $1 \leq n \leq m-2$, then $\mathrm{Tot}(Z) \leq P(m-2, m)$, because an $m$-nomial in $n$ variables can be considered as an $m$-nomial in $m-2$ variables with the particularity that the last $m-2-n$ variables are not actually involved in its formula.
  - If $m-1 \leq n$ and $d \leq m-2$, then $d \leq n-1$. By Proposition 2, $\mathrm{Tot}(Z) \leq 0 + P(d, m) \leq P(m-2, m)$.
  - If $m-1 \leq n$ and $d = m-1$, again by Proposition 2, $\mathrm{Tot}(Z) \leq 0 + 1 \leq P(m-2, m)$.

□

Proposition 1 ensures that, in order to study the number of connected components of the zero set of a 4-nomial in $n$ variables in the positive orthant, it is enough to study the zero set of a 4-nomial in 2 variables in the positive quadrant.

Finally, let us recall two classical results from topology that will be quite useful in the next section.

**Theorem 7** *(Connected curve classification.) Let $\Gamma$ be a differentiable manifold of dimension 1. Then, $\Gamma$ is diffeomorphic either to $S^1$ or to $\mathbb{R}$ depending on whether $\Gamma$ is compact or not.*

The proof of this theorem can be found in [6].

We will also use the next adaptation of Jordan's Lemma to the positive quadrant, which can be easily proved from its original statement (see, for example, [4]) upon an application of the exponential function.

**Lemma 2** *(Adaptation of Jordan's Lemma). Let $\Gamma$ be a curve in $\mathbb{R}^2_+$ homeomorphic to $S^1$. Then, $\mathbb{R}^2_+ \setminus \Gamma$ has two connected components, which we will call $\mathrm{Int}(\Gamma)$ and $\mathrm{Ext}(\Gamma)$, such that they are both open sets, $\mathrm{Int}(\Gamma)$ is bounded, $\overline{\mathrm{Int}(\Gamma)} = \mathrm{Int}(\Gamma) \cup \Gamma$ is compact and $\mathrm{Ext}(\Gamma)$ is unbounded.*

# 3 On 4-nomials in 2 variables.

Most of the results we will obtain in this section come from the study of the restriction of 4-nomials in 2 variables to curves of the type $\{x \in \mathbb{R}^2_+ \mid x^a = J\}$ with $a \in \mathbb{R}^2$ and $J \in \mathbb{R}_+$. Let us introduce the notation we will use.

**Notation 2** *Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be an $m$-nomial in 2 variables, $p = (p_1, p_2) \in \mathbb{R}^2_+$ and $u = (u_1, u_2) \in \mathbb{R}^2, u \neq 0$. By $h_{(p,u)}$ we will denote the following parametrization of $\{x \in \mathbb{R}^2_+ \mid x^u = p^u\}$:*

$$h_{(p,u)} : \mathbb{R}_+ \longrightarrow \mathbb{R}^2_+,$$

$$h_{(p,u)}(t) = (h^{(1)}_{(p,u)}(t), h^{(2)}_{(p,u)}(t)) = \begin{cases} (t, (p^u)^{1/u_2} t^{-u_1/u_2}) & \text{if } u_2 \neq 0, \\ (p_1, t) & \text{if } u_2 = 0. \end{cases}$$

*By $f_{(p,u)}$ we will denote the following function:*

$$f_{(p,u)} : \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad f_{(p,u)} = f \circ h_{(p,u)}.$$

**Remark 2**

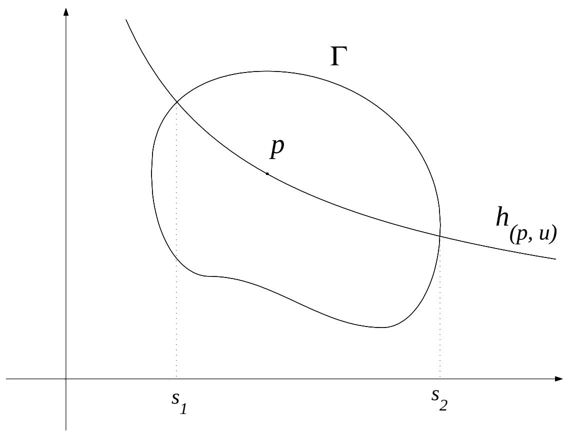
- *If $u_2 \neq 0$, $h_{(p,u)}(p_1) = p$ and if $u_2 = 0$, then $h_{(p,u)}(p_2) = p$.*

- *$f_{(p,u)}$ is an $m'$-nomial in 1 variable, with $m' \leq m$. The exponents of $f_{(p,u)}$ are proportional to the projections of the exponent vectors of $f$ on $\langle u \rangle^{\perp}$. For instance, if $u_2 \neq 0$ and $a = (a_1, a_2)$ is an exponent of $f$, then $a_1 - u_1 a_2 / u_2 = \langle a, (u_2, -u_1) \rangle u_2^{-1}$ is an exponent vector of $f_{(p,u)}$. The inequality $m' \leq m$ is due to the fact that different exponent vectors of $f$ may have the same projection on $\langle u \rangle^{\perp}$. So, some monomials in $f_{(p,u)}$ may be re-grouped together and make the number of monomials decrease.*

- *Suppose $p = (p_1, p_2)$ is a critical point of $f$ satisfying $f(p) = 0$ and $u = (u_1, u_2) \in \mathbb{R}^2$. If $u_2 \neq 0$, then $p_1$ is a degenerate zero of $f_{(p,u)}$, and if $u_2 = 0$, then $p_2$ is a degenerate zero of $f_{(p,u)}$. This is a consequence of the chain rule.*

Notice that for $p \in \mathbb{R}^2_+$ and $u \in \mathbb{R}^2$, the image of $h_{(p,u)}$ is an unbounded curve containing $p$. The following lemma will give us some information about the intersection between this curve and a compact connected component of the zero set of an $m$-nomial.

**Lemma 3** *Let $f$ be an $m$-nomial in 2 variables and let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$. Let $\Gamma$ be a compact connected component of $Z$ containing only regular points of $f$ (so $\Gamma$ is a differentiable submanifold of $\mathbb{R}^2_+$ diffeomorphic to $S^1$). Let $p = (p_1, p_2) \in \mathrm{Int}(\Gamma)$ and $u = (u_1, u_2) \in \mathbb{R}^2, u \neq 0$. Then, if $u_2 \neq 0$, $f_{(p,u)}$ has a zero $s_1 \in (0, p_1)$ and a zero $s_2 \in (p_1, +\infty)$ such that $h_{(p,u)}(s_i) \in \Gamma$ $(i = 1, 2)$. If $u_2 = 0$, $f_{(p,u)}$ has a zero $s_1 \in (0, p_2)$ and a zero $s_2 \in (p_2, +\infty)$ such that $h_{(p,u)}(s_i) \in \Gamma$ $(i = 1, 2)$.*




*Proof:* Let us suppose $u_2 \neq 0$. The lemma is a consequence of the fact that $\Gamma$ is a compact set, $p$ lies in $\mathrm{Int}(\Gamma)$ and the images of the intervals $(0, p_1]$ and $[p_1, \infty)$ under the function $h_{(p,u)}$ are unbounded. If $u_2 = 0$, similar arguments work. $\square$

Suppose now that $f$ is a 4-nomial in 2 variables. As explained before, by studying the restriction of $f$ to curves of a certain type we will obtain some information about its coefficients.

