Homogeneous spaces in the category of affine algebraic groups

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Preface

The material presented in these notes has been extracted from the manuscript of a book in preparation entitled "Invariant theory". The author would like to thank the participants of the seminar on Invariant theory of the Centro de Matemática for their many helpful corrections and suggestions. The author is gratefull for all the comments and corrections that the readers would like to suggest.

In what follows we fix a few notations

If G is an abstract group and f a function defined on G with values in the base field k, for any $x \in G$ we call $x \cdot f$ and $f \cdot x$ the functions defined pointwisely for any $y \in G$ as: $(x \cdot f)(y) = f(yx)$ and $(f \cdot x)(y) = f(xy)$. If M is a finite dimensional representation of G, $\alpha \in M^*$ an element of the linear dual of M and $m \in M$, we call $\alpha|m : G \to k$ the function $(\alpha|m)(x) = \alpha(x \cdot m)$. If G is an affine algebraic group defined over an algebraically closed field k and M is a finite dimensional G–module, we say that M is a rational G–module if for all $\alpha \in M^*$ and $m \in M$ we have that $\alpha|m \in k[G]$. Here $k[G]$ denotes the algebra of polynomial functions on G. If the module M is not necessarily finite dimensional, we say that it is rational if it is the sum of G-stable rational finite dimensional submodules of M. If M is a G–module with an action of G on the left, we denote by ${}^G M$ the subspace of M consisting of all G–fixed points.

If X is an abstract algebraic variety we denote as $\mathcal{O}_X$ the sheaf of regular functions on X and as $k(X)$ the field of regular functions on X. In the case X is affine $\mathcal{O}_X(X)$ is abbreviated as $k[X]$.

If D is a commutative integral domain we denote as $[D]$ the field of fractions of D.

CHAPTER 1

Homogeneous spaces

1. Introduction

This chapter deals with the geometric structure of a homogeneous space of the form G/H, where G is an affine algebraic k–group and H a closed k–subgroup. It is substantially harder to study homogeneous spaces in the category of algebraic groups than in the category of Lie groups. The basic general results concerning the existence of a natural structure of algebraic variety on G/H are due to Rosenlicht and Weil (see [**37**] and [**39**]). The fact that G/H is quasi-projective is due to Chow (see [**8**]).

We start by developing some preparatory material interrelating the representations of H with the representations of G –the basic arguments are due to Chevalley, see [**7**]–, we show that up to a character ρ of H, an arbitrary H–representation can be thought as a subrepresentation of a convenient representation of G. Then we proceed to describe in quite an explicit fashion the manner in which H can be cut away from G via a finite number of semiinvariant polynomials with the same weight. This description will be crucial for the main purpose of this chapter: the endowment of the coset space G/H with a natural structure of algebraic variety. This structure makes of G/H a quasi-projective variety. The projective variety where G/H lives is the closure of the G–orbit of a point in a projective space whose coordinates are the given semiinvariant polynomials.

We also show that in the particular case that H is normal the homogeneous space –that is now a group– is in fact an affine algebraic group.

The considerations about the relationship between the representations of H and those of G will also play an important role in the study of observable subgroups when in Chapters 2 and 3 we improve our understanding of the structure of G/H studying conditions for it to be quasi-affine or even affine.

In this chapter our presentation follows–with minor modifications–the general pattern of the standard reference on the subject: A. Borel "Linear Algebraic Groups" [**5**]. Some other presentations that simplify and compactify the algebraico-geometrical prerequisites of the theory are for example: [**24**] and [**26**].

Many of the results will be presented without proofs and the interested reader can consult the above mentioned references.

2. Representations of H and representations of G

Along this section G will be an affine algebraic k–group and H a closed k–subgroup.

Definition 1.1. A rational character of H is a group homomorphism of affine algebraic groups $\gamma : H \to k^*$. We denote as $\mathcal{X}(H)$ the group of all rational characters of H and as $\mathbf{1_H}$ (or if it is clear from the context just as $\mathbf{1}$) the trivial character –i.e. the character that takes the value 1 for all $x \in H$–.

Next definition singles out a concept that will be crucial in order to understand the relationship between the representations of H and of G.

DEFINITION 1.2. Let G be an affine algebraic group defined over k and let H be a closed k–subgroup. A rational character $\rho : H \to k^*$ is said to be *extendible* to G –or simply extendible if there is no danger of confusion about the groups involved– if there exists a non–zero element $f \in k[G]$ such that $x \cdot f = \rho(x)f$ for all $x \in H$. An element f as in the last definition is called an extension of ρ or a ρ semi–invariant polynomial or a semiinvariant polynomial of weight ρ. We will denote as $E_G(H) = \{\rho : H \to k^* : \rho \text{ is an extendible character}\}$ the set of extendible characters.

Writing explicitly the extendibility condition we see that f extends ρ iff for all $z \in G$, $x \in H$, $f(zx) = f(z)\rho(x)$. Assuming that $f(1) = 1$, we obtain that for all $x \in H, f(x) = \rho(x)$, i.e., the polynomial f is an extension of ρ as a function. Notice that the condition $f(1) = 1$ in not restrictive in the sense that if g is a ρ–semiinvariant such that $g(z) \neq 0$ for some $z \in G$, then $f = g \cdot z$ is a ρ–semiinvariant verifying $f(1) = g(z) \neq 0$.

THEOREM 1.1. *Let G be an affine algebraic k–group and H a closed k–subgroup. Let M be a finite dimensional rational H–module. There exists a finite dimensional rational G–module N, an extendible character $\rho : H \to k^*$ and an injective morphism $\iota : M \to (N|_H)_{\rho^{-1}}$. Here $(N|_H)_{\rho^{-1}}$ indicates that the space N is endowed with the H–module structure $\star$ given by the restriction of the G–structure twisted by ρ^{-1}. In other words if $x \in H$ and $n \in N$, $x \star n = \rho^{-1}(x)x \cdot n$. Moreover, if M is a simple H–module, the G–module N can be taken to be simple.*

COROLLARY 1.1. *In the situation above if M is a non-zero simple rational H-module, N can be taken to be a rational G-submodule of $k[G]$. Moreover, given $0 \neq m_0 \in M$ and $z \in G$ the injection $\iota : M \to (k[G]|_H)_{\rho^{-1}}$ can be selected as to verify that $\iota(m_0)(z) \neq 0$.*

COROLLARY 1.2. *Let G be an affine algebraic k–group , H be a closed k–subgroup and $0 \neq I \subset k[G]$ an H–stable ideal of $k[G]$. There exists an element $0 \neq f \in I$ and an extendible character ρ of H such that $x \cdot f = \rho(x)f$ for all $x \in H$.*

