SIMPLE CONNECTEDNESS AND HOMOLOGY OF SCHURIAN ALGEBRAS

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Introduction.

These notes are an expanded version of a mini-course given at the XV$^{\text{th}}$ Encuentro Rioplatense held at the University of Buenos Aires. The aim of this mini-course was to survey the recent developments in the study of strongly simply connected algebras, and their applications in the representation theory of associative algebras.

Strongly simply connected algebras were introduced in the schurian case by Dräxler [**29**], under the name of completely separating algebras, then generalised to the non-schurian case by Skowroński [**52**]. They are defined as follows. Let A be a finite dimensional algebra over an algebraically closed field k. Then there exists a (unique) quiver Q, and (at least) a surjective k-algebra morphism $\nu : kQ \to A$, where kQ is the path algebra of Q, so that, setting $I = \text{Ker }\nu$, we can write $A \cong kQ/I$. To each pair (Q, I), we can associate the fundamental group $\pi_1(Q, I)$, see [**43, 37**]. The algebra A is called simply connected if Q has no oriented cycles and, for each pair (Q, I), the group $\pi_1(Q, I)$ is trivial (see [**10**], and the survey [**2**]). It is called strongly simply connected if every full convex subquiver of A is simply connected.

Simply connected algebras have played an important role in representation theory. An algebra whose quiver has no oriented cycles is simply connected if and only if it has no proper Galois covering [**51, 43**]. For any representation-finite algebra B, the indecomposable B-modules can be lifted to indecomposable modules over a simply connected algebra A contained inside a certain Galois covering of the so-called standard form of B, see [**19, 33**]. Thus, covering techniques reduce many problems of the study of representation-finite algebras to the study of representation-finite simply connected algebras, explaining the importance of the latter.

The general problem of finding a criterion allowing to verify whether a given algebra is simply connected or not is undecidable. In contrast, the subclass of strongly simply connected algebras seems much easier to handle. Indeed, characterisations of strong simple connectedness were obtained, see [**29, 52, 8, 4, 5**], and the representation theory of the tame strongly simply connected algebras is largely known, see [**47, 53, 54**]. In the representation-finite case, it is known that any strongly simply connected algebra is simply connected, hence the latter are by now well-understood [**15, 18**].

In these notes, we present what we hope is a unified theory of strongly simply connected algebras. Since most of the recent results are obtained in the schurian case, we concentrate on the latter, but point out which definitions and results hold in general. We have also chosen not to repeat the results already surveyed in [**2**]. As we show, the main concepts here are those of crowns, already appearing in the combinatorics of posets (see, for instance, [**28, 30, 49**]) and its generalisation to schurian algebras, which we call quasi-crowns [**5**]. In keeping with the nature of this mini-course, we have tried to make these notes as self-contained as possible, and included proofs or sketches of proofs for most results. The techniques either rest on the combinatorics of the bound quiver, or are homological, and use a Mayer-Vietoris sequence for a one-point extension, as introduced in [**19**] or a Hochschild cohomology exact sequence, as in [**38**].

These notes are divided into the following sections.

1. Preliminaries: algebras, quivers and modules.
2. Schurian algebras.
3. Full convex subcategories.
4. The fundamental group and simple connectedness.
5. Hochschild cohomology groups.
6. Strongly simply connected algebras.
7. Quasi-crowns, strong simple connectedness and multiplicative bases.
8. Strongly simply connected schurian algebras.
9. Quotients of incidence algebras.
10. Dismantlability.
11. Simple connectedness of incidence algebras.

1. Preliminaries: algebras, quivers and modules.

1.1. Algebras. Throughout this survey, k denotes a fixed algebraically closed field. By algebra is always meant an associative, finite dimensional k-algebra with an identity, and by module is meant a finitely generated right module. Given an algebra A, we denote by mod A the category of A-modules. We are interested in the representation theory of A, thus in characterising A by properties of mod A. For this purpose, we may assume, without loss of generality, that A is basic (that is, $A/\operatorname{rad} A$ is a direct product of copies of k) and connected (that is, indecomposable as a ring).

An algebra A can equivalently be considered as a k-category of which the object class $A_0 = \{x_1, ...x_n\}$ is in bijection with a complete set of primitive orthogonal idempotents $\{e_{x_1}, ...e_{x_n}\}$ in A, and the set of morphisms $A(x, y)$ from x to y is the k-vector space $e_x A e_y$, see [**18**]. Thus, an algebra B is a full subcategory of A if there exist objects $x_1, ...x_t$ in A_0 such that, if $e = \sum_{i=1}^{t} e_{x_i}$, then $B = eAe$.

An algebra A is called *triangular* if there exists no sequence of objects $\{x_1, ...x_t, x_{t+1} = x_1\}$ in A_0, with $t \geq 1$, such that $A(x_i, x_{i+1}) \neq 0$ for every i. In this paper, we deal exclusively with triangular algebras.

1.2. Quivers. A (finite) *quiver* Q is a quadruple (Q_0, Q_1, s, t) consisting of two finite sets: the set of points Q_0 and the set of arrows Q_1, and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$. Thus, one may think of a quiver as of an oriented graph. The *path algebra* kQ of a quiver Q has as basis the set of all paths in Q (including, for each point of Q, the stationary path at this point) and the product of two paths is their composition if possible, and 0 otherwise. Two paths v and w in a quiver Q are *parallel*, or the pair (v, w) is a *contour*, if v and w have the same source and the same target. A *relation* in Q from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i with $1 \leq i \leq m$, $\lambda_i \in k$ is non-zero and w_i is a path of length at least two from x to y. If $m = 1$ (or $m = 2$), then ρ is called a *monomial* (or *binomial*, respectively) relation. A relation of the form $w_1 - w_2$ (where (w_1, w_2) is a contour) is a *commutativity relation.*

Let A be an algebra, and $\{e_1, ...e_n\}$ be a complete set of primitive orthogonal idempotents in A. The quiver Q_A of A is defined as follows. The points $\{1, ...n\}$ of Q_A are in bijection with the e_i, and the arrows from x to y in Q_A in bijection with vectors in a k-basis $\{\bar{v}_\alpha\}$ of $e_x \frac{\text{rad} A}{\text{rad}^2 A} e_y$. Thus, there exists a surjective k-algebra morphism $\nu : kQ_A \to A$ defined by sending the stationary path at x to the idempotent e_x and an arrow $\alpha : x \to y$ to a representative $v_\alpha \in e_x$ (rad A) e_y of the residual class $\bar{v}_\alpha$. Thus, $A \cong kQ_A/I_\nu$, with $I_\nu =$ Ker ν. It is easily seen that, while Q_A is an invariant of the algebra (that is, does not depend on the particular set of idempotents), the morphism ν (and hence the ideal I_ν) heavily depend on the choice of the bases above. The morphism ν (or the pair (Q_A, I_ν)) is called a *presentation* of A, see **[18, 32]**. In this case, the ideal I_ν is admissible in kQ_A, that is, is generated by finitely many relations. If I is an admissible ideal in the path algebra kQ of a quiver Q, the pair (Q, I) is called a *bound quiver*.

1.3. Modules. Let $A = kQ/I$ be an algebra. A (finite dimensional) representation M of Q is defined by assigning to each $x \in Q_0$ a (finite dimensional) k-vector space $M(x)$, and to each arrow $\alpha : x \to y$ a k-linear map $M(\alpha) : M(x) \to M(y)$. A representation M of Q is called *bound by* I if, whenever $\rho = \sum_{i=1}^m \lambda_i(\alpha_{i_1} \cdots \alpha_{i_t})$ is a relation (where the $\lambda_i \in k$ are non-zero and the α_{i_j} are arrows), then $\sum_{i=1}^m \lambda_i M(\alpha_{i_t}) \cdots M(\alpha_{i_1})$ $= 0$. A morphism $f : M \to N$ between representations is a family of k-linear maps $(f_x : M(x) \to N(x))_{x \in Q_0}$ such that, if $\alpha : x \to y$ is an arrow, then $N(\alpha) f_x = f_y M(\alpha)$. Thus, bound representations are just functors from the k-category A to mod k. We thus get a category of bound representations, which is equivalent to mod A, see **[18, 32]**. Accordingly, we identify these two categories and view modules as bound representations.

For an A-module M, its *support* Supp$M$ is the full subcategory of A generated by the points $x \in A_0$ such that $M(x) \neq 0$. For each $x \in A_0$, we denote by S_x the corresponding simple A-module, and by P_x (or I_x) the projective cover (or injective envelope, respectively) of S_x.

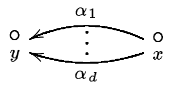
2. Schurian algebras.

2.1. Let A be an algebra. In order to understand its module category mod A, it suffices, in view of the classical Krull-Schmidt Unique Decomposition Theorem, to understand the indecomposable A-modules and the morphisms between them. The easiest case is that of representation-finite algebras (that is, admitting only finitely many isomorphism classes of indecomposable modules). Much is known about such algebras (see, for instance, **[16, 18, 32, 33]**) so we consider the following class.

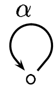
DEFINITION. *An algebra A is called schurian if, for every $x, y \in A_0$, we have* $\dim_k A(x, y) \leq 1$.

In terms of bound quivers, an algebra $A = kQ/I$ is thus schurian if and only if, for any contour (v, w) in Q, there exist scalars $\lambda, \mu \in k$ such that $\lambda v + \mu w \in I$. Thus, relations in Q are either monomial or binomial.

EXAMPLES. (a) Every triangular representation-finite algebra A is schurian. Indeed, if this is not the case, there exist $x, y \in A_0$ such that $\dim_k A(x, y) = d \geq 2$. Let $e = e_x + e_y$. The full subcategory eAe is the path algebra of the quiver

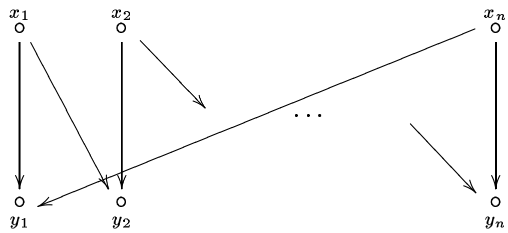


Since $d \geq 2$, it is representation-infinite. Since the functor $\mathrm{Hom}_A(eA, -)$ is a left quasi-inverse to $- \otimes_{eAe} eA : \mathrm{mod}\ eAe \to \mathrm{mod}\ A$, the latter is a full embedding. Hence A itself is representation-infinite, a contradiction. The triangularity assumption is here essential, as is shown by the non-schurian representation-finite algebra given by the quiver



bound by $\alpha^2 = 0$.

(b) There exist many classes of representation-infinite schurian algebras. An example is the path algebra of the quiver



with $n \geq 2$, called a *crown*.