**Lemma 4** *Let $f$ be a 4-nomial in 2 variables and $Z := f^{-1}(0) \subset \mathbb{R}^2_+$. Suppose that one of the following two conditions is satisfied:*

1. *$Z$ has a critical point $p = (p_1, p_2)$ and $Z \setminus \{p\} \neq \emptyset$.*
2. *$Z$ has a compact connected component $\Gamma$ and $Z \setminus \Gamma \neq \emptyset$.*

*Then, two of the coefficients of $f$ are positive and the other two are negative.*

*Proof:* Suppose $Z$ satisfies the first condition. Let $q = (q_1, q_2) \in Z \setminus \{p\}$.

If $p_1 \neq q_1$, then $p_1/q_1 \neq 1$. Let

$$v_1 := \frac{\log(q_2/p_2)}{\log(p_1/q_1)}$$

Then $p_1^{v_1} p_2 = q_1^{v_1} q_2$. Let $v \in \mathbb{R}^2$, $v := (v_1, 1)$. As it was explained in Remark 2, $p_1$ is a zero of $f_{(p,v)}$ with multiplicity at least 2. On the other hand,

$$f_{(p,v)}(q_1) = f(q_1, p^v q_1^{-v_1}) = f(q_1, q^v q_1^{-v_1}) = f(q) = 0,$$

because $q \in Z$. As $p_1 \neq q_1$, we know that $f_{(p,v)}$ has at least 3 zeroes (counting multiplicities) in $\mathbb{R}_+$. We know that $f_{(p,v)}$ is an $m'$-nomial with $m' \leq 4$. By Descartes' Rule of Signs, we know that the number of sign changes in $f_{(p,v)}$ is at least 3; thus, $m' = 4$ and among the 4 coefficients of $f_{(p,v)}$, there must be two positive and two negative. On the other hand, if

$$f(x) = \sum_{i=1}^{4} c_i x^{a_i},$$

then

$$f_{(p,v)}(x_1) = \sum_{i=1}^{4} c_i \, (p^v)^{a_{i2}} x_1^{a_{i1} - a_{i2} v_1}.$$

As the signs of the coefficients of $f_{(p,v)}$ are defined by the signs of the coefficients of $f$, then $f$ must have two positive and two negative coefficients.

If $p_1 = q_1$, as $p \neq q$, we will have $p_2 \neq q_2$. In this case, let us take $v := (1, 0)$ and proceed as above.

Let us suppose now that $Z$ satisfies the second condition, which is having a compact connected component $\Gamma$, and $Z \neq \Gamma$. If $Z$ has a critical point, then the first condition is also satisfied. If it does not have a critical point, we consider $\hat{p} := (\hat{p}_1, \hat{p}_2) \in \mathrm{Int}(\Gamma)$ and $\hat{q} := (\hat{q}_1, \hat{q}_2) \in Z \setminus \Gamma$.

If $\hat{p}_1 \neq \hat{q}_1$, in the same way we did before, we can find a vector $w \in \mathbb{R}^2$, $w = (w_1, 1)$ such that $\hat{p}^w = \hat{q}^w$. Then, $f_{(\hat{p},w)}$ has at least one zero $s_1$ in the interval $(0, \hat{p}_1)$ such that $h_{(\hat{p},w)}(s_1) \in \Gamma$ and at least one zero $s_2$ in the interval $(\hat{p}_1, +\infty)$ such that $h_{(\hat{p},w)}(s_2) \in \Gamma$. On the other hand,

$$f_{(\hat{p},w)}(\hat{q}_1) = f(\hat{q}_1, \hat{p}^w \hat{q}_1^{-w_1}) = f(\hat{q}_1, \hat{q}^w \hat{q}_1^{-w_1}) = f(\hat{q}) = 0,$$

because $\hat{q} \in Z$. Besides, due to the fact that $h_{(\hat{p},w)}(\hat{q}_1) = \hat{q} \in Z \setminus \Gamma$, $\hat{q}_1 \neq s_1$ and $\hat{q}_1 \neq s_2$. Then, we deduce that $f_{(\hat{p},w)}$ has at least three zeroes in $\mathbb{R}_+$, and then $f_{(\hat{p},w)}$ is also a 4-nomial with at least three sign changes. So, $f_{(\hat{p},w)}$ and $f$ have both two coefficients with each sign.

If $\hat{p}_1 = \hat{q}_1$, then $\hat{p}_2 \neq \hat{q}_2$, and the same argument works. $\square$

Due to the lemma above, we will focus our attention for a moment on 4-nomials with two coefficients of each sign. We will start relating some properties of the zero set of a 4-nomial in two variables of this form with its Newton polytope.

**Lemma 5** *Let $f$ be a 4-nomial in two variables with two positive and two negative coefficients, such that* $\dim \operatorname{Newt}(f) = 2$. *Let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$, and suppose one of the following two conditions is satisfied:*

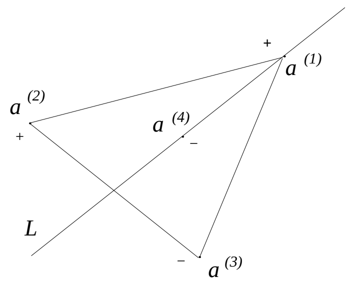
1. *$Z$ has a critical point $p = (p_1, p_2)$.*
2. *$Z$ has a compact connected component $\Gamma$.*

*Then* $\operatorname{Newt}(f)$ *is a quadrilateral without parallel sides and coefficients corresponding to adjacent vertices have opposite signs.*

*Proof:* Define $r = (r_1, r_2) \in \mathbb{R}^2_+$ as follows: if $Z$ satisfies the first condition, then $r = p$ and if $Z$ satisfies the second one but not the first one (so $\Gamma$ is diffeomorphic to $S^1$), then $r$ is any point in $\operatorname{Int}(\Gamma)$. By Remark 2 and Lemma 3, we know that for all $v = (v_1, v_2) \in \mathbb{R}^2$, $v \neq 0$, $f_{(r,v)}$ has at least two zeroes (counting multiplicities) in $\mathbb{R}_+$.