PROOF: Let $0 \neq V$ be a simple –and thus finite dimensional– H–submodule of I. As a first step we show that we can find a basis $\{v_1, v_2, \dots, v_n\}$ of V and an element $z \in G$ such that $v_1(z) \neq 0, v_2(z) = 0, \dots, v_n(z) = 0$. Let $V_z = V \cap \operatorname{Ker}(\varepsilon_z)$, with $\varepsilon_z : k[G] \to k$ the evaluation at z. If for all $z \in G$, $V_z = V$, then for all $z \in G, V \subset \operatorname{Ker}(\varepsilon_z)$ and that would imply that $V = 0$. Hence we can find $z \in G$ and $v_1 \in V$ such that $V_z \oplus kv_1 = V$. To finish our construction of an adequate basis just complete the vector v_1 to an arbitrary basis of V_z. Consider V^* the –also simple– representation contragradient to V and call $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ the basis of V^* dual to $\{v_1, v_2, \dots, v_n\}$. Next use Corollary 1.1 to find and H–equivariant injective morphism $\iota : V^* \to (k[G]|_H)_{\rho^{-1}}$ that verifies $\iota(\alpha_1)(z) \neq 0$. Next define $\Gamma : \operatorname{End}(V) \to I_{\rho^{-1}}$ as $\Gamma = m \circ (id \otimes \iota) \circ ev^{-1}$ where $m : I \otimes (k[G]|_H)_{\rho^{-1}} \to I_{\rho^{-1}}$ is the usual multiplication map and $ev : V \otimes V^* \to \operatorname{End}(V)$ is the standard identification. The map Γ is H–equivariant because it is the composition of H–equivariant morphisms. If we call $f = \Gamma(id)$ it is clear that being id an H–equivariant element of $\operatorname{End}(V)$ the polynomial f has to verify that $\rho^{-1}(x)x \cdot f = f$ for all $x \in H$. Then f is a ρ–semiinvariant and as $id = ev(\sum v_i \otimes \alpha_i)$ the polynomial f can be explicitely written as $f = \sum v_i\iota(\alpha_i)$ so that $f(z) = v_1(z)\iota(\alpha_1)(z) \neq 0$.

□

In the case that H is normal in G, Corollary 1.2 can be refined.

LEMMA 1.1. *Assume that G is an affine algebraic k–group, H is a closed normal k–subgroup and ρ is an extendible character. Then:*

1. *There exists G_ρ a closed subgroup of finite index that contains H such that if $f \in k[G]$ is a ρ–semiinvariant, then $\mathcal{S}(f)|_{G_\rho}$ is a ρ^{-1}–semiinvariant in G_ρ. Here as usual $\mathcal{S} : k[G] \to k[G]$ is the antipode in $k[G]$.*
2. *The character ρ^{-1} can be extended to G.*
3. *If G is connected and f is an extension of ρ, then $\mathcal{S}(f)$ is an extension of ρ^{-1}.*

In the case of a normal subgroup we can guarantee that the character in Corollary 1.2 can be taken to be one for all H–stable ideals of $k[G]$.

COROLLARY 1.3. *Let G and H be as before –H normal in G–. If $I \subset k[G]$ is an H–stable ideal, there exists an element $0 \neq f \in I$ such that $x \cdot f = f$ for all $x \in H$.*

PROOF: By Corollary 1.2 there exists an element $0 \neq f_0 \in I$ that is an extension to G of some character $\rho \in \mathcal{X}(H)$. By Lemma 1.1 there exists an extension g of ρ^{-1} from H to G. As $f_0 \neq 0$ we can find $z \in G$ such that $f_0(z) \neq 0$, and similarly we can find $w \in G$ such that $g(w) \neq 0$. The polynomial $f = (g \cdot wz^{-1})f_0 \in I$ is H-fixed and at $z \in G$ takes the value $f(z) = g(w)f_0(z) \neq 0$. □

OBSERVATION 1.1. In the language of Chapter 2, Corollary 1.3 asserts that normal subgroups are observable –see Definition 2.1–.

Next corollary shows that the invariant rational functions on a connected group can be written as quotients of semiinvariant polynomials of the same weight.

COROLLARY 1.4. *Let G be a connected affine algebraic k–group and H a closed k–subgroup. If $f \in {}^H[k[G]]$ there exists $u, v \in k[G]$, $v \neq 0$ and $\rho \in E_G(H)$ such that $f = u/v$, $x \cdot u = \rho(x)u$ and $x \cdot v = \rho(x)v$ for all $x \in H$. In particular H is a group with no non-trivial rational characters or if H is a normal subgroup then ${}^H[k[G]] = [{}^H k[G]]$.*

PROOF: If $f = 0$ the result is obvious. Assume $f \neq 0$. Considering the H–stable ideal $I = k[G]f \cap k[G] \subset k[G]$ we find $\rho \in E_G(H)$ and $0 \neq u \in I$ such that $x \cdot u = \rho(x)u$ for all $x \in H$. Writing $u = vf$ for some $v \in k[G]$ we have that $\rho(x)(vf) = \rho(x)u = x \cdot u = x \cdot (vf) = (x \cdot v)(x \cdot f) = (x \cdot v)f$. Hence $x \cdot v = \rho(x)v$. In the case that H is normal in G, we proceed in the same manner except that Corollary 1.3 guarantees that ρ can be taken to be one. Then u is H–invariant and so is v. □

In Exercise 1 we extend the conclusions of Corollary 1.4 to a finite number of invariant rational functions.

3. Definition of subgroups in terms of semiinvariants

In this section we use the results of Section 2 to prove that an arbitrary closed subgroup of an affine algebraic group can be defined in terms of semiinvariants, we also prove that when the given subgroup is normal it can be defined in terms of invariants. Our methodos of proof–that use systematically Corollaries 1.2 and 1.3 as well as Lemma 1.1– are slight variations of the standard ones –see for example [**5**], [**7**], [**24**], [**26**]–.

The proof of the result that follows can be found in the mentioned references.

THEOREM 1.2. *Let G be an affine algebraic k–group and H a closed k–subgroup. There is a finite set of non-zero polynomials $f_1, \dots, f_n \in k[G]$, an extendible character $\rho \in E_G(H)$ and an extension $f \in k[G]$ of ρ such that $H = \{w \in G : w \cdot f_i = f(w)f_i,\ i = 1, \dots, n\}$ and*

$\mathfrak{h} = \{\tau \in \mathfrak{g} : \tau \cdot f_i = \tau(f) f_i,\ i = 1, \dots, n\}$. *Moreover the polynomials* $f_i,\ i = 1, \dots, n$ *can be taken in such a way that* $H = \mathcal{Z}(f_1, \dots, f_n)$ *and* $\mathfrak{h} = \{\tau \in \mathfrak{g} : \tau(f_i) = 0,\ i = 1, \dots, n\}$.

There is a similar result for the case of a normal subgroup.

THEOREM 1.3. *Let* G *be an affine algebraic* k*–group and* H *a closed normal* k*–subgroup. There is a finite set of non zero polynomials* $g_1, \dots, g_m \in k[G]$ *such that* $H = \{w \in G : w \cdot g_i = g_i, i = 1, \dots, m\}$ *and* $\mathfrak{h} = \{\tau \in \mathfrak{g} : \tau \cdot g_i = 0, i = 1, \dots, m\}$.

COROLLARY 1.5. *Let* G *be an connected affine algebraic* k*–group and* H *a closed* k*–subgroup. Then* $H = \{w \in G \ :\ w \cdot f = f\ \forall f \in {}^H[k[G]]\}$ *and* $\mathfrak{h} = \{\sigma \in \mathfrak{g} : \sigma \cdot f = 0\ \ \forall f \in {}^H[k[G]]\}$.