(c) Let Σ be a partially ordered set (poset) ordered by $\leq$. Its *incidence algebra* $k\Sigma$ is the subalgebra of the full $n \times n$ matrix algebra with coefficients in k defined by

$$k\Sigma = \{[a_{ij}]_{i,j\in\Sigma_0} \in M_n(k) | a_{ij} = 0 \text{ if } i \not\leq j\}.$$

Clearly, $k\Sigma$ is schurian. The quiver Q_Σ of Σ is the (oriented) Hasse diagram of Σ: its points are the elements of Σ, and there exists an arrow $x \to y$ if and only if x covers y, that is, if $x > y$ and there exists no $z \in \Sigma$ such that $x > z > y$. Let I_Σ be the ideal of kQ_Σ generated by all the differences $v - w$, with (v, w) a contour in Q_Σ, then $k\Sigma \cong kQ_\Sigma / I_\Sigma$.

The quiver Q_Σ enjoys the following property: if $\alpha : x \to y$ is an arrow in Q_Σ, there is no path from x to y of length at least two (such a path is called a *bypass*). Conversely, if Q is a quiver having no bypass, then there exists a unique poset Σ such that $Q = Q_\Sigma$: indeed, we define an order on the set $\Sigma = Q_0$ by setting $x \geq y$ if and only if there exists a path from x to y, see, for instance, [**35**].

2.2. Let A be a schurian algebra. Following [**16**] (1.2), we say that a k-basis $\mathcal{B}$ of A is a *multiplicative* (or a *normed*) *basis* if:

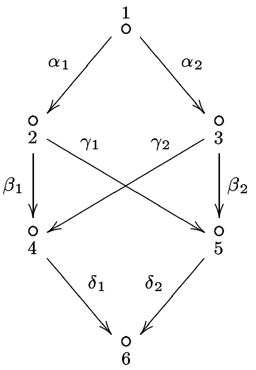
(a) $e_x \in \mathcal{B}$ for each $x \in A_0$.

(b) $\mathcal{B} \cap e_x\ (\mathrm{rad}\ {}^n A)\ e_y$ is a basis of $e_x\ (\mathrm{rad}\ {}^n A)\ e_y$ for all $x, y \in A_0$ and all $n \geq 1$.

(c) $b \in \mathcal{B} \cap e_x A e_y$ and $c \in \mathcal{B} \cap e_y A e_z$ imply $bc \in \mathcal{B}$ or $bc = 0$.

EXAMPLES. (a) A presentation $\nu : kQ_A \to A$ of an algebra A is called *normed* if $I_\nu = \text{Ker } \nu$ is generated by paths and differences $v - w$ of parallel paths v, w. If a presentation ν is normed, then the set $\{\nu(w) | w \text{ is a path such that } \nu(w) \neq 0\}$ is a multiplicative basis.

(b) The following example, due to Bongartz [**17**], plays a key role in the sequel. Let A_λ be given by the quiver



bound by $\alpha_1\beta_1 = \alpha_2\gamma_2$, $\alpha_1\gamma_1 = \alpha_2\beta_2$, $\beta_1\delta_1 = \gamma_1\delta_2$ and $\beta_2\delta_2 = \lambda \cdot \gamma_2\delta_1$, where $\lambda \in k \backslash \{0, 1\}$, and $\text{rad}^3 A = 0$. Then $A_{\lambda_1} \not\cong A_{\lambda_2}$ whenever $\lambda_1 \notin \{\lambda_2, \lambda_2^{-1}\}$. In particular, A_λ does not admit a multiplicative basis.

The existence of a multiplicative basis for an arbitrary representation-finite algebra was established in [**16**]. As a consequence, for any $d \geq 1$, there exist only finitely many isomorphism classes of representation-finite algebras of dimension d.

2.3.

DEFINITION. [**19, 21, 22**] *Let A be a schurian algebra. Its classifying space $S_\bullet A$ is the following simplicial complex:*

(a) *The set $S_0 A$ of 0-simplices is the set A_0 of objects of A.*

(b) *The set $S_n A$ of non-degenerate n-simplices is that of the sequences $(x_0, x_1, ...x_n) \in A_0^{n+1}$ such that the composition of morphisms*

$$A(x_0, x_1) \times A(x_1, x_2) \times ... \times A(x_{n-1}, x_n) \to A(x_0, x_n)$$

is non-zero.

For instance, $S_1 A$ is the set of all pairs $(x, y) \in A_0^2$ such that $A(x, y) \neq 0$.

The order in which the vertices of a simplex are listed is not important. Since $S_\bullet A$ is always a finite simplicial complex, it may be considered as embedded in an euclidean space of a suitable dimension.

Let $C_n A$ be the free abelian group with basis $S_n A$. We obtain a chain complex

$$\cdots \longrightarrow C_n A \xrightarrow{d_n} C_{n-1} A \xrightarrow{d_{n-1}} \cdots \longrightarrow C_1 A \xrightarrow{d_1} C_0 A,$$

where the differential d_n is defined by

$$d_n(x_0, x_1, ...x_n) = \sum_{i=0}^{n}(-1)^i(x_0, ...x_{i-1}, x_{i+1}, ...x_n).$$

The homology groups of this chain complex are denoted by $SH_n(A)$, and called the *simplicial homology groups* of A. If G denotes an arbitrary abelian group, then the cohomology groups of the complex $\mathrm{Hom}(C_\bullet A, G)$ are denoted by $SH^n(A, G)$ and called the *simplicial cohomology groups* of A with coefficients in G.

We give a sufficient condition for two schurian algebras with the same classifying space to be isomorphic. We denote by $k^\times$ the multiplicative group of the non-zero scalars.

PROPOSITION. [**19**] (2.3) *Let A be a schurian algebra.*

(a) *There exists a bijection between $SH^2(A, k^\times)$ and the set of isomorphism classes of schurian algebras B such that $S_\bullet A = S_\bullet B$.*
(b) *If $SH^2(A, k^\times) = 0$, and B is a schurian algebra such that $S_\bullet A = S_\bullet B$, then $B \cong A$. Moreover, A admits a multiplicative basis.*

Proof.

(a) For each pair $(x, y) \in S_1A$, we choose a basis vector v_{yx} of $A(x, y)$. Let $(x, y, z) \in S_2A$. Since $v_{zy}v_{yx} \neq 0$, there exists a non-zero scalar c_{zyx} such that
$$v_{zy}v_{yx} = c_{zyx}v_{zx}.$$
The map $S_2A \to k^\times$ defined by $(x, y, z) \mapsto c_{zyx}$ induces a group morphism $c_2A \to k^\times$.
This morphism is in fact a 2-cocycle: indeed, it suffices to verify that, for every $(x_0, x_1, x_2, x_3) \in S_3A$, we have $c_Ad_2(x_0, x_1, x_2, x_3) = 1$ and this equality only expresses the associativity of the multiplication of the basis vectors.
On the other hand, the residual class $\bar{c}_A \in SH^2(A, k^\times)$ does not depend on our choice of bases. Indeed, if, for $(x, y) \in S_1A$, the vector v'_{yx} is another basis vector of $A(x, y)$, there exists a unique scalar $g(x, y)$ such that $v'_{yx} = g(x, y)v_{yx}$. Let $c'_A : c_2A \to k^\times$ be the 2-cocycle associated to the new basis, and $g : c_1A \to k^\times$ be the group morphism induced by $(x, y) \mapsto g(x, y)$. It is easily verified that $c_A^{-1}c'_A = gd_2$ and so $\bar{c}_A = \bar{c'}_A$.
The map $A \mapsto \bar{c}_A$ furnishes the required bijection: if A, B are schurian such that $S_\bullet A = S_\bullet B$ and $\bar{c}_A = \bar{c}_B$, there exists $g : c_1A \to k^\times$ such that $c_A^{-1}c_B = gd_2$. Then g defines a base change of the sets $B(x, y)$, so that the structure constants of B become the same as those of A, thus $A \cong B$.
(b) Since the first statement follows from (a), we prove the second. Let indeed, for $(x, y) \in S_1A$, v_{yx} be a basis of $A(x, y)$, and c_A be the corresponding 2-cocycle. Since, by hypothesis, $\bar{c}_A = \bar{0}$, there exists a group morphism $g : c_1A \to k^\times$ such that $c_A = gd_1$. We define a new basis vector by $u_{yx} = g(x, y)^{-1}v_{yx}$ for $(x, y) \in S_1A$. It is not hard to check that the new basis $\{e_z, u_{yx} | z \in A_0, (x, y) \in S_1A\}$ is multiplicative. □

2.4. We look for conditions under which the assumption $SH^2(A, k^\times) = 0$ holds true. Since k is algebraically closed, the multiplicative group $k^\times$ is divisible and hence injective. If, moreover, char $k = 0$, then $k^\times$ is an injective cogenerator of the category of abelian groups (Indeed, let G be any abelian group, we must show that $\mathrm{Hom}_{\mathbb{Z}}(G, k^\times) \neq 0$. Since $k^\times$ is injective, it suffices to construct, for each non-zero $x \in G$, a non-zero morphism $u : x\mathbb{Z} \to k^\times$. If x is of finite order n, let $u(x) = \xi$, where ξ is a primitive n^{th} root of unity, while, if x is of infinite order, we set $u(x) = \zeta$, where ζ is an element of infinite multiplicative order: such elements exist in $k^\times$, since $\mathbb{Q}$ embeds in k).

LEMMA. *If $SH_2(A) = 0$, then $SH^2(A, k^\times) = 0$. If char $k = 0$, then the converse also holds.*

Proof. The Dual Universal Coefficients Theorem [**50**] (12.11) gives

$$SH^2(A, k^\times) \cong \text{Hom}_{\mathbb{Z}}(SH_2(A), k^\times) \oplus \text{Ext}^1_{\mathbb{Z}}(SH_1(A), k^\times).$$

If $SH_2(A) = 0$, the injectivity of $k^\times$ yields $SH^2(A, k^\times) = 0$. If char $k = 0$, the fact that $k^\times$ is an injective cogenerator implies easily the converse. □

2.5. We now consider another class of schurian algebras, very close to the incidence algebras.

DEFINITION. *A schurian triangular algebra $A = kQ/I$ is semi-commutative if, for every contour (v, w) of Q, we have $v \in I$ if and only if $w \in I$.*

Examples of semi-commutative algebras are the quotients of the incidence algebras, Indeed, it is easy to show that a schurian algebra $A = kQ/I$ is a quotient of an incidence algebra if and only if:

(i) there exists a poset Σ such that $Q = Q_\Sigma$, and
(ii) there exists an ideal J of kQ generated by monomials such that $I = I_\Sigma + J$.

Identifying J with the ideal I/I_Σ of $k\Sigma$, we have $A = k\Sigma/J$. Clearly, then, A is semi-commutative.

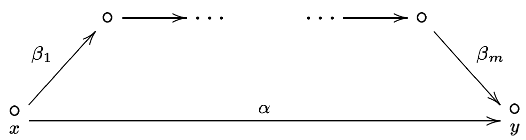
Conversely, we ask when a semi-commutative algebra is a quotient of an incidence algebra.

LEMMA. *Let A be a semi-commutative algebra. Then:*

(a) *There exists a unique poset Σ such that $Q = Q_\Sigma$.*
(b) *If $SH^2(A, k^\times) = 0$, then A is a quotient of $k\Sigma$.*

Proof.