Since $\dim \operatorname{Newt}(f) = 2$, the exponent vectors do not lie on a line. Suppose $f(x) = \sum_{i=1}^{4} c_i x^{a_i}$. Then the vertices of $\operatorname{Newt}(f)$ are among the vectors $a_1, a_2, a_3$ and $a_4$ and $\operatorname{Newt}(f)$ might either be a triangle or a quadrilateral. We will need to study four cases separately.

- Suppose $\operatorname{Newt}(f)$ is a triangle, whose vertices are the vectors $a_1, a_2$ and $a_3$ and that the vector $a_4$ lies in the interior of $\operatorname{Newt}(f)$. Assume $c_1$ and $c_2$ are positive and $c_3$ and $c_4$ are negative (by multiplying $f$ by $-1$ and reordering the monomials if necessary). Let $v := a_1 - a_2 \neq 0$ and $L$ the line through $a_1$ and $a_4$.



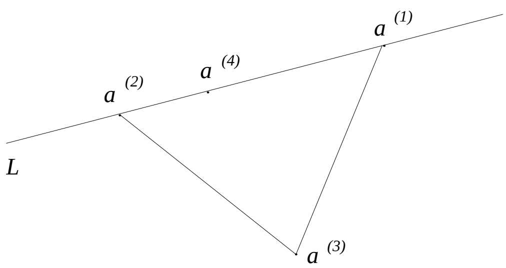

As $a_1$ and $a_4$ have the same projection on $\langle v\rangle^{\perp}$ and $a_2$ and $a_3$ are on opposite sides of the line $L$, we conclude that $f_{(r,v)}$ is a 3-nomial of the following type (if $v_2 \neq 0$):

$$f_{(r,v)}(x_1) = c_3(r^v)^{a_{32}/v_2} x_1^{a_{31}-a_{32}v_1/v_2} +$$

$$+(c_1(r^v)^{a_{12}/v_2} + c_4(r^v)^{a_{42}/v_2}) x_1^{a_{41}-a^{42}v_1/v_2} + c_2(r^v)^{a_{22}/v_2} x_1^{a_{21}-a_{22}v_1/v_2}.$$

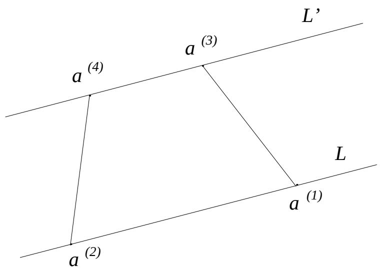
Even though we do not know at this point if in the above formula the terms are written in increasing or decreasing order, this 3-nomial has exactly one sign change, because the monomials of higher and lower exponent have distinct coefficient signs (we are supposing $c_2 > 0$ and $c_3 < 0$). By Descartes' Rule of Signs, it cannot have 2 zeroes (counting multiplicities) as we know it does. Then, we have a contradiction, and we conclude that the Newton polytope of $f$ cannot be a triangle having the remaining exponent vector in its interior. If $v_2 = 0$, the same procedure works.

- Let us suppose now that $\mathrm{Newt}(f)$ is a triangle whose vertices are the exponent vectors $a_1, a_2$ and $a_3$; and that the vector $a_4$ lies on one of the edges of $\mathrm{Newt}(f)$. Without loss of generality, we suppose that $a_4$ lies on the segment $a_1a_2$.



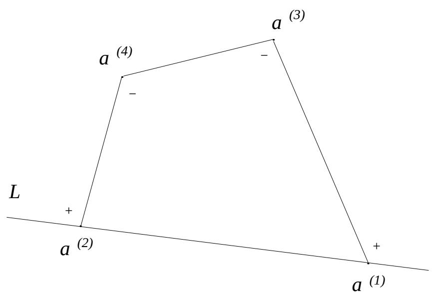

By taking again $v := a_1 - a_4$, we have that $a_1$, $a_2$ and $a_4$ have the same projection on $\langle v\rangle^{\perp}$. Thus, $f_{(r,v)}$ is a 2-nomial (because its first, second and fourth term can be re-grouped together in a single monomial) and, by Descartes' Rule of Signs, $f_{(r,v)}$ cannot have 2 zeroes (counting multiplicities) as we know it should. We then have a contradiction, which enables us to get rid of this case.

- Suppose Newt$(f)$ is a quadrilateral with a pair of parallel opposite sides. Without loss of generality, we suppose that the segments $a_1a_2$ and $a_3a_4$ are parallel.




  Let us take $v := a_1 - a_2$. As $a_1$ and $a_2$ have the same projection on $\langle v\rangle^\perp$, and $a_3$ and $a_4$ also do so, we can re-group the monomials in $f_{(r,v)}$ and form a 2-nomial, which again is impossible.

- Finally, suppose that Newt$(f)$ is a quadrilateral and that the coefficients of same sign correspond to adjacent vertices. Without loss of generality, let us suppose that $a_1$ and $a_2$ are adjacent, $a_3$ and $a_4$ are adjacent too, $c_1$ and $c_2$ are positive and $c_3$ and $c_4$ are negative.




  Let $v := a_1 - a_2$ and $L$ the line through $a_1$ and $a_2$. As $a_1$ and $a_2$ have the same projection on $\langle v\rangle^\perp$, then $f_{(r,v)}$ is a 3-nomial. But, as the two remaining exponent vectors (corresponding both to negative coefficients) lie in the same side of $L$, $f_{(r,v)}$ has just one sign change. For this reason, it cannot have two zeroes, and we get a contradiction.

We conclude that the lemma follows. $\square$

The next lemma shows the existence of a convenient change of variables for certain bivariate 4-nomials.

**Lemma 6** *Let $f$ be a bivariate 4-nomial having two positive coefficients and two negative coefficients and such that* $\mathrm{Newt}(f)$ *is a quadrilateral with no parallel opposite sides and coefficients corresponding to adjacent vertices having opposite signs. Then, there is an invertible change of variables $h$ such that $f \circ h$ is*

$$f \circ h(x_1, x_2) = 1 - x_1 - x_2 + A x_1^c x_2^d,$$

*with $A > 0$, $c, d > 1$, and $h$ is the composition of a monomial change of variables with a re-scaling of the variables.*

*Proof:* Suppose that we enumerate the vertices of $\mathrm{Newt}(f)$ in such a way that $a_1$ and $a_4$ are not adjacent. Because of Remark 1 we can suppose $f$ is of the following type:

$$f(x_1, x_2) = 1 + \sum_{i=2}^{4} c_i x^{a_i},$$

i.e., $a_1 = (0, 0)$. As coefficients with the same sign correspond to non-adjacent vertices of $\mathrm{Newt}(f)$, we know that $c_2, c_3 < 0$ and $c_4 > 0$. Consider the four triangles that can be formed with three of the four vertices of $\mathrm{Newt}(f)$. Among these triangles, there must be one having the minimal area. Suppose it is the triangle $a_1a_2a_3$ (this can be enforced by rotating indices if necessary). Because of the fact that $\mathrm{Newt}(f)$ does not have parallel opposite sides, this area is strictly less than the area of the triangles $a_1a_2a_4$ and $a_1a_3a_4$.