PROOF: By Theorem 1.2 we can find a finite set of polynomials $\{f_1, f_2, \dots, f_n\} \subset k[G]$ that are H–semiinvariant of a certain weight ρ and cut exactly H, i.e. $H = \{w \in G : w \cdot f_i = f(w) f_i, i = 1, \dots, n\}$, $f \in k[G]$ an extension of ρ. Let V be the finite dimensional G–submodule of $k[G]$ generated by $\{f_1, f_2, \dots, f_n\}$, and call $W = \bigoplus_{i=1}^n V$. Consider $u = (f_1, f_2, \dots, f_n) \in W$. Clearly $w \in H$ if and only if $w \cdot u = f(w)u$. If $z \notin H$ the elements $z \cdot u$ and u are linearly independent so we can find $\alpha, \beta \in W^*$ such that $\alpha(u) = \beta(u) = 1$, $\alpha(z \cdot u) = 0$, $\beta(z \cdot u) = 1$. Consider the polynomials $\alpha|u$, $\beta|u \in k[G]$ defined as $(\alpha|u)(z) = \alpha(z \cdot u)$ and similarly for $\beta|u$. For $x \in H$ we have that $x \cdot u = \rho(x)u$; then the polynomials $\alpha|u$ and $\beta|u$ are ρ–semiinvariant and $\dfrac{\alpha|u}{\beta|u} \in {}^H[k[G]]$. Next we show that $z \cdot \dfrac{\alpha|u}{\beta|u} \neq \dfrac{\alpha|u}{\beta|u}$. Indeed, $z \cdot \dfrac{\alpha|u}{\beta|u}(1) = \dfrac{(\alpha|u)(z)}{(\beta|u)(z)} = \dfrac{\alpha(z \cdot u)}{\beta(z \cdot u)} = 0$ and $\dfrac{\alpha|u}{\beta|u}(1) = \dfrac{(\alpha|u)(1)}{(\beta|u)(1)} = \dfrac{\alpha(u)}{\beta(u)} = 1$. Hence $z \notin \{w \in G : w \cdot f = f\ \ \forall f \in {}^H[k[G]]\}$.

For the Lie algebra we proceed in a similar manner. One easily shows –with the same notations than in the first part of the proof– that $\sigma \in \mathfrak{h}$ if and only if $\sigma \cdot u = \sigma(f)u$. Hence, if $\sigma \notin \mathfrak{h}$ the elements $\sigma \cdot u$ and u are linearly independent so we can find $\alpha, \beta \in W^*$ such that $\alpha(u) = \beta(u) = 1$, $\alpha(\sigma \cdot u) = 0$, $\beta(\sigma \cdot u) = 1$. If we compute $\sigma \cdot \dfrac{\alpha|u}{\beta|u} = \dfrac{(\alpha|\sigma \cdot u)\beta|u - \alpha|u(\beta|\sigma \cdot u)}{(\beta|u)^2}$ we observe that $(\sigma \cdot \dfrac{\alpha|u}{\beta|u})(1) = \dfrac{\alpha(\sigma \cdot u)\beta(u) - \alpha(u)\beta(\sigma \cdot u)}{(\beta(u))^2} = -1 \neq 0$. As $\dfrac{\alpha|u}{\beta|u}$ is an H–invariant rational function, we conclude that $\sigma \notin \{\sigma \in \mathfrak{g} : \sigma \cdot f = 0\ \ \forall f \in {}^H[k[G]]\}$. Conversely, assuming that If $\tau \in \mathfrak{h}$ it follows immediately that $\tau \cdot f = 0$ for all H–invariant rational functions. □

COROLLARY 1.6. *Let* G *be an connected affine algebraic* k*–group and let* H *be a closed* k*–subgroup. There exists a finite dimensional rational* G*–module* M *and a point* $p_0 \in \mathbb{P}(M)$ *–the projective space built on* M*– such that the stabilizer of* p_0 *with respect to the natural action of* G *on* $\mathbb{P}(M)$ *is* H*. Moreover, if we consider the orbit map* $\theta : G \to \mathbb{P}(M)$ *given as* $\theta(w) = w \cdot p_0$*, then* $\operatorname{Ker}(d_e\theta) = \mathfrak{h}$.

PROOF: Consider the subfield ${}^H[k[G]]$ of the k–finitely generated field $[k[G]]$. Take $\{f_1, \dots, f_n\}$ a finite set of field generators of ${}^H[k[G]]$. Using Exercise 2, we deduce the existence of an extendible character $\rho \in E_G(H)$ and $n+1$ semiinvariant polynomials of weight ρ, $u_0, u_1, \dots, u_n \in k[G], u_0 \neq 0$ such that $f_i = u_i/u_0, i = 1, \dots, n$. Let N be the finite-dimensional rational G–module generated by $u_0, u_1, \dots, u_n$ and call $M = \bigoplus_{i=0}^n N$. Consider the point $p_0 = [u_0 : u_1 : \cdots : u_n] \in \mathbb{P}(M)$ and factor the action of G to $\mathbb{P}(M)$; call $G_{p_0} = \{w \in G : w \cdot p_0 = p_0\}$. As the polynomials $u_i,\ i = 0, \dots, n$, are H–semiinvariants of the same weight, it follows that $H \subset G_{p_0}$. Conversely, take $w \in G$ such that $w \cdot p_0 = p_0 \in \mathbb{P}(M)$; we have that $w \cdot u_i = \lambda_w u_i, i = 0, \dots, n$, for some $\lambda_w \in k^*$. Then $w \cdot f_i = f_i$, $i = 1, \dots, n$ and as the elements f_i, $i = 1, \dots, n$, generate ${}^H[k[G]]$, we conclude that $w \cdot f = f$ for all $f \in {}^H[k[G]]$, and hence (see Corollary 1.5) that $w \in H$.

Next we prove the assertion about the Lie algebra. We write $\theta = Q \circ A$ where $A : G \to M$ is given as $A(w) = (w \cdot u_0, \dots, w \cdot u_n)$ and $Q : M \setminus \{0\} \to \mathbb{P}(M)$ is the canonical map. As we proved before, $T_{(u_0,\dots,u_n)}(M \setminus \{0\}) = M$, $T_{[u_0:\dots:u_n]}(\mathbb{P}(M)) = M/k(u_0, \dots, u_n)$ and $d_{p_0}Q = q$ where q is the canonical projection. Then $\operatorname{Ker}(d_{p_0}Q) = k(u_0, \dots, u_n)$.

As $d_e\theta = d_{p_0}Q \circ d_eA$ we conclude that $\operatorname{Ker}(d_e\theta) = \{\sigma \in \mathfrak{g} : d_eA(\sigma) \in k(u_0, \dots, u_n)\}$. As A is a linear action we have that $d_eA(\sigma) = \sigma \cdot (u_0, \dots, u_n) = (\sigma \cdot u_0, \dots, \sigma \cdot u_n) + (u_0$–see Exercise 6–. Hence, $\operatorname{Ker}(d_e\theta) = \{\sigma \in \mathfrak{g} : \sigma \cdot u_i = \mu_\sigma u_i, i = 0, \dots, n$ for some $\mu_\sigma \in k\}$. By the derivation rules we have that $\sigma \cdot f_i = \sigma \cdot \dfrac{u_i}{u_0} = \dfrac{(\sigma \cdot u_i)u_0 - u_i(\sigma \cdot u_0)}{u_0^2} = 0$. As f_i, $i = 1, \dots, n$, generate ${}^H[k[G]]$ we conclude as before–using Corollary 1.5–that $\sigma \in \mathfrak{h}$. All that remains is to prove that if $\sigma \in \mathfrak{h}$, then $d_e\theta(\sigma) = 0$. This is a consequence of the fact–already proved–that if for all $x \in H$, $x \cdot u_i = \rho(x)u_i$ then $\sigma \cdot u_i = \mu_\sigma u_i$ for some $\mu_\sigma \in k$. □

Next theorem will be of particular interest for the considerations about observability in Chapter 2.