(a) We must show that the quiver Q of A contains no bypass. Indeed, assume there exists a subquiver of Q of the form



and let $A \cong kQ/I$ be an arbitrary presentation of A. Since $\alpha \notin I$, we have $\beta_1\beta_2 \cdots \beta_m \notin I$. Since $I \subseteq \text{rad}\,^2 kQ$, there is no binomial relation linking α and $\beta_1\beta_2 \cdots \beta_m$. Then dim $_k A(x, y) \geq 2$, a contradiction.

(b) Let J be the ideal of $k\Sigma$ generated by all paths w in Q such that $w \in I$, and set $B = k\Sigma/J$. The semi-commutativity of A implies that, for any two points $x, y \in Q_0$, we have $A(x, y) \neq 0$ if and only if $B(x, y) \neq 0$. Therefore, A and B have the same classifying space. Applying (2.3) yields $A \cong B$. □

If, in particular, A is a semi-commutative algebra such that $SH_2(A) = 0$, then A is a quotient of an incidence algebra. On the other hand, Bongartz' example (2.2)(b) shows a semi-commutative algebra which is not a quotient of an incidence algebra.

3. Full convex subcategories.

3.1.

DEFINITION. *Let A be an algebra. A full subcategory C of A is convex if, for any path $x_0 \to x_1 \to \cdots \to x_t$ of A with $x_0, x_t \in C_0$, we have $x_i \in C_0$ for any i.*

Full convex subcategories of A are obtained by deleting successively sources and sinks.

LEMMA. *Let A be an algebra. A full subcategory C of A is convex if and only if there exists a sequence of full subcategories.*

$$C = C^{(0)} \subset C^{(1)} \subset ... \subset C^{(n)} = A$$

of A such that, for any j, $C_0^{(j)} \backslash C_0^{(j-1)}$ contains a unique point, which is a source or a sink of $C^{(j)}$.

Proof. Since the sufficiency is obvious, we prove the necessity by induction on the number of objects $|A_0|$ of A. If all sources and sinks of A are in C, then $C = A$. Otherwise, let $s \in A_0$ be a source or a sink such that $s \notin C_0$ and let $C^{(n-1)}$ be the full convex subcategory of A with object class $C_0^{(n-1)} = A_0 \backslash \{s\}$. Then $C \subseteq C^{(n-1)}$. Since $|C_0^{(n-1)}| = |A_0| - 1$, we just apply the induction hypothesis. $\square$

Let s be a source of A and define $A^{(s)}$ to be the full convex subcategory of A with object class $A_0 \backslash \{s\}$. We then say that A is the *one-point extension* of $A^{(s)}$ by the module $M = \text{rad } P_s$. In this case, the point s is called the *extension point* and A can be written as a triangular matrix algebra

$$A = \begin{bmatrix} A^{(s)} & 0 \\ M & k \end{bmatrix}$$

where the multiplication is induced from the $A^{(s)}$-module structure of M.

If s is a sink of A, we define dually $A_{(s)}$, and A is a *one-point coextension* of $A_{(s)}$ by the module I_s/S_s, with *coextension point* s.

The preceding lemma says that A is constructed starting from a full subcategory C by a sequence of one-point extensions or coextensions.

3.2. Let A be a schurian triangular algebra, and s be a source in A. Following [**19**], we define two sets

$$\Sigma^s = \{x \in A_0 | A(s,x) \neq 0\} \text{ and } \Sigma^{(s)} = \Sigma\ \{s\}.$$

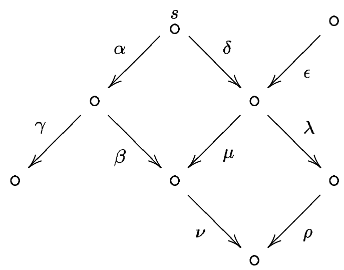
Thus, $x \in \Sigma^s$ if and only if x belongs to the support of the indecomposable projective module $P_s = e_s A$, that is, $P_s e_x \neq 0$. Similarly, $x \in \Sigma^{(s)}$ if and only if x belongs to the support of rad P_s.

We define a partial order on Σ^s by:

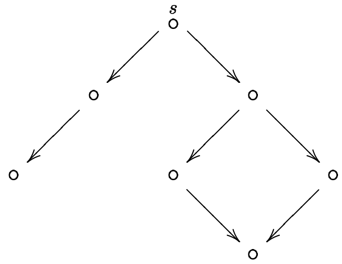
$$x \leq y \text{ if and only if } A(s,y)A(y,x) \neq 0$$

(that is, there exists a non-zero path from s to x, passing through y), and we give to $\Sigma^{(s)}$ the induced order. The incidence algebras $k\Sigma^s$ and $k\Sigma^{(s)}$ are thus subcategories, generally not full, of A.

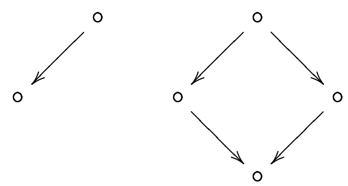
EXAMPLE. Let A be given by the quiver



bound by $\alpha\beta = 0$ and $\lambda\rho = \mu\nu$. Then Σ^s and $\Sigma^{(s)}$ are the posets with respective quivers



and



LEMMA. *Let s be a source in a schurian triangular algebra A, then we have a Mayer-Vietoris exact sequence*

$$\cdots \longrightarrow SH_2(A^{(s)}) \longrightarrow SH_2(A) \longrightarrow SH_1(k\Sigma^{(s)}) \longrightarrow SH_1(A^{(s)})$$
$$\longrightarrow SH_1(A) \longrightarrow SH_0(k\Sigma^{(s)}) \xrightarrow{j} SH_0(k\Sigma^s) \oplus SH_0(A^{(s)}) \longrightarrow SH_0(A).$$

Proof. We apply [**50**] (7.17) and the definitions of the classifying spaces of A, $A^{(s)}$, $k\Sigma^s$ and $k\Sigma^{(s)}$ to obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow SH_2(k\Sigma^s) \oplus SH_2(A^{(s)}) \longrightarrow SH_2(A) \longrightarrow SH_1(k\Sigma^{(s)})$$
$$\longrightarrow SH_1(k\Sigma^s) \oplus SH_1(A^{(s)}) \longrightarrow SH_1(A) \longrightarrow SH_0(k\Sigma^{(s)})$$
$$\xrightarrow{j} SH_0(k\Sigma^s) \oplus SH_0(A^{(s)}) \longrightarrow SH_0(A).$$

Now, the poset Σ^s admits s as unique maximal element, hence $k\Sigma^s$ is homeomorphic to a cone, so that $SH_n(k\Sigma^s) = 0$ for any $n \geq 1$. □

3.3. Recall that $SH_0(A)$ is the free abelian group having as rank the number of connected components of $S_\bullet A$ (or, equivalently, of A). Thus, the morphism j in the sequence of (3.2) is injective if and only if the number of connected components of $k\Sigma^{(s)}$ (that is, the number of indecomposable summands of rad P_s) equals the number of connected components of $A^{(s)}$.

DEFINITION. **[15, 42]** *A source s in a (non-necessarily schurian) algebra A is separating if the number of connected components of $A^{(s)}$ equals the number of indecomposable summands of rad P_s. The algebra A is called separated if every $x \in A_0$ is separating as a source in the full subcategory of A consisting of the non-predecessors of x. We define dually coseparating sinks and coseparated algebras.*

LEMMA. *In the sequence* (3.2) *above, the source s is separating if and only if the morphism j is injective.* □

In the example (3.2) above, the source s is not separating, hence the morphism j is not injective.

4. The fundamental group and simple connectedness.

4.1.

DEFINITION. *Let A be a schurian triangular algebra. The fundamental group $\pi_1(A)$ of A is defined to be that of its classifying space $S_\bullet A$. The algebra A is called simply connected if $S_\bullet A$ is a simply connected simplicial complex.*

We reformulate in the context of algebras the well-known standard technique for computing the fundamental group of a simplicial complex (see, for instance, **[50]** (7.34)). Let $A = kQ/I$ be a schurian triangular algebra, T be a maximal tree in the quiver Q, F be the free multiplicative group on the set Q_1 of arrows of Q, and K be the normal subgroup of F generated by all elements of one of the following two types:

(a) arrows in T, and
(b) expressions of the form $(\alpha_1\alpha_2\cdots\alpha_m)(\beta_1\beta_2\cdots\beta_n)^{-1}$ whenever $\alpha_1\alpha_2\cdots\alpha_m$ and $\beta_1\beta_2\cdots\beta_n$ are two non-zero parallel paths in Q.

Then $\pi_1(A) \cong F/K$, see **[21]** (2.1).

We give another expression for the fundamental group. A *walk* in a quiver Q is an expression of the form $\alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\cdots\alpha_t^{\epsilon_t}$ where the α_i are arrows, and $\epsilon_i \in \{1,-1\}$: that is, it is a composition of arrows and formal inverses of arrows. Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i$ (where the $\lambda_i \in k$ are non-zero and the w_i are paths of length at least two from x to y) is called *minimal* if $m \geq 2$ and for any non-empty proper subset $J \subsetneqq \{1, 2, ...m\}$, we have $\sum_{j\in J} \lambda_j w_j \notin I$. Let $\pi_1(Q)$ denote the fundamental group of Q (considered as a graph). It is well-known that $\pi_1(Q)$ is the free group in $\chi(Q)$ generators, where $\chi(Q) = 1 - |Q_0| + |Q_1|$ is the Euler characteristic of Q, see **[45]**. Let N be the normal subgroup of $\pi_1(Q)$ generated by all elements of the form $\gamma^{-1}u^{-1}v\gamma$, where γ is a walk from the base point to x and u, v are two paths from x to y occurring in the same minimal relation. The quotient $\pi_1(Q)/N$ is called the *fundamental group of the bound quiver* (Q, I), see **[43, 37]**. It was shown in **[14]** that, if the algebra $A = kQ/I$ is schurian, then the fundamental group of (Q, I) is an invariant of the algebra A. However, if $A = kQ/I$ is not schurian, it does heavily depend on the presentation, see, for instance **[23]**. The following result was simultaneously shown in **[48]** for incidence algebras and in **[21]** for a class of schurian algebras which includes all the triangular ones.

LEMMA. **[21]** (2.2) *Let $A = kQ/I$ be a schurian triangular algebra, then $\pi_1(A) \cong \pi_1(Q)/N$.* □

4.2. It follows that the Euler characteristic provides a lower bound for the number of binomial relations in a schurian simply connected algebra.

COROLLARY. [**4**] (4.2) *Let A be schurian, triangular and simply connected. Let S be a set of binomial relations and M be a set of monomials such that I is generated by $S \cup M$. Then $|S| \geq \chi(Q)$.*

Proof. Since A is simply connected, $\pi_1(Q) = N$. The result follows from the observation that $\pi_1(Q)$ is free in $\chi(Q)$ generators. □

4.3.