Let $B := \{a_2, a_3\}$ and $C$ be the matrix having the elements of $B$ as columns. As $\mathrm{Newt}(f)$ is a quadrilateral, $a_1 = (0, 0), a_2$ and $a_3$ do not lie on a line. Then $B$ is a basis of $\mathbb{R}^2$ and $C$ is non-singular. As in [5, Lemma 1], let $h$ be the composition of $h_{C^{-1}}$ and the linear re-scaling $(x_1, x_2) \mapsto \left(\frac{x_1}{|c_2|}, \frac{x_2}{|c_3|}\right)$. Let $(c, d) := C^{-1}a_4$. Then,

$$f \circ h(x_1, x_2) = 1 - x_1 - x_2 + c_4 \frac{1}{|c_2|^c} \frac{1}{|c_3|^d} x_1^c x_2^d.$$

Let $A$ be the last coefficient of the 4-nomial above. Then $A > 0$. On the other hand, the Newton polytope $\mathrm{Newt}(f \circ h_{C^{-1}})$ must also be a quadrilateral with vertices $(0, 0), (1, 0)$, $(0, 1)$ and $(c, d)$. As $a_1$ and $a_4$ are opposite vertices in $\mathrm{Newt}(f)$, then $(0, 0)$ and $(c, d)$ must be opposite vertices in $\mathrm{Newt}(f \circ h_{C^{-1}})$. Thus, $c, d > 0$. As the area of triangle $a_1a_2a_3$ is smaller than that of triangle $a_1a_2a_4$, the area of triangle $(0, 0)(1, 0)(0, 1)$ should be smaller than that of triangle $(0, 0)(1, 0)(c, d)$, and thus $d > 1$. In an analogous way, we can prove that $c > 1$. $\square$

Let us recall that, as $h$ is a diffeomorphism of $\mathbb{R}^2_+$, the zero sets of $f$ and $f \circ h$ have the same number of compact and non-compact connected components and critical points.

The following lemma will let us deal with the case when the zero set of the 4-nomial has a critical point.

**Lemma 7** *Let $f$ be a 4-nomial in 2 variables such that* $\dim \mathrm{Newt}(f) = 2$ *and let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$. Suppose that $p = (p_1, p_2) \in Z$ is a critical point of $f$, and also that $Z \setminus \{p\} \neq \emptyset$. Then $Z$ is a connected non-compact set; that is to say,* $\mathrm{Non}(Z) = 1$ *and* $\mathrm{Comp}(Z) = 0$.

*Proof:* The proof of this lemma will be done in four steps. In the first one, we will make use of the lemmas we proved before to make sure that it is enough to restrict our attention to 4-nomials of a very specific form. In the second one, we will study the sign of $f$ on some curves we will consider. In the third one, we will use the information we obtained to characterize a non-compact connected set $W$ where $f$ vanishes. In the last one, we will prove that, in fact, $f^{-1}(0) = W$.

*Step 1.* By Lemma 4, we know that, among the coefficients of $f$ there must be two positive and two negative ones, and by Lemma 5, $\mathrm{Newt}(f)$ must be a quadrilateral without parallel opposite sides, and with same sign coefficients corresponding to opposite vertices. Then, by Lemma 6, we can suppose $f$ is of the following type:

$$f(x_1, x_2) = 1 - x_1 - x_2 + A x_1^c x_2^d,$$

with $A > 0$ and $c, d > 1$.

*Step 2.* Let $v := (c, d-1)$. This vector has the nice property that $(0,1)$ and $(c,d)$, which are exponent vectors in $f$, have the same projection on $\langle v \rangle^{\perp}$. Because of this fact, $f_{(p,v)}$ is a 3-nomial. In fact,

$$f_{(p,v)}(x_1) = \big( -(p^v)^{1/(d-1)} + A\,(p^v)^{d/(d-1)} \big) x_1^{-c/(d-1)} + 1 - x_1.$$

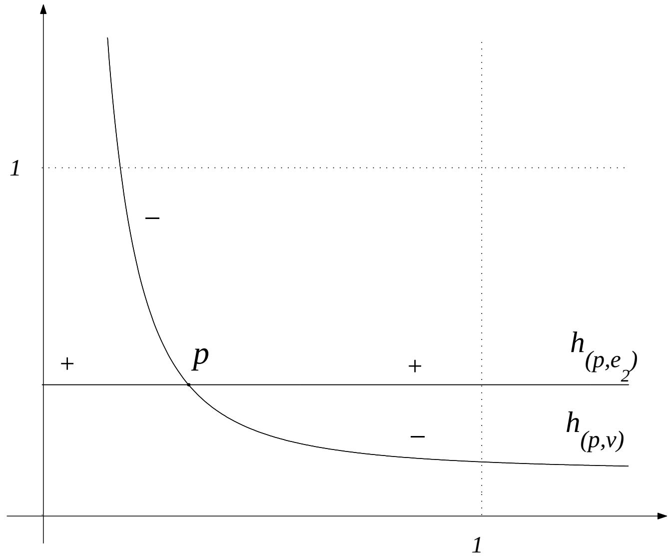
Notice that $-c/(d-1) < 0$ because $c, d > 1$. By Remark 2, $p_1$ is a zero of $f_{(p,v)}$ of multiplicity greater than or equal to 2. By Descartes' Rule of Signs, the 3-nomial $f_{(p,v)}$ must have at least two sign changes, $p_1$ is a zero of multiplicity exactly 2 and $f_{(p,v)}$ does not have other zeroes. As the unique zero of $f_{(p,v)}$ has an even multiplicity, and its leading exponent coefficient is negative, we know that $f_{(p,v)}(x_1) \leq 0$ for all $x_1 \in \mathbb{R}_+$ and $f_{(p,v)}(x_1) < 0$ if $x_1 \neq p_1$.

As $(0,0)$ and $e_2 := (0,1)$ are both vector exponents in $f$, it can be proved in an analogous way that $f_{(p,e_2)}(x_1) \geq 0$ for all $x_1 \in \mathbb{R}_+$ and $f_{(p,e_2)}(x_1) > 0$ if $x_1 \neq p_1$. Moreover,

$$f_{(p,e_2)}(x_1) = f(x_1, (p^{e_2})^1 x_1^0) = (1 - p_2) - x_1 + A p_2^d x_1^c,$$

and as it has a zero of multiplicity equal to two, it has two sign changes and then we deduce that $p_2 < 1$. In the same way we can also prove that $p_1 < 1$.