Theorem 1.4. *Let G be a connected affine algebraic k–group and let H be a closed k–subgroup. Then $H = \{w \in G : w \cdot f = f \ \ \forall f \in {}^Hk[G]\}$ if and only if ${}^H[k[G]] = [{}^Hk[G]]$ and this implies that $\mathfrak{h} = \{\sigma \in \mathfrak{g} : \sigma \cdot f = 0 \ \ \forall f \in {}^Hk[G]\}$. In particular if H is normal in G then we have that $H = \{w \in G : w \cdot f = f \ \ \forall f \in {}^Hk[G]\}$ and $\mathfrak{h} = \{\sigma \in \mathfrak{g} : \sigma \cdot f = 0 \ \ \forall f \in {}^Hk[G]\}$.*

Proof: Assume ${}^H[k[G]] = [{}^Hk[G]]$. If $z \cdot f = f$ for all $f \in {}^Hk[G]$ then $z \cdot g = g$ for all $g \in [{}^Hk[G]] = {}^H[k[G]]$. Using Corollary 1.5 we conclude that $z \in H$ and that $H = \{w \in G : x \cdot f = f \ \ \forall f \in {}^Hk[G]\}$.

Assume now that $H = \{w \in G : w \cdot f = f \ \ \forall f \in {}^Hk[G]\}$. We will prove that ${}^H[k[G]] \subset [{}^Hk[G]]$ –the other inclusion being obvious–.

Let $f \in {}^H[k[G]]$ and write it in the form $f = u/v$ for $u, v \in k[G]$ and $v \neq 0$. If we denote as $k[G]_v$ the localization of the integral domain $k[G]$ with respect to the multiplicative set of all powers of v, it is clear that $f \in k[G]_v$.

The relationship between the different algebras and fields we consider are displayed in the following diagram of field and ring extensions.

$$\begin{tikzcd}
{}^Hk[G] \arrow[r, hook] \arrow[d, hook] & {[{}^Hk[G]]} \arrow[r, hook] & {}^H[k[G]] \arrow[d, hook] \\
k[G] \arrow[r, hook] & k[G]_v \arrow[r, hook] & {[k[G]]}
\end{tikzcd}$$

A standard commutative algebra result guarantees that for a situation like the one variefied by the pair of k–algebras ${}^Hk[G] \subset k[G]_v$ if an element ξ of the larger algebra has the property that all pairs of k–algebra morphisms $\sigma, \tau : k[G]_v \to k$ that coincide on ${}^Hk[G]$ coincide also on ξ , then ξ is algebraic and purely inseparable over $[{}^Hk[G]]$. It follows immediately from Hilbert's Nullstellensatz that an arbitrary k–algebra morphism from $k[G]_v$ into k is given by the evaluation at a point $z \in G$ such that $v(z) \neq 0$.

Take $z, z' \in G$ that are not zeroes of v and with the property that the evaluations $\epsilon_z, \epsilon_{z'} : k[G]_v \to k$ coincide when restricted to ${}^Hk[G]$. For a given $g \in {}^Hk[G]$ and for an arbitrary $w \in G$, the polynomial $g \cdot w \in {}^H k[G]$, hence $(z \cdot g)(w) = g(wz) = (g \cdot w)(z) = (g \cdot w)(z') = g(wz') = (z' \cdot g)(w)$. So that for all $g \in {}^H k[G]$, $z \cdot g = z' \cdot g$ and then for some $h \in H$, $z' = zh$. Then $z \cdot f = z' \cdot f$, and evaluating at 1 we conclude that $\epsilon_z(f) = \epsilon_{z'}(f)$. Using the result mentioned above we conclude that f is purely inseparable algebraic over $[{}^H k[G]]$.

Thus, any $f \in {}^H[k[G]]$ is purely inseparable algebraic over $[{}^H k[G]]$. In the case that the base field has characteritic zero we are done. Otherwise, if p is the characteristic of the field k, we can find a positive integer n such that $f^{p^n} \in [{}^H k[G]]$, i.e. there exist $u, v \in {}^H k[G]$ such that $f^{p^n} = \frac{u}{v}$. Hence $vf^{p^n} \in {}^H k[G]$ and also $(vf)^{p^n} \in {}^H k[G]$. As the element $u' = vf$ verifies $u'^{p^n} \in {}^H k[G]$ we conclude from Exercise 2 that $u' \in k[G]$. It also follows easily that h' is H–fixed. Then $f \in [{}^H k[G]]$.

Next we prove the result concerning the Lie algebra. Assume that $[{}^H k[G]] = {}^H[k[G]]$, if we take $\sigma \in \mathfrak{g}$ such that $\sigma \cdot f = 0$ for all $f \in {}^H k[G]$, then $\sigma \cdot f = 0$ for all $f \in {}^H[k[G]] = [{}^H k[G]]$. We conclude from Corollary 1.5 that $\sigma \in \mathfrak{h}$. The fact that if $\sigma \in \mathfrak{h}$, then $\sigma \cdot f = 0$ for all $f \in {}^H k[G]$ is very easy to prove. $\square$

Next result is a refinament of Corollary 1.6 and its proof is very similar. It will be crucial in order to deal with finer aspects of the structure of G/H–see Section 6 of this same chapter and also Chapter 2–.

Corollary 1.7. *Let G be a connected affine algebraic k–group and H a closed k–subgroup with the property that ${}^H[k[G]] = [{}^H k[G]]$. There exists a finite dimensional rational G–module M and a point $m_0 \in M$ such that the stabilizer of m_0 is H and the annihilator of m_0 is $\mathfrak{h}$–recall that the annihilator is defined as* $\operatorname{Ann}(m_0) = \{\sigma \in \mathfrak{g} : \sigma \cdot m_0 = 0\}$.

Observation 1.2. The hypothesis ${}^H[k[G]] = [{}^H k[G]]$ of Corollary 1.7 is verified in many important cases, for example in the case that H is a normal subgroup of G. Also if H is a closed subgroup of G that has only one character–the trivial one– it follows from Corollary 1.4 that in this case we are also in the above hypothesis. Examples of such groups are the unipotent groups or groups of the form $[K, K]$ for some affine group K. The condition ${}^H[k[G]] = [{}^H k[G]]$ will be one of the characterizations of "observable subgroups" and this concept will be studied extensively in Chapter 2.

4. The homogeneous space G/H as an algebraic variety

The main result of this section is the endowment of the algebraic homogeneous space G/H with an structure of algebraic variety. The basic reference for the results that follow is [**5**]. At some points we have followed it closely.

The corresponding quotient problem in the general situation–i.e. in the case of a group H acting on an arbitrary variety X– will not be treated here but in order to clarify the particular situation of homogeneous spaces and to unify the methods and techniques, we introduce some terminology and prove a few elementary results corresponding to the general set up.

Definition 1.3. Let X be an algebraic variety defined over k and let G be an affine algebraic k–group. A regular action of G on X is a morphism of varieties $\Gamma : G \times X \to X$ that is also an action of the abstract group on the underlying set of X. In this situation it is customary to say that X is an algebraic G–variety, or simply that X is a G–variety.