THEOREM. [**31**] (7.2) *Let G be any finitely presented group. Then there exists a poset Σ such that $\pi_1(k\Sigma) \cong G$.*

Proof. It is well-known that, for any finitely presented group G, there exists a finite simplicial complex S such that $\pi_1(S) \cong G$, see [**50**] (7.45). Let Σ be the set of all non-degenerate simplices of S, ordered by inclusion. Then $S_\bullet(k\Sigma)$ is the first barycentric subdivision of S, so that $\pi_1(k\Sigma) = \pi_1(S_\bullet(k\Sigma)) \cong \pi_1(S) \cong G$. □

Posets obtained from a simplicial complex as in the proof above have been characterised in [**24**].

4.4. One consequence is that the problem of verifying whether an algebra is simply connected or not is undecidable. We have however the following theorem, of which the first part is [**52**] (2.3) and the second part is in [**15**]. A triangular non-schurian algebra A is *simply connected* if, for every presentation $A = kQ/I$ of A, the fundamental group of the bound quiver (Q, I) is trivial [**12**].

THEOREM. *If an algebra is separated, then it is simply connected. If it is representation-finite, then the converse also holds.* □

4.5. We end this section with a computational tool, obtained in [**3**]. Let s be a source in a schurian triangular algebra $A = kQ/I$, and suppose $A^{(s)} = \prod_{i=1}^{c} A_i^{(s)}$ with all $A_i^{(s)}$ connected. We agree to set $\pi_1(A^{(s)}) = \prod_{i=1}^{c} \pi_1(A_i^{(s)})$ and, accordingly, $A^{(s)}$ is simply connected if and only if so is each $A_i^{(s)}$. Let $s^{\rightarrow}$ denote the set of all arrows in A of source s, and $\approx$ be the least equivalence relation on $s^{\rightarrow}$ such that $\alpha \approx \beta$ if there exist non-zero parallel paths αu and βv, see [**10**] (2.1). Let $t(Q, I)$ denote the number of equivalence classes for $\approx$, and t the number of indecomposable summands of rad P_s. By [**10**] (2.2), we have $c \leq t(Q, I) \leq t$.

THEOREM. [**3**] *The cokernel of the morphism $\pi_1(A^{(s)}) \to \pi_1(A)$ induced by the inclusion is the free group in $t(Q, I) - c$ generators.* □

4.6.

COROLLARY. [**3**] *Let A be a triangular schurian algebra, and s be a source such that $A^{(s)}$ is simply connected. Then A is simply connected if and only if the point s is separating.*

Proof. Indeed, rad P_s is indecomposable if and only if $t = 1$, thus if and only if $t(Q, I) = 1$ for every presentation $A = kQ/I$ (see [**10**] (2.2)), or if and only if $\pi_1(A)$ is trivial. □

5. Hochschild cohomology groups.

5.1. There is a strong relationship between the simple connectedness of a schurian algebra A, and the vanishing of the first Hochschild cohomology group of A with coefficients in the bimodule ${}_AA_A$. The Hochschild complex $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$ is defined as follows: $C^i = 0$, $d^i = 0$ for every $i < 0$, $C^0 =_A A_A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$ (where $A^{\otimes i}$ stands for the i^{th} tensor power $A \otimes_k \cdots \otimes_k A$ of A), $d^0 : A \to \mathrm{Hom}_k(A, A)$ is such that $(d^0x)(a) = ax - xa$ (for $a, x \in A$) and $d^i : C^i \to C^{i+1}$ is given by

$$(d^i f)(a_1 \otimes \ldots \otimes a_{i+1}) = a_1 f(a_2 \otimes \ldots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{i+1})$$
$$+(-1)^{i+1} f(a_1 \otimes \ldots \otimes a_i)a_{i+1}$$

for $f \in C^i$ and $a_1, \ldots a_{i+1} \in A$. Then $HH^i(A) = H^i(C^\bullet)$ is called the i^{th} Hochschild cohomology group of A with coefficients in ${}_AA_A$, see **[25]**.

We denote by k^+ the additive group of the field k.

THEOREM. **[46]** *Let A be a schurian triangular algebra, then*

$$HH^1(A) \cong \; Hom \; (\pi_1(A), k^+).\square$$

5.2. This implies that, for schurian triangular algebras, the first Hochschild cohomology group is a simplicial cohomology group.

COROLLARY. *Let A be a schurian triangular algebra. Then*

$$HH^1(A) \cong SH^1(A, k^+).$$

Proof. By **[50]** (4.29), $SH_1(A)$ is the abelianisation of $\pi_1(A)$, hence

$$HH^1(A) \cong \text{ Hom } (\pi_1(A), k^+) \cong \text{ Hom } (SH_1(A), k^+) \cong SH^1(A, k^+).$$

where the last isomorphism follows from the Dual Universal Coefficients theorem. □

5.3. Another consequence of (5.1) is that the simple connectedness of a schurian algebra implies the vanishing of its first Hochschild cohomology group. The converse is not true, even for incidence algebras. Indeed, let G be any simple non-abelian group. By (4.3), there exists a poset Σ such that $\pi_1(k\Sigma) \cong G$. Since G is simple, its abelianisation $SH_1(k\Sigma)$ is zero. Hence $HH^1(k\Sigma) \cong$ Hom $(\pi_1(k\Sigma), k^+) \cong$ Hom $(SH_1(k\Sigma), k^+) = 0$.

It is an interesting problem to identify the classes of - not necessarily schurian - algebras A such that A is simply connected if and only if $HH^1(A) = 0$. In **[52]**, Skowroński conjectures that this equivalence holds true for tame algebras. This is the case for representation-finite algebras.

THEOREM. **[20]** *A representation-finite algebra A is simply connected if and only if $HH^1(A) = 0$.* □

5.4. A related conjecture is formulated in [**6**]. Recall from [**39**] that a - not necessarily schurian - algebra A is *quasi-tilted* if gl.dim.$A \leq 2$ and, for every indecomposable A-module M, we have pd $M_A \leq 1$ or id $M_A \leq 1$. It is conjectured that a quasi-tilted algebra A is simply connected if and only if $HH^1(A) = 0$.

THEOREM. [**6**] *A tame quasi-tilted algebra A is simply connected if and only if $HH^1(A) = 0$.* □

5.5. A slight generalisation of (5.4) is known. Recall from [**27**] that a - not necessarily schurian - algebra A is *weakly shod* if the length of any path from an indecomposable injective to an indecomposable projective is bounded. Any quasi-tilted algebra is weakly shod.

THEOREM. [**7**] *A tame weakly shod algebra A is simply connected if and only if $HH^1(A) = 0$.* □

5.6. In the spirit of (5.2), Martins and de la Peña construct in [**44**] for a schurian algebra A and for any $n \geq 2$, a monomorphism $SH^n(A, k^+) \to HH^n(A)$. It was shown by Gerstenhaber and Schack [**36**] that this monomorphism is an isomorphism whenever A is an incidence algebra, a result generalised by Bustamante to the case where A is semi-commutative.

THEOREM. [**22**] (6.5) *Let A be a semi-commutative algebra. Then, for any n, we have $HH^n(A) \cong SH^n(A, k^+)$.* □

5.7. We need another result on the Hochschild cohomology of a non-necessarily schurian algebra A. Let s be a source, and $A^{(s)}$ be the full convex subcategory of A with object class $A_0 \backslash \{s\}$. The following sequence plays for Hochschild cohomology a role similar to the Mayer-Vietoris sequence of (3.2) for simplicial cohomology.

THEOREM. [**38**] (5.3) *Let s be a source in an algebra A, and $M = rad\ P_s$, then we have an exact sequence*

$$0 \longrightarrow HH^0(A) \longrightarrow HH^0(A^{(s)}) \longrightarrow End\ M/k \longrightarrow HH^1(A)$$
$$\longrightarrow HH^1(A^{(s)}) \longrightarrow Ext^1_{A^{(s)}}(M, M) \longrightarrow \cdots \longrightarrow Ext^{i-1}_{A^{(s)}}(M, M)$$
$$\longrightarrow HH^i(A) \longrightarrow HH^i(A^{(s)}) \longrightarrow \cdots . \square$$

5.8.

COROLLARY. [**52**] (3.2) *If $HH^1(A) = 0$, then every source of A is separating.*

Proof. Let s be a source, then write $A^{(s)} = \prod_{i=1}^{c} A_i^{(s)}$ connected, and $M = \oplus_{i=1}^{c} M_i$, where each M_i is an $A_i^{(s)}$-module. We have a short exact sequence:

$$0 \to HH^0(A) \to HH^0(A^{(s)}) \to \text{ End } M/k \to 0.$$

Since $HH^0(A) \cong k$ and $HH^0(A^{(s)}) \cong k^c$, we have $\dim_k$ End $M = c$. Thus, for each i, we have End $M_i = k$: each M_i is an indecomposable $A_i^{(s)}$-module. □

5.9.

COROLLARY. **[5]** (5.2) **[10]** (2.6) *Let A be a schurian algebra such that $SH_1(A) = 0$ (for instance, let A be simply connected). Then every source of A is separating.*

Proof. By (5.1), the hypothesis implies $HH^1(A) = 0$. □

6. Strongly simply connected algebras.

6.1.

DEFINITION. **[29, 52]** *A triangular algebra A is called strongly simply connected if every full convex subcategory of A is separated.*

This definition can be extended to define in the same way the strong simple connectedness of a locally bounded category A, in the sense of **[18]**, which is interval-finite (that is, such that, for any $x, y \in A_0$, the set $\{z \in A_0 | A(x,z)A(z,y) \neq 0\}$ is finite), compare with **[8]**.

Obvious examples of strongly simply connected algebras are algebras whose quivers are trees. In fact, a hereditary (or monomial) algebra is strongly simply connected if and only if its quiver is a tree. The algebra of Bongartz' example (2.2)(b) is not strongly simply connected: indeed, the full convex subcategory obtained by deleting the unique sink is not separated. However, it is simply connected (so, there exist simply connected algebras which are not strongly simply connected). We give several characterisations of strongly simply connected schurian algebras **[29]** (2.4) **[52]** (4.1).

THEOREM. *Let A be a schurian triangular algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *Every full convex subcategory of A is simply connected.*
(c) *For every full convex subcategory C of A, we have $SH_1(C) = 0$.*
(d) *For every full convex subcategory C of A, we have $HH^1(C) = 0$.*
(e) *For every full convex subcategory C of A, and every abelian group G, we have $SH^1(C, G) = 0$.*

Proof. (a) implies (b): this is just (4.4).

(b) implies (c): this is trivial.

(c) implies (d): by (5.1), $HH^1(C) \cong \text{Hom}\,(\pi_1(C), k^+) \cong \text{Hom}\,(SH_1(C), k^+) = 0$.

(d) implies (a): this is proven by induction on the number of objects $|A_0|$ of A. Since A is triangular, there exists a source s in A. Write $A^{(s)} = \prod_{i=1}^{c} A_i^{(s)}$, with the $A_i^{(s)}$ connected. By (5.8), s is separating. Moreover, the $A_i^{(s)}$ are proper full convex subcategories of A, thus, by the induction hypothesis, are separated. Hence so is A.

(e) is equivalent to (c): by the Dual Universal Coefficients Theorem, $SH^1(C, G) \cong \text{Hom}\,(SH_1(C), G)$, so ($c$) implies ($e$). The converse follows upon setting $G = SH_1(C)$. □

The equivalence of (a)(b)(d) still holds in the non-schurian case.