It can easily be checked that for $x_1 \in (0, p_1), h^{(2)}_{(p,e_2)}(x_1) < h^{(2)}_{(p,v)}(x_1)$, and for $x_1 \in (p_1, +\infty), h^{(2)}_{(p,e_2)}(x_1) > h^{(2)}_{(p,v)}(x_1)$. To illustrate the situation, in the following figure we have drawn the curves $h_{(p,e_2)}$ and $h_{(p,v)}$, together with some signs indicating the sign of $f$ on them:




Finally, for a fixed $\alpha \in \mathbb{R}_+$, let us analyze the function $f(\alpha, x_2)$ in the variable $x_2$:

$$f(\alpha, x_2) = (1-\alpha) - x_2 + A\alpha^c x_2^d.$$

Let us notice that for every fixed $\alpha \in (0,1)$, $\lim\limits_{x_2 \to 0+} f(\alpha, x_2) = 1 - \alpha > 0$.

*Step 3.* In order to study how many times the line $\{x_1 = \alpha\}$ intersects $Z$ for a fixed $\alpha \in \mathbb{R}_+$, we will continue studying the function $f(\alpha, x_2)$. As $A\alpha^c > 0$ and $d > 1$, if $\alpha < 1$, this function is a 3-nomial with two sign changes. Because of Descartes' Rule of Signs, it will have either no zeroes or two (counted with multiplicity) in $\mathbb{R}_+$. If $\alpha = 1$, this function is a 2-nomial with just one sign change and, finally, if $\alpha > 1$, it is a 3-nomial with one sign change. In both cases, it has exactly one zero in $\mathbb{R}_+$.

For a fixed $\alpha \in (0, p_1)$ the function (in the variable $x_2$) $f(\alpha, x_2)$ must have an odd number of zeroes (counted with multiplicity) in the interval $(h^{(2)}_{(p,e_2)}(\alpha), h^{(2)}_{(p,v)}(\alpha))$. As it has at most two zeroes in $\mathbb{R}_+$, then it has just one zero in that interval. Let us call it $g(\alpha)$.

In an analogous way, for a fixed $\alpha \in (p_1, 1)$ the function $f(\alpha, x_2)$ must have at least one zero in the interval $(0, h^{(2)}_{(p,v)}(\alpha))$, which we will call $t(\alpha)$, and another one in the interval $(h^{(2)}_{(p,v)}(\alpha), h^{(2)}_{(p,e_2)}(\alpha))$, which we will call $g(\alpha)$. As this function has at most two zeroes in $\mathbb{R}_+$, then it has no other zeroes.

For a fixed $\alpha \in [1, +\infty)$, the function $f(\alpha, x_2)$ must have an odd number of zeroes (counted with multiplicity) in the interval $(h^{(2)}_{(p,v)}(\alpha), h^{(2)}_{(p,e_2)}(\alpha))$. As this function has at most one zero in $\mathbb{R}_+$, then it has just one zero in that interval. Again, let us call it $g(\alpha)$.

Finally, let us define $g(p_1) = p_2$, and let us prove that the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ we have just defined is continuous.

As $p$ is a critical point of $f$, we know that

$$\frac{\partial f}{\partial x_2}(p) = -1 + dAp_1^c p_2^{d-1} = 0,$$

and this implies that $p^v = 1/dA$.

Suppose there exists $x_1 \in \mathbb{R}_+, x_1 \neq p_1$ such that $\frac{\partial f}{\partial x_2}(x_1, g(x_1)) = 0$; then

$$g(x_1) = (1/dA)^{1/(d-1)} x_1^{-c/(d-1)} = (p^v)^{1/(d-1)} x_1^{-c/(d-1)} = h^{(2)}_{(p,v)}(x_1),$$

and this is impossible because of the definition of $g$. Then, for all $x_1 \neq p_1$, $\frac{\partial f}{\partial x_2}(x_1, g(x_1)) \neq 0$.

Let us fix $\alpha \neq p_1$ and see that $g$ is continuous in $\alpha$. Suppose that $\alpha > p_1$ (if $\alpha < p_1$ the proof can be done in the same way). We know that $h^{(2)}_{(p,v)}(\alpha) = (p^v)^{1/d-1}\alpha^{-c/d-1} < g(\alpha) < p_2$ and $\frac{\partial f}{\partial x_2}(\alpha, g(\alpha)) \neq 0$. Then, by the Implicit Function Theorem, there is a continuous function, let us call it $s$, defined in an interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ with $\alpha - \varepsilon > p_1$, such that $s(\alpha) = g(\alpha)$ and $f(x_1, s(x_1)) = 0$ for all $x_1$ in the interval of definition. Moreover, choosing a suitable value of $\varepsilon$, we can suppose that, for all $x_1$ in $(\alpha - \varepsilon, \alpha + \varepsilon)$, $s(x_1)$ lies in $((p^v)^{1/(d-1)} x_1^{-c/(d-1)}, p_2)$. As $x_2 = g(x_1)$ is the unique value in this interval such that $f(x_1, x_2) = 0$, we have $g \equiv s$ in $(\alpha - \varepsilon, \alpha + \varepsilon)$ and therefore $g$ is continuous in $\alpha$.

To prove that $g$ is continuous in $p_1$, notice that if $x_1 > p_1$, then

$$(p^v)^{1/(d-1)} x_1^{-c/(d-1)} < g(x_1) < p_2$$

and

$$\lim_{x_1 \to p_1^+} (p^v)^{1/(d-1)} x_1^{-c/(d-1)} = (p^v)^{1/(d-1)} p_1^{-c/(d-1)} = p_2.$$

So, we have $\lim_{x_1 \to p_1^+} g(x_1) = p_2$. Analogously, we prove that $\lim_{x_1 \to p_1^-} g(x_1) = p_2$.

Now, let us consider $w \in \mathbb{R}^2, w := (c-1, d)$. In the same way we proved the existence of the function $g$, we can prove that there exists a function $k : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

- For all positive $x_2$, we have $f(k(x_2), x_2) = 0$.
- $k$ is continuous.
- If $x_2 < p_2$ then $k(x_2) \in (p_1, (p^w)^{1/(c-1)} x_2^{-d/(c-1)})$, and $x_1 = k(x_2)$ is the unique value in that interval such that $f(x_1, x_2) = 0$.
- $k(p_2) = p_1$.
- If $x_2 > p_2$ then $k(x_2) \in ((p^w)^{1/(c-1)} x_2^{-d/(c-1)}, p_1)$, and $x_1 = k(x_2)$ is the unique value in that interval such that $f(x_1, x_2) = 0$.