The theorem that follows is an important technical device in order to deal with homogeneous spaces. The proof can be found for example in [**5**]

THEOREM 1.5. *Assume that G is a connected affine algebraic k–group and that X is an algebraic G–variety. Fix $x \in X$ and consider the morphism $\pi : G \to X$ given as $\pi(w) = w \cdot x$ –that will frequently be considered as a map $\pi : G \to O(x)$ with $O(x) = G \cdot x = \pi(G)$–. Consider also the stabilizer of x, $G_x = \{w \in G : w \cdot x = x\}$. Then $O(x)$ is an algebraic variety without singular points and $\dim O(x) = \dim G - \dim G_x$. Also $\mathrm{Lie}(G_x) \subset \mathrm{Ker}(d_e\pi)$. Moreover, the following four conditions are equivalent.*

1. $\pi : G \to O(x)$ *is a separable morphism.*
2. $d_e\pi : \mathrm{Lie}(G) \to T_x(O(x))$ *is surjective.*
3. $\mathrm{Ker}(d_e\pi) = \mathrm{Lie}(G_x)$.
4. *The map $\pi : G \to O(x)$ is open and for an arbitrary open subset $U \subset O(x)$, the corresponding map $\pi^* : \mathcal{O}_{O(x)}(U) \to {}^{G_x}\mathcal{O}_G(\pi^{-1}(U))$ is an algebra isomorphism. Here ${}^{G_x}\mathcal{O}_G(\pi^{-1}(U))$ denotes as usual the algebra of regular functions on $\pi^{-1}(U)$ that are constant on the fibers of π, i.e. on the G_x–orbits.*

In Section 5 a pair $(O(x), \pi)$ when it satisfies the conditions 1)–4) above is called a geometric quotient of G.

The main general results about the structure of homogeneous spaces are concentrated in the theorem that follows.

THEOREM 1.6. *Let G be an affine algebraic k–group and H a closed k–subgroup. The set $G/H = \{wH : w \in G\}$ of all left cosets of G modulo H can be endowed with a structure of quasi-projective variety defined over k that verifies:*

1. *The canonical projection $\pi : G \to G/H$ and the map $\Pi : G \times G/H \to G/H$ given as $\Pi(w, zH) = wzH$ are morphisms of algebraic varieties.*
2. $\dim(G/H) + \dim(H) = \dim(G)$.
3. *The map π is open and separable.*
4. *If U is an open subset of G/H the map $\pi_U^* : \mathcal{O}_{G/H}(U) \to \mathcal{O}_G(\pi^{-1}(U))$ is injective and its image is ${}^H\mathcal{O}_G(\pi^{-1}(U))$.*
5. *Let $\phi : G \to W$ be a morphism of algebraic varieties constant on the cosets mod H– i.e. such that for all $w \in G$, $x \in H$ then $\phi(wx) = \phi(w)$–. The set theoretical map $\Phi : G/H \to W$ induced by ϕ on the quotient is a morphism of algebraic varieties.*
6. *Let $Z \subset G$ be a closed and H–saturated set –saturated means that $ZH = Z$–. Then $\pi(Z) \subset G/H$ is closed.*
7. *If G is irreducible, the field of rational functions on G/H, i.e. $k(G/H)$ is isomorphic to ${}^H[k[G]]$.*

PROOF: We can assume that G is connected. The general case is obtanied by patching together a finite number of varieties corresponding to the connected components of the original group.

Take M and $p_0 \in \mathbb{P}(M)$ as in Corollary 1.6 and call $X = \mathbb{P}(M)$ the projective space associated to the k–linear space M and Γ the natural action of G on X–i.e. the action induced by the linear action–. In this situation X acquires an structure of algebraic G–variety. By construction the stabilizer of p_0 is H. In the category of sets G/H and the orbit $G \cdot p_0 \subset X$ are isomorphic. Call $\theta : G \to X$ the orbit map, i.e. $\theta(w) = w \cdot p_0$ and $Y = \overline{G \cdot p_0}$. As Y is closed in X it inherits an structure of projective variety and being the closure of an orbit it is well known that $G \cdot p_0 \subset Y$ is open relatively to Y. We endow to G/H with the structure of quasi–projective variety transplanted from $G \cdot p_0$ via the isomorphism associated to θ. By construction G/H is quasi–projective and θ induces an isomorphism from G/H onto $G \cdot p_0$.

(1) If $\widehat{\theta} : G/H \to G \cdot p_0$ denotes de map induced by θ, the canonical projection $\pi : G \to G/H$ equals $\pi = (\widehat{\theta})^{-1} \circ \theta$. By definition it is a morphism of algebraic varieties.

The diagram below is commutative:

$$\begin{tikzcd}
G \times G/H \arrow[r, "\Pi"] \arrow[d, "id\times\widehat{\theta}"'] & G/H \arrow[d, "\widehat{\theta}"] \\
G \times G \cdot p_0 \arrow[r, "\Gamma|_{G\times G\cdot p_0}"] \arrow[d, hook] & G \cdot p_0 \arrow[d, hook] \\
G \times X \arrow[r, "\Gamma"] & X
\end{tikzcd}$$

As Γ is a morphism and $\widehat{\theta}$ an isomorphsim, it follows immediately that Π is a morphism of algebraic varieties.

(2) This assertion follows immediately from Theorem 1.5.

3.4. Call $\mathfrak{g} = \mathrm{Lie}(G)$ and similarly $\mathfrak{h} = \mathrm{Lie}(H)$. We start by observing that $d_e\pi : \mathfrak{g} \to T_H(G/H)$ is surjective. As $\theta = \widehat{\theta} \circ \pi$ and $\widehat{\theta}$ is an isomorphism, $\mathrm{Ker} d_e\theta = \mathrm{Ker} d_e\pi$. Corollary 1.6 asserts that $\mathrm{Ker} d_e\pi = \mathfrak{h}$. As $\dim_k T_H(G/H) = \dim G/H = \dim G - \dim H = \dim_k\mathfrak{g} - \dim_k\mathfrak{h}$ and as $d_e\pi : \mathfrak{g} \to T_H(G/H)$ is a linear map with kernel equal to $\mathfrak{h}$, we conclude that $d_e\pi$ is surjective. Hence, using 1.5 we deduce that the morphism π is separable as well as the rest of of 3. and 4.

5. As $\phi = \Phi \circ \pi$ is continuous and π is surjective and open, it is clear that Φ is continuous. Next we prove that the composition by Φ sends regular functions into regular functions–i.e., that Φ is a morphism–. Let $U \subset W$ be an open subset of W and let $f : U \to k$ be a regular function on U. As $\phi : G \to W$ is a morphism, the function $f \circ \phi : \phi^{-1}(U) \to U \to k$ belongs to $\mathcal{O}_G(\phi^{-1}(U))$. As $\phi(wx) = \phi(w)$ for all $x \in H$ and $w \in W$ we deduce that $(f \circ \phi)(wx) = (f \circ \phi)(x)$, i.e. $f \circ \phi \in {}^H\mathcal{O}_G(\phi^{-1}(U)) = {}^H\mathcal{O}_G(\pi^{-1}(\Phi^{-1}(U)))$. Using assertion 4. of this theorem we can guarantee the existence of a regular function $g : \Phi^{-1}(U) \to k$ such that $g \circ \pi = f \circ \phi = f \circ \Phi \circ \pi$. Then, $g = f \circ \Phi$ and consequently $\Phi^*(\mathcal{O}_W(U)) = \mathcal{O}_{G/H}(\Phi^{-1}(U))$.