As a trivial consequence of the above theorem, an algebra A is strongly simply connected if and only if so is the opposite algebra A^{op}.

6.2. Much attention has been given to the study of the bound quiver of strongly simply connected algebras. The main result [**29**] (2.7) [**8**] (2.4) says that a schurian strongly simply connected algebra is a quotient of an incidence algebra (in particular, admits a multiplicative basis). In order to prove it, we need the main structure theorem from [**8**].

We need some terminology. Let $A = kQ/I$ be a triangular algebra. A contour (p, q) in Q from x to y is *interlaced* if the paths p and q have a common point besides x and y. A contour (p, q) is *irreducible* if there exists no sequence of paths $p = p_0, p_1, ...p_t = q$ from x to y such that, for each i, the contour (p_i, p_{i+1}) is interlaced. Let C be a (simple, non-oriented) cycle which is not a contour. Let $\sigma(C)$ be the number of sources (equivalently, of sinks) in C. The cycle C is *reducible* if there exist two points x, y in C and a path $p : x \to ... \to y$ in Q such that C consists of two walks w_1, w_2 from x to y, both $w_1 p^{-1}$ and $w_2 p^{-1}$ are cycles and finally $\sigma(w_1 p^{-1}) < \sigma(C)$, $\sigma(w_2 p^{-1}) < \sigma(C)$. The cycle C is *irreducible* if either it is an irreducible contour, or it is not reducible in the above sense. Equivalently, C is irreducible if either it is an irreducible contour, or it verifies the following condition and its dual: for any source x in C, no proper successor of x in Q is a source in C, and exactly two successors of x in Q are sinks of C.

A typical example of an irreducible cycle which is not a contour is the crown of example (2.1)(b).

Let (Q, I) be a bound quiver. Two parallel paths p, q are called *naturally homotopic* (see [**22**]) if either $p = q$ or there exist a sequence of parallel paths $p = p_0, p_1, ...p_t = q$ and, for any i, paths u_i, v_{i1}, v_{i2}, w_i such that $p_i = u_i v_{i1} w_i$, $p_{i+1} = u_i v_{i2} w_i$, with v_{i1}, v_{i2} appearing in the same minimal relation (in the sense of (4.1)). A contour (p, q) is called *naturally contractible* if p and q are naturally homotopic, compare with the formulation in [**8**].

For instance, in Bongartz' example (2.2)(b), all contours are naturally contractible. The next theorem is valid whether the algebra is schurian or not.

THEOREM. [**8**] (1.3) *A triangular algebra A is strongly simply connected if and only if, for any presentation $A \cong kQ/I$, the following conditions are satisfied:*

(a) *Any irreducible cycle in Q is an irreducible contour.*

(b) *Any irreducible contour in (Q, I) is naturally contractible.*

Proof. Necessity. Assume that $A \cong kQ/I$ is strongly simply connected. Let w be an irreducible cycle which is not an irreducible contour. Then $w = p p_1^{-1} v q_1 q^{-1}$, where x is a source on w, $p : x \to ... \to a$, $p_1 : c_1 \to a_1 \to ... \to a$ are paths , $v : c_1 - c_2 - ... - c_n$ is a walk with c_1, c_n sources on w, $q_1 : c_n \to b_1 \to ... \to b$, $q : x \to ... \to b$ are paths.

Since w is irreducible, there is no path from x to any c_i. Also, since Q is finite, we may suppose that no path from x to a intersects the paths from x to b. It is easily seen that x is not separating as a source in the convex envelope of w in Q.

Suppose now that there exists an irreducible contour (p, q) in Q from x to y that is not naturally contractible. Define a partial order on the contours in Q as follows. Let (p_1, q_1) and (p_2, q_2) be contours from x_1 to y_1 and x_2 to y_2, respectively. We set $(p_1, q_1) \leq (p_2, q_2)$ if either $(p_1, q_1) = (p_2, q_2)$ or $(x_1, y_1) \neq (x_2, y_2)$ and then x_1 is a successor of x_2 and y_1 is a predecessor of y_2. We may assume (p, q) to be minimal with respect to this order. Let $B = kQ'/I'$ be the convex envelope of x, y in A. We show that x is not separating as a source in B. Let W_1 (or W_2) denote the set of non-trivial paths in Q' of source x which are contained in a path naturally homotopic to p (or not naturally homotopic to p, respectively) in (Q', I'). The minimality of (p, q)

implies that $W_1 \cap W_2 = \emptyset$ and each path in Q' that is reducible to p lies in W_1. Then rad $P_x = R_1 + R_2$ where, for each i, R_i is the k-vector space with basis the residual classes modulo I' of the paths of W_i. Since any two paths $p_1 \in W_1$, $p_2 \in W_2$ are not simultaneously involved in a relation, we have $R_1 \cap R_2 = 0$. Moreover, two paths $p_1 \in W_1$ and $p_2 \in W_2$ are not reducible to each other, thus they have no common points besides x and y. Thus, if $p_i : x \to ... \to y$ is a path in the k-basis of R_i, and α is an arrow in Q', then either $p_i'\alpha = 0$ or $p_i\alpha$ lies in the k-basis of R_i. Therefore, the R_i are B-submodules of rad P_x. Thus x is not separating in B.

Sufficiency. Let $A \cong kQ/I$ be a presentation satisfying our two conditions. It suffices to show that A is simply connected. Let w be a cycle in Q. If $\sigma(w) = 1$, then w is a contour, so its sides are naturally homotopic. If $\sigma(w) > 1$, then w is not irreducible, so w is homotopic to some composition w_1w_2, where w_1, w_2 are cycles with $\sigma(w_1) < \sigma(w)$, $\sigma(w_2) < \sigma(w)$. By induction, w is naturally contractible. It follows easily that any closed walk in Q is contractible. □

6.3.

COROLLARY. [8] (2.1) *Let $A \cong kQ/I$ be a schurian strongly simply connected algebra. Then A is semi-commutative.*

Proof. Let (p, q) be a contour, which we assume minimal in the partial order of the proof of (6.2), such that $p \notin I$ while $q \in I$. If (p, q) is reducible, there exist paths $p = p_0, p_1, ...p_t = q$ such that, for each i, (p_i, p_{i+1}) is interlaced. The minimality of (p, q) implies that $p_1 \notin I$ and, inductively, $q \notin I$, a contradiction. Therefore, (p, q) is irreducible. Then, by (6.2), it is naturally contractible, so there exist paths $p = p_0, p_1, ...p_t = q$ such that, for each i, (p_i, p_{i+1}) is a contour and p_i, p_{i+1} contain subpaths q_i, q_{i+1}, respectively, which are involved in the same binomial relation in (Q, I). If $q_1 \neq p_1$, then (p_0, p_1) is interlaced. Hence $p_1 \notin I$ by the minimality of (p, q). If $q_1 = p_1$, then $p = p_0$ and p_1 are involved in the same binomial relation, hence $p_1 \notin I$. Inductively, $q \notin I$, a contradiction. □

6.4.

COROLLARY. [4] (4.4) *Let $A = kQ/I$ be a schurian strongly simply connected algebra. Then:*

(a) *There exists a unique poset Σ such that $Q = Q_\Sigma$.*
(b) *The incidence algebra $k\Sigma$ is strongly simply connected.*

Proof. (*a*) This follows from (6.3) and (2.5).

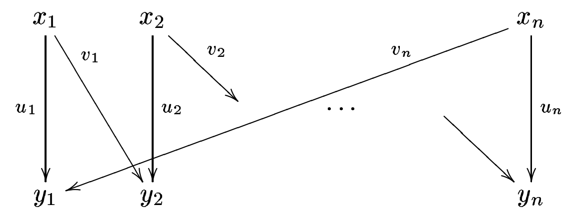
(*b*) Since $Q = Q_\Sigma$, any irreducible cycle in Q_Σ is an irreducible contour, by (6.2). Since any contour in $k\Sigma$ is commutative, the statement follows from (6.2). □

7. Quasi-crowns, strong simple connectedness and multiplicative bases.

7.1. Let A be a schurian triangular algebra. Let $x, y \in A_0$ be such that $A(x, y) \neq 0$. We define the *interval* $[x, y]$ to be the full subcategory having as objects all $z \in A_0$ lying on a non-zero path from x to y, that is, such that the composition $A(x, z) \times A(z, y) \to A(x, y)$ is non-zero.

If, for instance, A is an incidence algebra, then all paths from x to y are non-zero, and $[x, y]$ coincides with the convex envelope of x and y in A.

Definition. *Let C be a full subcategory of A having $2n$ objects $\{x_1, ...x_n, y_1, ...y_n\}$ and $2n$ non-zero morphisms $\{u_1, ...u_n, v_1, ...v_n\}$ with $n \geq 2$ and of the form*



We say that C is a crown (of width n) in A if:

(a) $[x_i, y_j] \cap [x_h, y_l] = \emptyset$ *if and only if* $j = i$ *and* $(h, l) \in \{(i, i), (i-1, i), (i, i+1)\}$ *or* $j = i + 1$ *and* $(h, l) \in \{(i, i), (i, i+1), (i+1, i+1)\}$.

(b) *The intersection of three distincts* $[x_i, y_j]$ *is empty.*

(c) *For each* i, $[x_i, y_i] \cap [x_i, y_{i+1}] = \{x_i\}$ *and* $[x_i, y_i] \cap [x_{i-1}, y_i] = \{y_i\}$.

Here, and in the sequel, we agree to set $x_0 = x_n$, $x_1 = x_{n+1}$, $y_0 = y_n$, $y_1 = y_{n+1}$.

Lemma. **[11]** (6.2) *Let $k\Sigma$ be an incidence algebra which is the convex envelope of a crown C. Then $HH^1(k\Sigma) \neq 0$. In particular, $k\Sigma$ is not simply connected.*

Proof. Let Σ' be the poset obtained from Σ by adding two elements a, b such that $a \geq x \geq b$ for all $x \in \Sigma$. By **[41]** (1.2) , **[26]** (2.1), $HH^1(k\Sigma) \cong \text{Ext}^3_{k\Sigma'}(S_a, S_b)$.

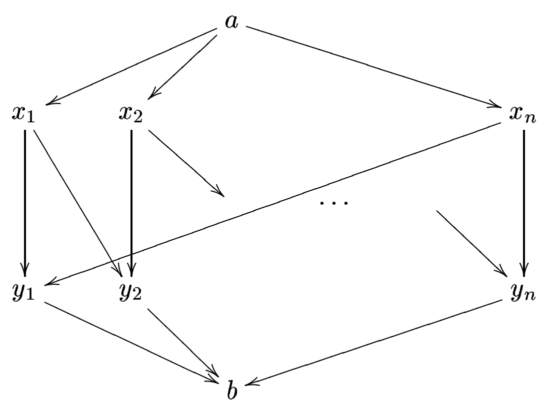
Considering the short exact sequences

$$0 \to \text{ rad } P_a \to P_a \to S_a \to 0 \quad \text{and} \quad 0 \to S_b \to I_b \to I_b/S_b \to 0.$$

we deduce that $HH^1(k\Sigma') \cong \text{Ext}^1_{k\Sigma'}(\text{rad}P_a, I_b/S_b)$ by dimension shifting. We thus only need to construct a non-split short exact sequence

$$0 \to I_b/S_b \to E \to \text{ rad } P_a \to 0$$

in mod $k\Sigma'$.