Let us define $W_1 = \{(x_1, g(x_1)) \mid x_1 \in \mathbb{R}_+\} \subset \mathbb{R}^2_+$, $W_2 = \{(k(x_2), x_2) \mid x_2 \in \mathbb{R}_+\} \subset \mathbb{R}^2_+$ and $W = W_1 \cup W_2$. As the functions $g$ and $k$ are continuous, $W_1$ and $W_2$ are connected. As $g(p_1) = p_2$ and $k(p_2) = p_1$, it follows that $p \in W_1 \cap W_2$, and then $W$ is connected. Moreover, it is an unbounded set.

*Step 4.* Let us prove now that $W = Z$, and therefore, that $\mathrm{Non}(Z) = 1$ and $\mathrm{Comp}(Z) = 0$.

Due to the fact that, for all $x_1$ and $x_2$ in $\mathbb{R}_+$, $f(x_1, g(x_1)) = 0$ and $f(k(x_2), x_2) = 0$, it is clear that $W \subset Z$. Let $q := (q_1, q_2) \in Z$.

Suppose that $q_1 < p_1$ and $q_2 < p_2$. Let

$$z_1 := \frac{\log(q_2/p_2)}{\log(p_1/q_1)}$$

So, $p_1^{z_1} p_2 = q_1^{z_1} q_2$. As $p_1/q_1 > 1$ and $q_2/p_2 < 1$, then $z_1 < 0$. Let $z \in \mathbb{R}^2$, $z := (z_1, 1)$. We know that $p_1$ is a zero of multiplicity at least 2 of $f_{(p,z)}$. On the other hand,

$$f_{(p,z)}(q_1) = f(q_1, p^z q_1^{-z_1}) = f(q_1, q^z q_1^{-z_1}) = f(q_1, q_2) = 0,$$

because $q \in Z$. Then, $f_{(p,z)}$ has at least 3 zeroes (counted with multiplicity) and, by Descartes' Rule of Signs, at least 3 sign changes. As $c, d > 1$ and $z_1 < 0$, then $0 < -z_1 < c - dz_1$ and $0 < 1 < c - dz_1$, and we have that

$$f_{(p,z)}(x_1) = 1 - x_1 - p^z x_1^{-z_1} + A(p^z)^d x_1^{c-dz_1}$$

has just two sign changes. Then, it cannot happen that $q_1 < p_1$ and $q_2 < p_2$ at the same time.

Suppose now that $q_1 \geq p_1$. Consider the following cases:

- $q_1 \geq 1$: as we have shown at the beginning of this Lemma, the line $\{x_1 = q_1\}$ intersects $Z$ in a single point, which is $(q_1, g(q_1))$. Then, it must be $q_2 = g(q_1)$ and then $q \in W_1$.

- $p_1 < q_1 < 1$: we know that the line $\{x_1 = q_1\}$ intersects $Z$ in two points: $(q_1, g(q_1))$ and $(q_1, t(q_1))$, with $g(q_1) \in ((p^v)^{1/(d-1)} x_1^{-c/(d-1)}, p_2)$ and $t(q_1) \in (0, (p^v)^{1/(d-1)} q_1^{-c/(d-1)})$. If $q_2 = g(q_1)$ then $q \in W_1$. If $q_2 = t(q_1)$, then

$$q_2 < (p^v)^{1/(d-1)} q_1^{-c/(d-1)} = (p_1/q_1)^{c/d-1} p_2 < p_2,$$

and therefore $x_1 = k(q_2)$ is the unique value of $x_1$ in the interval $(p_1, (p^w)^{1/(c-1)} q_2^{-d/(c-1)})$ such that $f(x_1, q_2) = 0$. Since the previous inequalities imply

$$q_1 < p_1 p_2^{(d-1)/c} q_2^{-(d-1)/c} < p_1 p_2^{d/(c-1)} q_2^{-d/(c-1)} = (p^w)^{1/(c-1)} q_2^{-d/c-1},$$

we conclude that $q_1 = k(q_2)$ and so $q \in W_2$.

- If $q_1 = p_1$, let us see that $\{x_1 = q_1\}$ intersects $Z$ only in $p$. Let us consider $e_1 = (1, 0)$. We know that $p_2$ is a zero of multiplicity at least 2 of $f_{(p,e_1)}$, but
$$f_{(p,e_1)}(x_2) = (1 - p_1) - x_2 + Ap_1^c x_2^d$$
is a 3-nomial with two sign changes. By Descartes' Rule of Signs, $p_2$ has multiplicity equal to 2 and $f_{(p,e_1)}$ has no other zeroes. Then $\{x_1 = q_1\} \cap Z = \{p\}$, and then $q = p \in Z$.

If $q_2 \geq p_2$, we proceed in an analogous way. Thus, we conclude that $Z = W$, and that $Z$ has a unique connected component, which is unbounded. □

We can now give a proof of the following theorem.

**Theorem 8** *Let $f$ be a 4-nomial in 2 variables and let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$. If $Z$ has a compact connected component $\Gamma$, then $Z = \Gamma$.*

*Proof:* Suppose $Z \setminus \Gamma \neq \emptyset$. By Lemma 4, we know that among the coefficients of $f$ there are two positive and two negative. On the other hand, we know that $\dim \mathrm{Newt}(f) = 2$, otherwise $Z$ could not have compact connected components. Then, by Lemma 5, $\mathrm{Newt}(f)$ is a quadrilateral without parallel sides and coefficients of the same sign correspond to opposite vertices. By Lemma 7, $Z$ does not have critical points; otherwise it would have only a unique non-compact connected component. Finally, because of Lemma 6, we can suppose $f$ is of the following type:

$$f(x_1, x_2) = 1 - x_1 - x_2 + Ax_1^c x_2^d,$$

with $A > 0; c, d > 1$.

Again, in order to study how many times the line $\{x_1 = \alpha\}$ intersects $Z$ for a fixed $\alpha$ in $\mathbb{R}_+$, let us define a function $g_\alpha$ in the variable $x_2$ as the restriction of $f$ to that line, i.e.:
$$g_\alpha(x_2) = f(\alpha, x_2).$$

Then

$$g'_\alpha(x_2) = -1 + Ad\alpha^c x_2^{d-1} < 0 \iff x_2 < \left(\frac{1}{Ad}\right)^{1/(d-1)} \alpha^{-c/(d-1)}.$$

Let $J := (1/Ad)^{1/(d-1)}$. Then $J > 0$ and the function $g_\alpha$ has a minimum in $x_2 = J\alpha^{-c/(d-1)}$. For $x_1 \in \mathbb{R}_+$, let $\ell_1(x_1)$ be the minimum of the function $g_{x_1}$ and let $\ell(x_1) := (x_1, \ell_1(x_1))$. Then,

$$f \circ \ell(x_1) = (-J + AJ^d)x_1^{-c/(d-1)} + 1 - x_1,$$

so $f \circ \ell$ turns out to be a 3-nomial.