6. If $V = VH$ and we denote as V^c the complement of V, it is clear that $\pi(V) \cap \pi(V^c) = \emptyset$ and also that $\pi(V) \cup \pi(V^c) = G/H$. As π an open map the set $\pi(V^c)$ is open and $\pi(V)$ is closed.

7. The proof of this assertion follows from 4. and from that fact that an arbitrary H–fixed element of $[k[G]]$ is the quotient of a couple of semi–invariant polynomials–see Exercise 4–.

□

Corollary 1.8. *Let G be an affine algebraic k–group and H a closed k–subgroup. With respect to the natural structures of algebraic varieties the homogeneous space G/H is quasi–affine if and only if $G_1/H \cap G_1$ is quasi–affine and G/H is affine if and only if $G_1/H \cap G_1$ is affine.*

PROOF: Consider G_1H that is a closed subgroup of finite index of G and write $G = z_1G_1H \cup z_2G_1H \cup \cdots \cup z_rG_1H$. The variety G/H is a disjoint union of translates of G_1H/H that are isomorphic to $G_1/G_1 \cap H$. $\square$

5. Categorical quotients, orbit spaces and geometric quotients

As mentioned before we will not deal with the general theory of quotients where H is an affine algebraic k–group acting by morphisms in an algebraic variety X defined over k. Here we touch briefly upon some of the more formal aspects of the theory.

DEFINITION 1.4. Let X be an H-variety, a categorical quotient of X by H is a pair (Y,π) where Y is an algebraic k–variety and $\pi : X \to Y$ is a morphism such that: a) The map π is is surjective and constant on the orbits, i.e. if $w \cdot x = y$ for $x, y \in X$, $w \in H$, then $\pi(x) = \pi(y)$; b) If Z is an arbitrary algebraic k–variety and $\phi : X \to Z$ is a morphism of varieties that is constant on the orbits, there exists a unique morphism $\Phi : Y \to Z$ such that $\Phi \circ \pi = \phi$.

OBSERVATION 1.3. Notice that condition a) implies that for any $y \in Y$ the subset $\pi^{-1}(y)$ is a union of orbits and condition b) implies that if a certain morphism $\phi : X \to Z$ is constant along the orbits it takes also the same values along two orbits generated by points $x, x' \in X$ belonging to the same fibre of π.

Condition a) implies that for any $y \in Y$ the subset $\pi^{-1}(y)$ is a union of orbits.

DEFINITION 1.5. In the situation above, if the map π verifies the additional hypothesis that for all $y \in Y$ the subset $\pi^{-1}(y)$ is exactly one orbit, we say that (Y,π) is an orbit space.

DEFINITION 1.6. Let X be an arbitrary algebraic k–variety, a geometric quotient of X is a pair (Y,π) where Y is an algebraic k–variety and $\pi : X \to Y$ is a morphism such that: a) The map π is surjective and open ; b) If $U \subset Y$ is open, the map $\pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ is injective and its image is the set of regular functions $f : \pi^{-1}(U) \to k$ that are constant on the fibers of π.

The proof of next lemma is completely routinary.

LEMMA 1.2. *A categorical quotient if it exists it is unique up to isomorphism.*

In order to relate the concepts of categorical and geometric quotient we need a generalization of some of the arguments in Theorem 1.6 part 5. As the proofs are similar we omit them.

LEMMA 1.3. *Let X be an algebraic k–variety and (Y,π) a geometric quotient. If Z is an arbitrary k–variety and $\phi : X \to Z$ a morphism of varieties constant along the fibres of π, the unique set theoretical map $\Phi : Y \to Z$ that factors ϕ, i.e. that verifies $\Phi \circ \pi = \phi$ is a morphism of k–varieties.*

COROLLARY 1.9. *Assume that X is an H–variety for some affine algebraic group H. Suppose that the set X admits a geometric quotient (Y,π) with the property that the orbits of H on X are exactly the fibers of π. Then (Y,π) is an orbit space and in particular a categorical quotient.*

COROLLARY 1.10. *Let G be an affine algebraic k–group and H a k–subgroup. The homogeneous space G/H with its natural structure–the one defined in Theorem 1.6– is a geometric quotient for the H–space G a categorical quotient and also an orbit space.*

OBSERVATION 1.4. The above Corollary and Lemma 1.2 remedies the non-naturality of the explicit construction of the variety structure on G/H.

6. Quasi–affine quotients and the case of a normal sugbroup

In this section we consider some special cases where one can pin down further the structure of the homogeneous space G/H. Our main concern is the special case where H is a closed normal k–subgroup of the original affine algebraic group G. In this situation it can be proved that G/H is an affine algebraic k–group and that the canonical projection $\pi : G \to G/H$ is a homomorphism in the corresponding category.

The first step in order to prove the affiness of G/H is to observe –using the methods of Section 4– that it is quasi-affine and this in turn will follow easily from the following general assertion: if G is irreducible and ${}^{H}[k[G]] = [{}^{H}k[G]]$ then G/H is quasi-affine.

To finish with the case of a normal subgroup we prove that a quasi–affine algebraic group is always affine.

The general consideration of the algebraico–geometric structure of the quotient of an affine group by a normal closed subgroup appeared for the first time in [**7**]. A treatment of the same problem but using purely Hopf algebra techniques can be found for example in [**23**] or [**24**].

Next theorem is a refinament of Theorem 1.6

THEOREM 1.7. *Let G be an irreducible affine algebraic k–group and H a closed k–subgroup. Suppose that ${}^{H}[k[G]] = [{}^{H}k[G]]$, then the homogeneous space G/H endowed with the structure of algebraic variety defined before is quasi–affine. In particular if H has no non-trivial characters the homogeneous space G/H is quasi affine.*

PROOF: Consider M and $m_0 \in M$ as in Corollary 1.7. Call $\theta : G \to M$ the orbit map. We know that $H = \{z \in G : z \cdot m_0 = m_0\}$ and that $\mathfrak{h} = \operatorname{Ker}(d_e(\theta)) = \{\sigma \in \mathfrak{g} : \sigma \cdot m_0 = 0\}$. Using Theorem 1.5 we conclude that $(G \cdot m_0, \theta)$ is a geometric quotient and then a categorical quotient. From the uniqueness guaranteed by Lemma 1.2 we deduce that the homogeneous space G/H is isomorphic to $G \cdot m_0$ that is an orbit in an affine space and consequently quasi-affine. $\square$

Next result is an easy and expected consequence of Theorem 1.6.

LEMMA 1.4. *Let G be an affine algebraic k–group and H a closed normal k–subgroup. The maps $m : G/H \times G/H \to G/H$ and $\iota : G/H \to G/H$–given respectively as the quotient of the multiplication and the inverse of G– endow the variety G/H with a structure of algebraic k–group with identity $eH \in G/H$.*

Next theorem illustrates the influence of the algebraic over the geometric structure on a variety. Examples of the converse situation also abound. The interested reader may look for example at the proof that a projective algebraic group has to be abelian in [**32**].