Consider in $k\Sigma'$ all the paths $u_{n_1}, u_{n_2}, ...u_{n_r}$ from x_n to y_n and, for each i, let γ_{n_i} be the unique arrow of u_{n_i} with source x_n. Since C is a crown, no γ_{n_i} occurs in a path of $Q_{\Sigma'}$ different from the u_{n_j}. Let $\lambda \in k$ and define a $k\Sigma'$-module E_λ by

$$\begin{array}{lcll}
E_\lambda(a) & = & k & = \; E_\lambda(b) \\
E_\lambda(x) & = & k^2 & \text{for any } x \notin \{a,b\} \\
E_\lambda(\alpha) & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for any arrow } \alpha \text{ of source } a \\
E_\lambda(\beta) & = & \begin{pmatrix} 0 & 1 \end{pmatrix} & \text{for any arrow } \beta \text{ of target } b \\
E_\lambda(\gamma_{n_i}) & = & \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} & \text{for any } i \in \{1,...r\} \\
E_\lambda(\gamma) & = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for any other arrow.}
\end{array}$$

It is easily checked that E_λ is indeed a $k\Sigma'$-module and that, if $\lambda \neq \mu$, then $E_\lambda \ncong E_\mu$.

We now define $f_\lambda : I_b/S_b \to E_\lambda$ by $f_{\lambda,a} = 1$, $f_{\lambda,b} = 0$ and $f_{\lambda,x} = \binom{1}{0}$ for all $x \notin \{a,b\}$. Then f_λ is easily seen to be a monomorphism of cokernel rad P_a. Thus, for each $\lambda \in k$, we have an extension

$$0 \longrightarrow I_b/S_b \xrightarrow{f_\lambda} E_\lambda \longrightarrow \text{rad } P_a \longrightarrow 0$$

Since the E_λ are pairwise non-isomorphic, these extensions, except perhaps one of them, do not split. This shows that $HH^1(k\Sigma) \neq 0$. The second statement follows from (5.1). □

7.2.

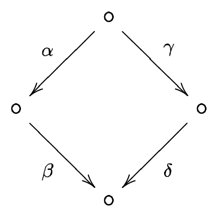
THEOREM. **[29]** (3.3) *An incidence algebra $k\Sigma$ is strongly simply connected if and only if it contains no crowns.*

Proof. Assume first $k\Sigma$ to be strongly simply connected. Since any full convex subcategory of $k\Sigma$ is simply connected, the convex hull of any crown in $k\Sigma$ would yield a contradiction by (7.1). Therefore $k\Sigma$ contains no crowns.

Conversely, if $k\Sigma$ is not strongly simply connected, then, since every contour in $k\Sigma$ is commutative, it follows from (6.2) that there exists an irreducible cycle C in $k\Sigma$ which is not a contour. We can assume, without loss of generality, that C is of least length. It is then easily (but tediously) verified that the sources and the sinks of C generate a crown in $k\Sigma$. □

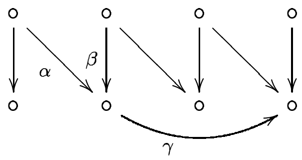
7.3. For schurian algebras which are not incidence algebras, the absence of crowns does not suffice to characterise strong simple connectedness.

EXAMPLES. (a) Let A be given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Then A has no crown but is not (strongly) simply connected.

(b) Let A be given by the quiver



bound by $\alpha\gamma = 0$, $\beta\gamma = 0$. Then A has no crown but is not (strongly) simply connected. On the other hand, A contains a "non-full" crown.

DEFINITION. *A full subcategory C of a schurian triangular algebra $A = kQ/I$ is a quasi-crown if there exists a set of arrows $\{\alpha_1, ...\alpha_r\}$ in Q such that, if R denotes the ideal of A generated by the classes of arrows $\alpha_i + I$ (with $1 \leq i \leq r$), then:*

(a) *The canonical surjection $C \to C/C \cap R$ is a retraction, whose section is an algebra morphism.*

(b) *$C/C \cap R$ is a crown in A/R.*

Condition (a) expresses that C is a split extension of $C/C \cap R$ in the sense of [**9, 13**]. There exists a criterion allowing to verify whether a schurian triangular algebra is a split extension or not, see [**5**] (3.2). Clearly, in an incidence algebra, a quasi-crown is a crown. However, even in quotients of incidence algebras, these two concepts do not coincide, as is shown by example (b) above (take R to be the ideal generated by the class of the arrow γ).

LEMMA. [**5**] (3.5) *Let A be a schurian strongly simply connected algebra. Then A contains no quasi-crown.*

Proof(Sketch). Assume that A contains a quasi-crown. By (6.2), the cycle C defined by this quasi-crown is reducible. Denoting by $x_1, ...x_n$ the sources of C, and $y_1, ...y_n$ its sinks, there must exist a path w from x_i to y_j (with $j \neq i$, $i+1$) or from x_i to x_j (with $j \neq i$). We may assume $j \geq i$ and $j - i$ minimal. If $j > i+1$, the new path w yields in each case an irreducible cycle which is not a contour, a contradiction to (6.2). Therefore $j = i+1$, so we get a contour which we may assume by duality to be from x_i to x_{i+1}, this contour being (v_i, wu_{i+1}). Since A is semi-commutative by (6.3) and v_i is non-zero, then wu_{i+1} is non-zero. However, C being a split extension of $C/C \cap R$ forces wu_{i+1} to be zero, a contradiction. □

7.4.

LEMMA. [**5**] (3.6) *Let s be a source in a schurian triangular algebra A. If the points $\{x_1, ...x_n, y_1, ...y_n\}$ induce a crown in $k\Sigma^s$, then the same points induce a quasi-crown in A.*

Proof (Sketch). Let R be the ideal of A generated by the classes of all the arrows lying on non-zero paths between the x_i and the y_j which are not in $k\Sigma^s$, and show that this ideal and the given crown in $k\Sigma^s$ verify the definition of quasi-crown. □

7.5.

PROPOSITION. **[5]** (6.2) *Let A be a schurian triangular algebra containing no quasi-crown. Then:*

(a) $SH_n(A) = 0$ *for all* $n \geq 2$.
(b) $SH^n(A, G) = 0$ *for all* $n \geq 3$ *and all abelian groups* G.

Proof. (a) This is done by induction on $|A_0|$. The result being clear if $|A_0| = 1$, assume that $|A_0| > 1$ and let s be a source in A. Consider the Mayer-Vietoris sequence of (3.2)

$$\cdots \longrightarrow SH_3(A^{(s)}) \longrightarrow SH_3(A) \longrightarrow SH_2(k\Sigma^{(s)}) \longrightarrow SH_2(A^{(s)})$$
$$\longrightarrow SH_2(A) \longrightarrow SH_1(k\Sigma^{(s)}) \longrightarrow \cdots$$

The algebra $A^{(s)}$ is schurian and contains no quasi-crowns. Since $|A_0^{(s)}| < |A_0|$, the induction hypothesis yields $SH_n(A^{(s)}) = 0$ for all $n \geq 2$. Moreover, by (7.4), the incidence algebra $k\Sigma^{(s)}$ contains no crown. By (7.2), it is (strongly) simply connected, so $SH_1(k\Sigma^{(s)}) = 0$. On the other hand, $|k\Sigma_0^{(s)}| < |A_0|$ so the induction hypothesis also implies $SH_n(k\Sigma^{(s)}) = 0$ for all $n \geq 2$. Therefore, $SH_n(A) = 0$ for all $n \geq 2$.

(b) This follows from (a) and the Dual Universal Coefficients Theorem **[50]** (12.11) $SH^n(A, G) \cong \mathrm{Hom}_{\mathbb{Z}}(SH_n(A), G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(SH_{n-1}(A), G)$. □

7.6. Parts (a)(b) of the next corollary are **[29]** (2.6), part (c) is **[34]** (see also **[5]** (6.9) (6.10)).

COROLLARY. *Let A be a schurian strongly simply connected algebra. Then:*

(a) $SH_n(A) = 0$ *for all* $n \geq 1$.
(b) $SH^n(A, G) = 0$ *for all* $n \geq 1$ *and all abelian groups* G.
(c) $HH^n(A) = 0$ *for all* $n \geq 1$.

Proof. (a) By (7.3) and (7.5), we have $SH_n(A) = 0$ for all $n \geq 2$. On the other hand, $SH_1(A) = 0$ because A is simply connected.

(b) By (7.3) and (7.5), we have $SH^n(A, G) = 0$ for all $n \geq 3$. The simple connectedness of A implies that $SH^1(A, G) = 0$. Finally, $SH^2(A, G) \cong \mathrm{Hom}_{\mathbb{Z}}(SH_2(A), G) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(SH_1(A), G) = 0$.

(c) By (5.6), the semi-commutativity of A implies that $HH^n(A) \cong SH^n(A, k^+) = 0$ because of (b). □

As a consequence of (c), the Hochschild cohomology ring $HH^\star(A)$ of A equals k.

7.7.

THEOREM. **[5]** (6.3) *Let A be a schurian triangular algebra containing no quasi-crowns. Then A admits a multiplicative basis.*

Proof. By (7.5), $SH_2(A) = 0$. The result follows from (2.4) and (2.3). □

7.8.

COROLLARY. **[17]** *Let A be a triangular representation-finite algebra. Then A admits a multiplicative basis.* □

7.9.

COROLLARY. [**5**] (6.4) *For each natural number d, there exist only finitely many isomorphism classes of schurian triangular algebras of dimension d, not containing quasi-crowns.* □

7.10.

COROLLARY. [**29, 8**] *Let A be a schurian strongly simply connected algebra, then A admits a multiplicative basis.* □

8. Strongly simply connected schurian algebras.

8.1.

THEOREM. [**29**] (2.7) *Let A be a schurian strongly simply connected algebra. Then there exists a unique poset Σ such that A is a quotient of $k\Sigma$. Furthermore, $k\Sigma$ is strongly simply connected.*

Proof. By (6.4), there exists a unique poset Σ such that Q_Σ is the quiver of A and, moreover, $k\Sigma$ is strongly simply connected. The result follows from (2.5) because, by (7.6), $SH^2(A, k^\times) = 0$. □

8.2. As a consequence, we obtain that a schurian strongly simply connected algebra admits a normed presentation, in the sense of example (2.2)(a).