As $\Gamma$ is a compact set, the function $x_1$ reaches its minimum (let us call it $m$) and its maximum (let us call it $M$) on $\Gamma$. Let us prove that $m \neq M$: as $\Gamma$ is a differentiable manifold of dimension 1, then $\Gamma$ has an infinite number of points. If $m = M$, then $\Gamma \subset \{x_1 = m\}$, and the 3-nomial $g_m$ has infinitely many zeroes, which is impossible.

Let $p := (p_1, p_2)$ and $q := (q_1, q_2)$ in $\Gamma$ such that $m = p_1$ and $M = q_1$. Let us see that $p_2$ is a zero of multiplicity 2 of $g_{p_1}$. As $p \in \Gamma$, $g_{p_1}(p_2) = f(p_1, p_2) = 0$. On the other hand, as $p$ is the minimum of $x_1$ in $\Gamma$, using Lagrange multipliers,

$$g'_{p_1}(p_2) = \frac{\partial f}{\partial x_2}(p) = 0.$$

By Descartes' Rule of Signs, we know that $g_{p_1}$ must have at least two sign changes. As we know that

$$g_{p_1}(x_2) = (1 - p_1) - x_2 + A p_1^c x_2^d,$$

then it must be $p_1 < 1$, and so $g_{p_1}$ has no zeroes other than $p_2$. As the unique zero of $g_{p_1}$ has an even multiplicity and its leading coefficient is positive, for all $x_2 \neq p_2$, $g_{p_1}(x_2) > 0$. Then, $f \circ \ell(p_1) = 0$. In an analogous way, we can prove that $q_2$ is the unique zero of the function $g_{q_1}$ and $f \circ \ell(q_1) = 0$. We conclude that $p_1 = m$ and $q_1 = M$ are two different zeroes of $f \circ \ell$.

As $f \circ \ell(x_1) = (-J + AJ^d)x_1^{-c/(d-1)} + 1 - x_1$ is a 3-nomial, it has no zeroes other than $m$ and $M$ which have multiplicity 1. As its leading coefficient is negative, we know that $f \circ \ell(x_1) < 0$ for all $x_1 \in (0, m) \cup (M, +\infty)$ and $f \circ \ell(x_1) > 0$ for all $x_1 \in (m, M)$. Let $s \in (m, M)$. Then, for all $x_2 \in \mathbb{R}_+$, $f(s, x_2) \geq f \circ \ell(s) > 0$, . Then, $\Gamma \cap \{x_1 = s\} = \emptyset$ and the open sets $\{x_1 < s\}$ and $\{x_1 > s\}$ disconnect $\Gamma$, which is a contradiction. $\square$

Now we can give a proof of Theorem 4, which is the main goal of this section.

*Proof:*

1. The inequality $P_{comp}(2, 4) \leq 1$ is a consequence of Theorem 8. In the following example the equality holds:

   $$f_1(x_1, x_2) = x_2^2 - 4x_1^3 x_2 + x_1^8 + 3x_1^4.$$

   In fact, $f_1(x_1, x_2) = 0$ if and only if $x_2 = 2x_1^3 \pm x_1^2\sqrt{1 - (x_1^2 - 2)^2}$, and the set of positive values of $x_1$ where the polynomial under the square root symbol is non-negative is the interval $[1, \sqrt{3}]$.

2. Let $f$ be a 4-nomial in 2 variables and let $Z := f^{-1}(0) \subset \mathbb{R}_+^2$.

   If $\dim \mathrm{Newt}(f) = 1$, then by Proposition 2 and Descartes' Rule of Signs, we know that $\mathrm{Non}(Z) \leq P(1, 4) \leq 3$.

If $\dim \mathrm{Newt}(f) = 2$ and 0 is a regular value of $f$, then by [5, Theorem 3], $\mathrm{Non}(Z) \leq 2$.

If $\dim \mathrm{Newt}(f) = 2$ and 0 is not a regular value of $f$, there is a critical point $p$ in $Z$. If $Z = \{p\}$, then $\mathrm{Non}(Z) = 0$. If $Z \neq \{p\}$, by Lemma 7, $\mathrm{Non}(Z) = 1$.

The equality holds in the following example:

$$f_2(x_1, x_2) = (x_1 - 1)(x_1 - 2)(x_1 - 3) = x_1^3 - 6x_1^2 + 11x_1 - 6.$$

3. Let $f$ be a 4-nomial in 2 variables, and let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$.

   If $Z$ has any compact connected component $\Gamma$, by Theorem 8, $Z = \Gamma$ and then $\mathrm{Tot}(Z) = 1$. If it does not, because of the previous item we have that $\mathrm{Tot}(Z) \leq 3$ and the same example shows that the equality holds.

4. Let $f$ be a 4-nomial in 2 variables such that $\dim \mathrm{Newt}(f) = 2$, and let $Z := f^{-1}(0) \subset \mathbb{R}^2_+$.

   If $Z$ has any compact connected component $\Gamma$, again by Theorem 8, $Z = \Gamma$ and then $\mathrm{Tot}(Z) = 1$. If it does not, as it was shown in the second item of this theorem, $\mathrm{Non}(Z) \leq 2$, and $\mathrm{Tot}(Z) \leq 2$.

   The equality holds in the following example:

$$f_3(x_1, x_2) = x_1x_2 - 2x_1 - x_2 + 1.$$

   In fact, $f(x_1, x_2) = 0$ is an implicit equation for the hyperbola $x_2 = \frac{1}{x_1 - 1} + 2$.

□

# 4 On $m$-nomials in $n$ variables

In this section we will prove Theorems 2 and 3. Theorem 2 gives us an explicit upper bound for the number of connected components of the zero set of an $m$-nomial in $n$ variables in the positive orthant and Theorem 3 is an auxiliary theorem for Theorem 2, but it also will be used in the next section. Let us give now a proof of Theorem 3.