THEOREM 1.8. (1) *Let K be a quasi–affine algebraic k–group. Then K is affine.*
(2) *Let G be an affine algebraic k–group and H a closed normal k–subgroup. The homogeneous space G/H endowed with its natural structure of algebraic variety is an affine algebraic group.*

PROOF:

(1) It is clear that we can assume that K is connected–denote as K_1 the connected component of the identity of K; if K is quasi-affine so is K_1 and if K_1 is affine so

is K–. Consider $\mathcal{O}_K(K)$ the algebra of regular functions on K. In the same fashion than for affine algebraic groups there is a natural locally finite action of K on $\mathcal{O}_K(K)$ by left translations. Indeed, the multiplication morphism $m : K \times K \to K$ induces an algebra homomorphism $m^* : \mathcal{O}_K(K) \to \mathcal{O}_K(K \times K) \cong \mathcal{O}_K(K) \otimes \mathcal{O}_K(K)$–for this isomorphism see Exercise 14 or else prove it directly–. Writing $m^*(f) = \sum_{i=1}^r h_i \otimes k_i$ then for all $x, y \in K$, $f(xy) = \sum h_i(x)k_i(y)$ and $y \cdot f = \sum k_i(y)h_i$. If we call W the subspace of $\mathcal{O}_K(K)$ generated by $\{k_i, i = 1, \dots, r\}$ from Exercise 9 we deduce we can find a basis $\{g_i, i = 1, \dots, n\}$ of W and elements $y_j \in K$, $j = 1, \dots, n$ such that $g_i(y_j) = \delta_{ij}$. Using this basis to express $m^*(f) = \sum_{j=1}^n h'_j \otimes g_j$, we conclude that the elements h'_j, $j = 1, \dots, n$ generate the k–linear space generated by the orbit of f.

Let X be an affine variety that contains K as a dense an open subvariety. We have an injection of $k[X] = \mathcal{O}_X(X) \subset \mathcal{O}_K(K)$. For future use we observe that $[\mathcal{O}_K(K)] \cong k(X) \cong k(K)$. Indeed if we take $0 \neq f \in k[X]$ such that $f(X \setminus K) = 0$, it is clear that the principal open subset X_f verifies: $X_f \subset K \subset X$. Then: $k[X] \subset \mathcal{O}_K(K) \subset k[X_f]$. Hence being X_f an open subset of K we conclude that $k(K) = k(X_f) = k(X) = [\mathcal{O}_K(K)]$. Considering again the inclusion $k[X] \subset \mathcal{O}_K(K)$ it follows that the elements of $k[X]$ separate the points of X and consequently separate the points of K. As $k[X]$ is finitely generated as a k–algebra, if we take a finite set of its generators we have a finite number of elements of $\mathcal{O}_K(K)$ that separate the points of K. Call this set $\{f_1, \dots, f_n\}$. Denote as V the finite dimensional rational K–submodule of $\mathcal{O}_K(K)$ generated by $\{f_1, \dots, f_n\}$.

The action of K on V defines a morphism of algebraic groups $\kappa : K \to GL(V)$ that is injective. The injectivity can be proved as follows: if for some $x \in G$, $x \cdot v = v$ for all $v \in V$, then $x \cdot f_i = f_i$ for all i. Evaluating at $e \in G$, we conclude that $f_i(x) = f_i(e)$ for all i and that implies $x = e$. The image inside of $GL(V)$ of K is a constructible abstract subgroup of an affine algebraic group–$GL(V)$ and as such is affine. Call $H = \kappa(K)$ and split κ as in the diagram.

$$\begin{tikzcd} K \arrow[r, hook, "\kappa"] \arrow[d, "\kappa'"'] & GL(V) \\ H \arrow[ur, hook] & \end{tikzcd}$$

where κ' is a bijective morphism from K onto H.

The corresponding diagram at the level of the regular functions is:

$$\begin{tikzcd} k[GL(V)] \arrow[r, "\kappa^*"] \arrow[d, "p"'] & \mathcal{O}_K(K) \\ k[H] \arrow[ur, hook, "{\kappa'}^*"'] & \end{tikzcd}$$

Notice that p is surjective and ${\kappa'}^*$ injective–as κ' is surjective–. Next we check that $[\mathrm{Im}(\kappa^*)] = [\mathrm{Im}(\kappa'^*)] = k(K)$. First observe that $k[X] \subset \mathrm{Im}(\kappa^*)$. For this we take a basis $\{e_1, \dots, e_s\}$ of V and write $x \cdot e_i = \sum_j \alpha_{ij}(x)e_j$ for $x \in G$. Identifying $GL(V)$ with $GL_n(k)$ we have that $\kappa(x) = (\alpha_{ij}(x))_{1 \leq i,j \leq s}$ and then $\alpha_{ij} \in \mathrm{Im}(\kappa^*)$ for $1 \leq i, j \leq s$. Writing $e_i(yx) = \sum_j e_j(y)\alpha_{ij}(x)$ for all $x, y \in K$ we conclude that $e_i = \sum e_j(e)\alpha_{ij}$. Hence, for $1 \leq i \leq s$ $e_i \in \mathrm{Im}(\kappa^*)$ and as the elements $f_i : i = 1, \cdots n$

are in V and $\{e_1, \cdots e_s\}$ is a basis of V it follows that , $f_i \in \operatorname{Im}(\kappa'^*) = \operatorname{Im}(\kappa^*)$ for $i = 1, \ldots, n$. Hence, $k[X] \subset \operatorname{Im}(\kappa'^*) = \operatorname{Im}(\kappa^*) \subset \mathcal{O}_K(K)$ and from what we proved before we deduce that $\kappa'*(k(H)) = [\operatorname{Im}(\kappa^*)] = [\operatorname{Im}(\kappa'^*)] = [\mathcal{O}_K(K)] = k(K)$. We have thus proved that the bijective morphism κ' is a birational equivalence and as such an isomorphism. As H is affine so is K.

(2) This part follows easily from Corollary 1.4, Theorem 1.7 and what we just proved. □

Observation 1.5. Let Y be an irreducible quasi-affine variety and let X be an affine variety where Y is imbedded as a dense and open subset. Then $k(X) = [\mathcal{O}_Y(Y)] = k(Y)$. The proof of this assertion follows along the same lines than the proof of the corresponding part of Theorem 1.8.

7. Applications and examples

In this section we present a few applications of the tools developed so far and exhibit some illustrative examples. In Chapters 2 and 3 we will illustrate more thoroughly the use of the methods of this chapter by studying how to improve our control–under special hypothesis–of the geometric structure of a homogeneous space.

We start with a standardt result that can be viewed as a particular case of the situation of a quotient of an arbitrary variety by a finite group. Its proof will be omitted.

Theorem 1.9. *Let G be an affine algebraic k–group and H a closed k–subgroup. If the homogeneous space G/H_1 is affine then G/H is affine. Here as usual H_1 denotes the connected component of the identity of H.*

Corollary 1.11. *Let G be an affine algebraic k–group and H a finite subgroup, then the homogeneous space G/H is an affine variety.*

Proof: If G is finite $H_1 = e$ and the results follows immediately from Theorem 1.9. □

Next we recall a few results about solvable affine algebraic groups.

Let G be a connected solvable affine algebraic group and call G_u the set of its unipotent elements. Then G_u is a closed normal connected subgroup of G and if T is a maximal torus, G is the semi-direct product $G = G_u \rtimes T$ with respect to the action by conjugation of T on G_u. Moreover all maximal tori are conjugate and if $S \subset T$ is an inclusion of tori there exists a torus $L \subset T$ such that $T = S \times L$.

Theorem 1.10. *Let G be a solvable affine algebraic k–group and let H be a connected closed subgroup. The homogeneous space G/H is affine.*

Proof: Using Theorem 1.9 we may assume that H is also connected. The general considerations recalled above guarantee that we can write $G = G_u \rtimes T$ and $H = H_u \rtimes S$ with $S \subset T$ tori of G. First we show that it is enough to prove the theorem in the case $S = T$. Indeed, as $T = S \times L$ for some torus L we have that $G/H \cong (G_u \rtimes S/H_u \rtimes S) \times L$.