THEOREM. [**8**] (2.4) *A triangular algebra A is schurian strongly simply connected if and only if there exists a presentation $A \cong kQ/I$ such that:*

(a) *Any irreducible cycle in Q is an irreducible contour.*
(b) *For any irreducible contour (p, q) in Q, we have $p, q \notin I$ and $p - q \in I$.*

Proof. Since the sufficiency follows from (6.2) and the definition of schurian, we just need to prove the necessity. Again, by (6.2), any irreducible cycle is an irreducible contour, and any irreducible contour is naturally contractible. We take the presentation of A as a quotient of an incidence algebra. Then all binomial relations are commutativity relations. Let (p, q) be an irreducible contour from x to y, and assume $p \in I$. We may also assume that (p, q) is minimal in the partial order of contours defined in the proof of (6.2). Since (p, q) is naturally contractible, there exist paths $p = p_0, p_1, ...p_m = q$ in Q from x to y such that, for each i, p_i and p_{i+1} contain subpaths q_i and q_{i+1}, respectively, which are involved in the same binomial relation. Since (p, q) is irreducible, there exists j such that p_j is reducible to p, while p_{j+1} is not. The minimality of (p, q) forces $p_j \in I$. Since p_{j+1} is not reducible to p, the paths p_j, p_{j+1} have a common point besides x and y. Thus p_j and p_{j+1} are involved in the same binomial relation, and this is impossible. Hence neither p nor q lies in I. □

8.3.

COROLLARY. [**4**] (4.5) *An algebra A is schurian strongly simply connected if and only if there exists a poset Σ such that $k\Sigma$ is strongly simply connected and $A \cong k\Sigma/J$, where the ideal J of $k\Sigma$ is generated by paths which are not entirely contained in irreducible contours.*

Proof. This follows from (8.1) and (8.2). □

8.4.

THEOREM. **[5]** (5.7) *Let A be a schurian triangular algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *A is separated and contains no quasi-crowns.*
(c) *A is simply connected and contains no quasi-crowns.*
(d) *$SH_1(A) = 0$ and A contains no quasi-crowns.*
(e) *$SH^1(A, G) = 0$ for all abelian groups G, and A contains no quasi-crowns.*

Proof. That (a) implies (b) follows from the definition and (7.3), that (b) implies (c) follows from (4.4), that (c) implies (d) and that (d) is equivalent to (e) are trivial. Thus, we just have to prove that (d) implies (a) and we do this by induction on $|A_0|$. Since the statement is clear for $|A_0| = 1$, assume it holds for all schurian algebras B without quasi-crowns such that $|B_0| < |A_0|$ and $SH_1(B) = 0$.

Let s be a source in A. By (7.4), the incidence algebra $k\Sigma^{(s)}$ contains no crown, hence it is simply connected, and $SH_1(k\Sigma^{(s)}) = 0$. The Mayer-Vietoris sequence

$$0 = SH_1(k\Sigma^{(s)}) \to SH_1(A^{(s)}) \to SH_1(A) = 0$$

yields $SH_1(A^{(s)}) = 0$. Since $A^{(s)}$ contains no quasi-crowns, it is strongly simply connected by the induction hypothesis.

This entails two consequences. Firstly, since s is separating (by (5.9)), then A is simply connected, by (4.6). Secondly, we have shown that any full convex subcategory A' of A obtained by deleting a source (or, dually, a sink) is strongly simply connected. The conclusion follows from the observation that any proper full convex subcategory of A is contained in such a subcategory A'. □

8.5.

COROLLARY. **[19]** (2.9) *A representation-finite algebra is strongly simply connected if and only if it is simply connected.* □

8.6. The last statement of the next corollary is **[1]** (1.2).

COROLLARY. *Let A be a schurian simply connected algebra which is a one-point extension of an algebra B containing no quasi-crown. Then B is strongly simply connected. If, in particular, B is representation-finite, then it is simply connected.*

Proof. Let s denote the extension point, so that $B = A^{(s)}$. By (7.4), $k\Sigma^{(s)}$ contains no crowns, so $SH_1(k\Sigma^{(s)}) = 0$. Since A is simply connected, $SH_1(A) = 0$. The Mayer-Vietoris sequence gives $SH_1(B) = 0$. Since B has no quasi-crown, it is strongly simply connected. □

8.7. The following corollary follows directly from (8.4), (7.5) and (7.6).

COROLLARY. *Let A be a schurian triangular algebra containing no quasi-crowns. Then A is strongly simply connected if and only if $SH_n(A) = 0$ for all $n \geq 1$.* □

9. Quotients of incidence algebras.

9.1. Predictably, we get much better results for quotients of incidence algebras. These results rest on the following combinatorial lemma, whose proof is omitted.

LEMMA. [**5**] (3.7) *Let A be a quotient of an incidence algebra and assume that, for some source s in A, $k\Sigma^{(s)}$ contains a crown. Then A contains a crown.* □

9.2.

PROPOSITION. [**5**] (5.8) *Let A be a quotient of an incidence algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *A is separated and contains no crowns.*
(c) *A is simply connected and contains no crowns.*
(d) *$SH_1(A) = 0$ and A contains no crowns.*
(e) *$SH^1(A, G) = 0$ for any abelian group G, and A contains no crowns.*

Proof. We repeat the proof of (8.4), taking (9.1) into account. □

9.3. We can show that, for strongly simply connected incidence algebras, the lower bound of (4.2) is attained. We again omit the proof.

LEMMA. [**4**] (3.6) *Let $k\Sigma = kQ_\Sigma/I_\Sigma$ be a strongly simply connected incidence algebra, and let (p, q) be a given irreducible contour in Q_Σ. Then there exists a set of generators of I_Σ, of cardinality $\chi(Q_\Sigma)$, consisting of the commutativity relations of irreducible contours, and containing the relation $p - q$.* □

9.4.

THEOREM. [**4**] (4.7) *Let $A = k\Sigma/J$ be a quotient of an incidence algebra. Then A is strongly simply connected if and only if A is simply connected and $k\Sigma$ is strongly simply connected.*

Proof. Since the necessity follows from (6.4), we prove the sufficiency. If A is not strongly simply connected then, by (8.2), there exists an irreducible contour (p, q) in the quiver Q of A such that $p \in J$. By (9.3), there exists a set of generators $\{\rho_1 = p - q, ...\rho_{\chi(Q)}\}$ of I_Σ. Let S be the subset of this set corresponding to those contours in Q which are non-zero in A. Thus, $|S| < \chi(Q)$ because $\rho_1 \notin S$. Hence, $I_\Sigma + J$ is generated by S together with a set of monomials $\{m_1, ...m_r\}$ where we may assume that m_1, m_2 are subpaths of p, q respectively (thus, $r \geq 2$). By (4.1), $\pi_1(A) \cong \pi_1(Q)/N$, where N is a normal subgroup of $\pi_1(Q)$ generated by $|S|$ elements. Since $|S| < \chi(Q)$, (4.2) yields a contradiction to the simple connectedness of A. □

9.5. It turns out that, among the schurian strongly simply connected algebras, the incidence algebras are characterised by having a generating set of cardinality equal to the Euler characteristic.

THEOREM. [**4**] (4.8) *Let $A = kQ/I$ be a schurian strongly simply connected algebra, given its normed presentation, and Σ be the associated poset. Then $A \cong k\Sigma$ if and only if I has a generating set of cardinality $\chi(Q)$.* □

9.6.

PROPOSITION. [**5**] (5.6) *Let A be a connected quotient of an incidence algebra containing no crowns. Then A is simply connected if and only if $HH^1(A) = 0$.*

Proof. By (5.1), it suffices to prove the sufficiency, and we do this by induction on $|A_0|$. Assume that $|A_0| > 1$ and $HH^1(A) = 0$. Up to duality, there exists a source s such that $A^{(s)}$ is connected. Since A is a quotient of an incidence algebra, so is $A^{(s)}$. Let $M = \text{rad } P_s$. Since $A^{(s)}$ is connected and $HH^1(A) = 0$, Happel's sequence (5.7) gives End $M = k$. Thus, M is indecomposable.

The support C of M is convex in A: indeed, let $x_0 \to x_1 \to ... \to x_t$ be a path in A with x_0, $x_t \in C_0$. Since A is semi-commutative, $A(s, x_0) \neq 0$ and $A(s, x_t) \neq 0$ imply $A(s, x_i) \neq 0$ for all i. Therefore, C is a quotient of an incidence algebra. Actually, C is even an incidence algebra (so that $C = k\Sigma^{(s)}$): if $a, b \in C_0$ are such that $A(a, b) \neq 0$ and $A(b, c) \neq 0$, then the existence of $M = \text{rad } P_s$ implies that $A(a, c) \neq 0$. Then we may define a partial order on C by $a \leq b$ whenever $A(a, b) \neq 0$, and this makes C an incidence algebra.

Since A contains no crown, neither does C, which therefore is strongly simply connected. By [**34**] (2.2), $\text{Ext}^1_C(M, M) = 0$. The convexity of C in $A^{(s)}$ implies that $\text{Ext}^1_{A^{(s)}}(M, M) = 0$. Happel's sequence (5.7)

$$0 = HH^1(A) \to HH^1(A^{(s)}) \to \text{ Ext }^1_{A^{(s)}}(M, M) = 0$$

yields $HH^1(A^{(s)}) = 0$. The induction hypothesis says that $A^{(s)}$ is simply connected. Since $HH^1(A) = 0$, then s is separating, by (5.8). By (4.6), A is simply connected. □

9.7.

COROLLARY. [**5**] *A schurian triangular algebra A is strongly simply connected if and only if A is a quotient of an incidence algebra, $HH^1(A) = 0$ and A contains no crowns.*

Proof. This follows from (9.6) and (8.4). □

10. Dismantlability.

10.1. It is shown in [**49**] that a poset is dismantlable if and only if its quiver contains no crown, that is, its incidence algebra is strongly simply connected. Our objective in this section is to introduce the corresponding notions and prove the corresponding results for schurian triangular algebras.

A point x in a schurian algebra A is called *doubly irreducible* [**30**] if there are at most one arrow of target x, and at most one arrow of source x.

For such a point $x \in A_0$, we define a new category $B = A(x)$. Suppose first

$$y \xrightarrow{\alpha} x \xrightarrow{\beta} z.$$

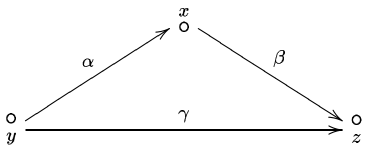
If $\alpha\beta \neq 0$, we let B be the full subcategory of A with object class $A_0 \backslash \{x\}$. If $\alpha\beta = 0$, we let B be the category with object class $A_0 \backslash \{x\}$ and the arrows of B are those of A except α and β which are replaced by a new arrow $\alpha' : y \to z$; moreover the relations in B are exactly those of A, except $\alpha\beta = 0$, which disappears. We define similarly B in case

$$x \xrightarrow{\beta} z$$

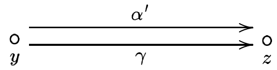
or

$$y \xrightarrow{\alpha} x.$$

Note that B is generally not schurian: if A is given by the quiver



bound by $\alpha\beta = 0$, then B is given by the quiver



LEMMA. [**5**] (4.1) *Let x be a doubly irreducible in A such that $A(x)$ is schurian. Then $\pi_1(A) \cong \pi_1(A(x))$.* □

10.2.