*Proof:* As observed earlier in Remark 1, we can assume $f$ is of the following form:

$$f(x) = c_m + \sum_{i=1}^{m-1} c_i x^{a_i}.$$

As $\dim\langle a_1, \ldots, a_{m-1}\rangle = n$, without loss of generality, we can suppose that $B := \{a_1, \ldots, a_n\}$ is a basis of $\mathbb{R}^n$. Let $A$ be the matrix having the elements of $B$ as columns, let $g$ be the $m$-nomial $f \circ h_{A^{-1}}$ and let $W := g^{-1}(0) \subset \mathbb{R}^n_+$. As $h_{A^{-1}}$ is a diffeomorphism, $\mathrm{Non}(W) = \mathrm{Non}(Z)$ and $\dim \mathrm{Newt}(g) = \dim \mathrm{Newt}(f)$. Moreover, we have that

$$g(x) = c_m + \sum_{i=1}^{m-1} c_i x^{A^{-1}a_i} = c_m + \sum_{i=1}^{n} c_i x_i + \sum_{i=n+1}^{m-1} c_i x^{A^{-1}a_i}.$$

Suppose $W$ has $t$ non-compact connected components and let $\{p_1, \ldots, p_t\}$ be a set of points intersecting each and every non-compact connected component of $W$. Suppose, for $1 \leq i \leq t$, $p_i = (p_{i1}, \ldots, p_{in})$. For $1 \leq j \leq n$, let us consider $M_j, m_j \in \mathbb{R}_+$ such that $M_j > \max\{p_{ij}, 1 \leq i \leq t\}$ and $m_j < \min\{p_{ij}, 1 \leq i \leq t\}$, and define $S_j = \{x \in \mathbb{R}^n_+ \mid x_j = M_j\}$ and $T_j = \{x \in \mathbb{R}^n_+ \mid x_j = m_j\}$. Let us prove that each non-compact connected component of $W$ intersects at least one of the sets $S_1, \ldots, S_n, T_1, \ldots, T_n$.

Let $X$ be a non-compact connected component of $W$. If $X$ is not bounded, then there exists $j_0, 1 \leq j_0 \leq n$, such that $X \cap S_{j_0}$ is not empty.

If $X$ is bounded, then it is not closed. Let $T := \cap_{j=1}^n \{x \in \mathbb{R}^n_+ \mid x_j \geq m_j\}$. If $X \subseteq T$, then it is a connected component of $W \cap T$. As $W = g^{-1}(0) \subset \mathbb{R}^n_+$ and $g$ is a continuous function, there exists a closed set $F \subset \mathbb{R}^n$ such that $W = F \cap \mathbb{R}^n_+$. Then

$$W \cap T = F \cap \mathbb{R}^n_+ \cap T = F \cap T,$$

and $W \cap T \subset \mathbb{R}^n_+$ is closed because it is an intersection of closed sets. It follows that $X$ is closed because it is a connected component of a closed set. This is a contradiction, and then $X \nsubseteq T$, and this implies that there exists $j_1$, $1 \leq j_1 \leq n$, such that $X \cap T_{j_1} \neq \emptyset$.

In this way, we have found $2n$ sets $(S_1, \ldots, S_n, T_1, \ldots, T_n)$ such that each non-compact connected component of $W$ has a non-empty intersection with one of them. Thus,

$$\mathrm{Non}(W) \leq \sum_{j=1}^{n} \mathrm{Tot}(W \cap S_j) + \sum_{j=1}^{n} \mathrm{Tot}(W \cap T_j).$$

Each of these $2n$ intersections has at most $P(n-1, m-1)$ connected components, because they can be regarded as zero sets of $m'$-nomials in $n-1$ variables, with $1 \leq m' \leq m-1$. For example, the set $W \cap S_n$ can be described as the zero set of the following function:

$$\hat{g} : \mathbb{R}^{n-1}_+ \to \mathbb{R},$$

$$\hat{g}(x_1, \ldots, x_{n-1}) = (c_m + c_n M_n) + \sum_{i=1}^{n-1} c_i x_i + \sum_{i=n+1}^{m-1} c_i (x_1, \ldots, x_{n-1}, M_n)^{A^{-1}a_i}.$$

We have thus proved that

$$\mathrm{Non}(Z) = \mathrm{Non}(W) \leq 2n\, P(n-1, m-1),$$

which is our first assertion.

Finally, note that for the function $\hat{g}$ defined above, $\dim \mathrm{Newt}(\hat{g}) = n - 1$. Proceeding inductively, we get the second inequality. $\square$

Let us prove now theorem 2.

*Proof:* Let us proceed by induction on $n$.

If $n = 1$ by Descartes' Rule of Signs, we know that

$$P(1, m) \leq m - 1 < 2^{m-1} 2^{1+(m-1)(m-2)/2}.$$

Suppose now that $n > 1$. Given an $m$-nomial $f$ in $n$ variables, let $d := \dim \mathrm{Newt}(f)$ and $Z := f^{-1}(0) \subset \mathbb{R}^n_+$.

If $d < n$, by the first item of Proposition 2 and the induction hypothesis,

$$\mathrm{Tot}(Z) \leq P(d, m) \leq (d+1)^{m-1} 2^{1+(m-1)(m-2)/2} \leq (n+1)^{m-1} 2^{1+(m-1)(m-2)/2}.$$

If $d = n$, as $m - 1 \geq d$, we have that $m \geq n + 1$. If $m = n + 1$, by the second item of Proposition 2, $\mathrm{Tot}(Z) \leq 1$. If $m \geq n + 2$, by the second item of Theorem 3, the first item of Theorem 6 and Theorem 1, we have

$$\mathrm{Tot}(Z) \leq \sum_{i=0}^{n-1} 2^i \frac{n!}{(n-i)!} (n - i + 1)^{m-i-1} 2^{(m-i-1)(m-i-2)/2}.$$

Now, we use the following inequality, valid for all $i, n, m \in \mathbb{N}$ such that $m \geq n + 2$ and $0 \leq i \leq n - 1$, that can be easily proved by induction on $i$:

$$2^i \frac{n!}{(n-i)!} (n - i + 1)^{m-i-1} 2^{(m-i-1)(m-i-2)/2} \leq \frac{1}{2^i} (n+1)^{m-1} 2^{(m-1)(m-2)/2}.$$

Then, we conclude that

$$\mathrm{Tot}(Z) \leq \sum_{i=0}^{n-1} \frac{1}{2^i} (n+1)^{m-1} 2^{(m-1)(m-2)/2} < (n+1)^{m-1} 2^{1+(m-1)(m-2)/2},$$

which completes the proof. $\square$

## 5 On 5-nomials in 3 variables

As a consequence of what has been proved in the previous sections, we get Theorem 5:

*Proof:* By Theorem 3, $\mathrm{Non}(Z) \leq 6P(2,4) = 18$. Nevertheless, in the proof of that theorem, we have shown the existence of six 4-nomials in two variables, let us call them $g_1, \ldots, g_6$, such that for $i = 1, \ldots, 6$, $\dim \mathrm{Newt}(g_i) = 2$ and

$$\mathrm{Non}(Z) \leq \sum_{i=1}^{6} \mathrm{Tot}(g_i^{-1}(0)).$$

By the fourth item of Theorem 4, we know that $\mathrm{Tot}(g_i^{-1}(0)) \leq 2$, and then we conclude that $\mathrm{Non}(Z) \leq 12$. $\square$

This bound is significantly sharper than the best previously known one, which was 10384.

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