COROLLARY. [**5**] (4.2) *Let x be a doubly irreducible in a strongly simply connected algebra A, then $A(x)$ is strongly simply connected.*

Proof. Assume that

$$y \xrightarrow{\alpha} x \xrightarrow{\beta} z$$

(the other cases are similar) and that C is a full convex subcategory of $A(x)$. If y, z do not simultaneously lie in C, then C is a full convex subcategory of A, hence is simply connected. Otherwise, A contains $C(x)$ as full convex subcategory. Since $C(x)$ is simply connected, the result follows from (10.1). □

10.3. A schurian algebra A is called *dismantlable* if there exists an order $\{x_1, x_2, ...x_n\}$ of all objects in A such that x_1 is doubly irreducible in A and, for each $i \geq 1$, $A(x_1, ...x_i) = A(x_1, ...x_{i-1})(x_i)$ is schurian and admits x_{i+1} as doubly irreducible.

THEOREM. [**5**] (4.5) (4.7) *A schurian triangular algebra is strongly simply connected if and only if it is dismantlable.*

Proof. Assume the algebra A to be dismantlable. By (10.1) and induction, A is simply connected. It thus suffices to show that every full convex subcategory of A is dismantlable. This is done by induction on $|A_0|$. Assume that A is dismantlable and let C be a non-dismantlable full convex subcategory of A. Since $C \neq A$, there exists, up to duality, a source $s \in A_0$ not in C. Then C is a full convex subcategory of $A^{(s)}$. Since $|A_0^{(s)}| < |A_0|$, the induction hypothesis implies that $A^{(s)}$ is not dismantlable. Since A is so, there exists an order $\{x_1, ...x_n\}$ as in the definition above. Since $A^{(s)}$ is not dismantlable, $x_1 \neq s$. If $x_1 \notin C_0$, then C is a full convex subcategory of $A(x_1)$. Since $|A(x_1)_0| < |A_0|$, the induction hypothesis yields a contradiction. Hence $x_1 \in C_0$, and $C(x_1)$ is a full convex subcategory of $A(x_1)$. Again, induction says that $C(x_1)$ is dismantlable. Hence so is C, another contradiction.

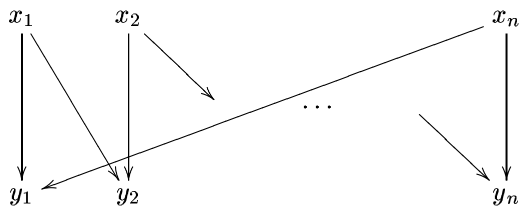
Conversely, let A be strongly simply connected. By (8.1), there exists a strongly simply connected incidence algebra $k\Sigma$ having A as a quotient. Since, by (7.2), $k\Sigma$ contains no crown, Σ is dismantlable by **[30]** (2.3). In particular, $k\Sigma$ (hence A) contains a doubly irreducible x. Also, $A(x)$ is schurian: this is clear if $A(x)$ is a full subcategory of A; otherwise there exist arrows

$$y \xrightarrow{\alpha} x \xrightarrow{\beta} z$$

such that $\alpha\beta = 0$ so, the statement is clear if x belongs to no cycle while, if it does, then we can assume that there exists an irreducible cycle containing α and β, a contradiction to (8.2). By (10.2), $A(x)$ is strongly simply connected. By induction, $A(x)$ is dismantlable. Hence so is A. □

11. Simple connectedness of incidence algebras.

11.1. In this section, we assume that $A = k\Sigma$ is an incidence algebra. A full subcategory C of A is a *weak crown* if it consists of $2n$ points $\{x_1, ...x_n, y_1, ...y_n\}$ and $2n$ non-zero morphisms of the form



(with $n \geq 2$) and satisfies conditions (a) and (b) of definition (7.1). Again, n is the *width* of C.

Let C be a weak crown in A. A point $x \notin C_0$ *suspends* C if it is a proper predecessor of two non-comparable points of C and no proper successor of x precedes the same points of C. A suspending point x of C is a *top* of C if it is an immediate predecessor of all maximal points of C. Let x be a suspending point of a weak crown C, then the *suspension* C^x of C is the full subcategory of A generated by x, all minimal points of C, and those of the maximal points which are not comparable to x. The dual notions are of points which *sustain* C, which lie at its *bottom*, and the *sustension* C_x.

A *circumference* of C is a cyclic walk $w = w_1^{\epsilon_1} \cdots w_n^{\epsilon_n}$ where, for each i, we have $\epsilon_i \in \{1, -1\}$ and w_i parallel to one of the paths $u_1, ...u_n, v_1, ...v_n$ and such that, moreover, each u_i and each v_i is parallel to exactly one of the w_j.

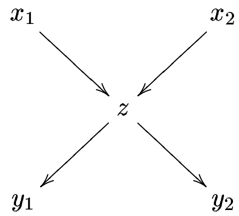
There are many circumferences of C starting and ending at a given point $x \in C_0$. If, for instance, $x = x_1$, two examples are $v_1 u_2^{-1} v_2 \cdots u_n^{-1} v_n u_1^{-1}$ and $u_1 v_n^{-1} u_n \cdots v_2^{-1} u_2 v_1^{-1}$. It is easy to prove that, for each i, each circumference of C in x_i is homotopic to a conjugate of a circumference at x_{i-1} (and also of a circumference at y_i).

LEMMA. **[11]** (3.2) *The convex hull of a weak crown C contains a crown as a full subcategory, with a circumference homotopic to a circumference of C.*

Proof. If C is not a crown, we may assume that $[x_1, y_1] \cap [x_1, y_2]$ contains a point $z \notin \{x_1, y_1, y_2\}$ or $[x_1, y_1] \cap [x_n, y_1]$ contains a point $z \notin \{x_1, x_n, y_1\}$. In the first case, the points $z, x_2...x_n, y_1...y_n$ generate a weak crown C'. In the second case, the points

$x_1, x_2 ... x_n, z, y_2 ... y_n$ generate a weak crown C'. In both cases, C' has a circumference homotopic to a circumference of C. If C' is not a crown, we iterate the procedure. □

11.2. Full subcategories of the form $x \to y$ are called *sticks*, and full subcategories of the form

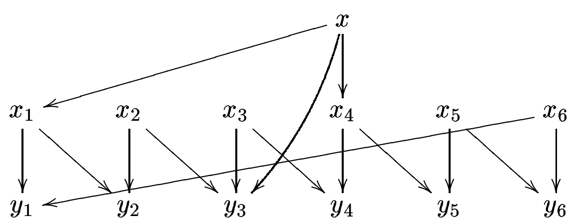


are called *crosses*. We refrain from proving the next proposition, but we illustrate it on an example.

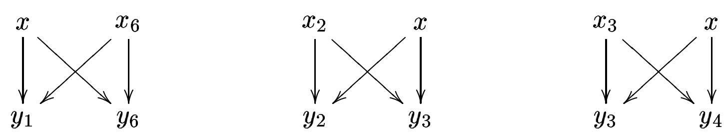
PROPOSITION. **[11]** (3.6) *Let x suspend a weak crown C.*

(a) *The suspension C^x uniquely decomposes as a union of weak crowns, crosses and sticks having in common the point x.*
(b) *The width of each weak crown in the decomposition of* (a) *is strictly smaller than the width $w(C)$ of C, except if x precedes no maximal point of C and x precedes exactly two minimal points which are consecutive.*
(c) *The product of circumferences, all starting and ending in x, of the weak crowns in the decomposition of* (a)*, is homotopic to a conjugate of a circumference of C.* □

EXAMPLE. Let



be a full subcategory of an incidence algebra, and let C be the weak crown with points $x_1, ... x_6, y_1, ... y_6$. The suspension C^x is easily seen to be the union of the stick $x \to y_5$ and the weak crowns

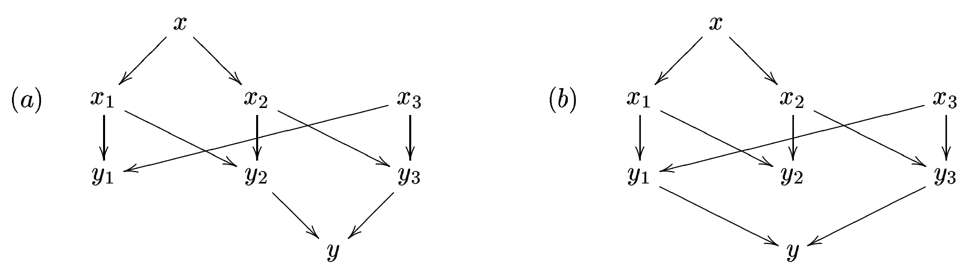


11.3.

DEFINITION. *Let C be a weak crown.*

(a) *If $w(C) = 2$, then C is called complete if there exists a point which suspends C and precedes its 2 maximal points or, dually, a point which sustains C and succedes its two minimal points.*
(b) *If $w(C) > 2$, then C is called complete if:*
   (i) *there exists a point x which suspends C and precedes at least 2 maximal points of C, and, moreover, each weak crown in the decomposition (11.2)(a) of C^x is complete, or dually*
   (ii) *there exists a point x which sustains C and succedes at least 2 minimal points of C, and, moreover, each weak crown in the decomposition (11.2)(a) of C_x is complete*

EXAMPLES. Consider the following incidence algebras



In (a), the shown crown is not complete, while in (b) it is complete.

LEMMA. **[11]** (4.3) *Let C be a complete weak crown. Then any circumference of C is homotopic to the trivial walk.*

Proof. This follows easily from the definition, using (11.2)(c). □

11.4. Recall that the problem of determining whether a schurian algebra is simply connected or not is undecidable. We give however here a sufficient, and also a necessary condition, for simple connectedness.

THEOREM. **[11]** *Let A be an incidence algebra, which is not strongly simply connected. Then*

(a) *If every crown in A is complete, then A is simply connected.*
(b) *If A is simply connected, then A contains a complete crown.*

Proof (Sketch).

(a) If A is not simply connected, it contains cyclic walks which are not contractible. Consider among these walks those with the least number of sources (or sinks) and among these latter, choose one of least length, denoted by w. One shows easily that the sinks and sources of w generate a weak crown C of A. Minimality of length implies that C is a crown.
(b) This is done by induction on $|A_0|$. We may assume $|A_0| > 4$. Since A is not strongly simply connected, it contains a crown C. If all sources and sinks of A lie in C, then A is the convex envelope of C, hence a contradiction, by (7.1). Assume, up to duality, that s is a source of A not in C. Assume moreover that

no crown of A is complete. Since $C \subseteq A^{(s)}$ and is connected, there exists a connected component B of $A^{(s)}$ containing C. Then B is not strongly simply connected. On the other hand, s is separating, by (5.9). Letting A' be the full subcategory of A with object class $B_0 \cup \{s\}$, the restriction to A' of rad P_s is thus indecomposable. By (4.6), A' is simply connected.

Since no crown in A is complete, no crown in B is complete. Therefore s is top of no weak crown in B (indeed, if s were top of a weak crown C' in B, then, by (11.1), C' contains a crown C'' and s would be the top of C'', a contradiction). By the induction hypothesis, B is not simply connected. Since s tops no weak crown in B, one can show that $\pi_1(B) \cong \pi_1(A')$, see **[11]** (5.3), and then the simple connectedness of A' yields a contradiction. □

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