Hence, we may assume that $G = G_u \rtimes S$ and $H = H_u \rtimes S$. It is clear that G/H and G_u/H_u are isomorphic so that we have reduced our theorem to the case that the original group G is unipotent. As a unipotent group has no characters using Corollary 1.7 and Observation 1.2 we deduce that there exists a finite-dimensional G–module M and an element $m_0 \in M$ such that $H = G_{m_0}$ and $\mathfrak{h} = \{\sigma \in \mathfrak{g} : \sigma \cdot m_0 = 0\}$. Using Theorem 1.5 we conclude that G/H is isomorphic to $G \cdot m_0$ that is a G-orbit on an affine space. Being G unipotent $G \cdot m_0$ is closed and then G/H is affine.

□

The next two theorems are due to A. Borel and were first published in [**4**]. They also appear–with slight differences in the books mentioned in the introduction (see [**24**] and [**26**]). Our presentation goes along the same lines.

The following important result goes under the name of Borel's fixed point theorem.

THEOREM 1.11. *Let X be a complete variety and G a connected solvable group acting regularly on X. The set ${}^{G}X = \{p \in X : z \cdot p = p \ \forall z \in G\}$ is a non-empty closed subset.*

PROOF: The fact that ${}^{G}X$ is closed is left as an exercise. We start by considering an orbit $O(p)$ in X of minimal dimension. Call the orbit map $\pi : G \to O(p)$. Being $O(p)$ of minimal dimension it is closed in X and consequently it is a complete variety.

The morphism $\widehat{\pi} : G/G_p \to O(p)$ is bijective and G–equivariant with irreducible domain and codomain. Being $O(p)$ complete, general results about complete varieties guarantee that G/G_p is also complete. Recall that we proved in Theorem 1.10 that being G solvable the homogeneous space G/G_p is affine. Being complete and affine it has to be trivial. And then we deduce that $G = G_p$ and hence that $p \in {}^{G}X$. □

OBSERVATION 1.6. In Exercise 9 we ask the reader to prove that the converse of Theorem 1.12 is true.

DEFINITION 1.7. Let G be an irreducible affine algebraic group defined over k. A closed k–subgroup is said to be a Borel subgroup if it is connected solvable and maximal with respect to these properties.

The systematic consideration of Borel subgroups in the theory of affine algebraic groups was initiated in [**4**]. The name Borel subgroup is due to C. Chevalley.

THEOREM 1.12. *Let G be an affine algebraic k–group and B a Borel subgroup. Then G/B is a projective variety and any other Borel subgroup is conjugate to B.*

PROOF: Let C be a Borel subgroup of maximal dimension –after the theorem is proved we will know that all such subgroups have the same dimension–. By Theorem 1.6 there exists a finite-dimensional rational G–module M and a point $p_0 \in \mathbb{P}(M)$ such that G/C is isomorphic to the orbit of p_0 in $\mathbb{P}(M)$. Let km_0 be the line of M corresponding to p_0. As the group C stabilizes the line km_0 we can consider the representation of C on M/km_0 and as C is solvable, it is the stabilizer of a full flag on M/km_0. Hence, there is a full flag $\mathcal{F}$ on M that has C as stabilizer. Call $\mathcal{P}$ the projective variety of all full flags of M and call $\theta : G/C \to G \cdot \mathcal{F}$ the orbit map. Using Theorem 1.5 we deduce that $\dim\, G \cdot \mathcal{F} = \dim(G) - \dim(C)$. Now assume that $\mathcal{G}$ is another full flag and call D its stabilizer. As D is solvable the definition of C guarantees that $\dim\, C \geq \dim\, D$. Then $\dim\, G \cdot \mathcal{G} = \dim(G) - \dim(D) \geq \dim(G) - \dim(C) = \dim\, G \cdot \mathcal{F}$. Hence $\dim\, G \cdot \mathcal{F}$ is minimal and thus $G \cdot \mathcal{F}$ is closed and consequently complete. As θ is a G–equivariant bijection, general results on complete varieties guarantee that G/C is complete and being quasi-projective it has to be a projective. Let B be an arbitrary Borel subgroup and consider the action of B on G/C by translations. By Theorem 1.11 there is a fixed point for the action, i.e. an element $zC \in G/C$ such that for all $b \in B$, $bzC = zC$. Then $z^{-1}bz \in C$ for all $b \in B$, in other words $z^{-1}Bz \subset C$. As between Borel subgroups there aren't any non-trivial inclusion relations and clearly the conjugate of a Borel subgroup is again a Borel subgroup, we conclude that there exists an element $z \in G$ such that $z^{-1}Bz = C$. □

Next we consider some examples of the possible geometric structures of homogeneous spaces.

From a general viewpoint Theorems 1.8 and 1.10 as well as 1.12 provide examples of homogeneous spaces that are respectively affine and projective. Conditions for a given homogeneous space to be quasi–affine or affine will be given in Chapters 2 and 3.

Concerning more concrete examples, in Exercise 7 of Chapter 2 the reader is asked to prove that the homogeneous space G/H where $G = \mathrm{SL}_2(k)$ and

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in k \right\}$$

is isomorphic to $k^2 - (0,0)$. This is an example of a quasi-affine homogeneous space that is not affine.

Next we exhibit an example of a group G and a subgroup H with the property that the homogeneous space is a quasi-projective variety that is neither projective nor quasi-affine.

Take

$$G = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \epsilon \end{pmatrix} : \alpha\delta - \beta\gamma \neq 0,\ \epsilon \neq 0 \right\} = \mathrm{GL}_2(k) \times k^*$$

and

$$H = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha\delta \neq 0 \right\} = \mathrm{B}_2(k) \times\ 1.$$

Clearly $G/H \cong (\mathrm{GL}_2(k)/\,\mathrm{B}_2(k)) \times\ k^*$ –see Exercises 10, 11 and 12–.

Still concerning the above situation we exhibit an example of an explicit transitive action of G on a quasi–projective variety, such that H is obtained as the stabilizer of a certain point. Take the action of G on $\mathbb{P}^3 = \{[x : y : z : t] : x, y, z, t \in k,\ (x, y, z, k) \neq (0, 0, 0, 0)\}$ given as follows:

$$\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \epsilon \right) \cdot [x : y : z : t] = [\alpha x + \beta z : (\alpha y + \beta t)\epsilon : \gamma x + \delta z : (\gamma y + \delta t)\epsilon]$$

Consider now the cuadric ruled surface $Q \subset \mathbb{P}^3$ defined as $Q = \{[x : y : z : t] \in \mathbb{P}^3 : xt = yz\}$ and the lines $L_1 = \{[0 : y : 0 : t] \in \mathbb{P}^3\}$, $L_2 = \{[x : 0 : z : 0] \in \mathbb{P}^3\}$ contained in Q. Denote as Q_0 the quasi-projective variety $Q_0 = Q \setminus (L_1 \cup L_2)$. A direct computation shows that the above action can be restricted to Q_0. Consider the point $p_0 = [1 : 1 : 0 : 0] \in Q_0$ and denote as $g = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \epsilon \right)$ a generic point of the group G. We have that $g \cdot p_0 = [\alpha : \alpha\epsilon : \gamma : \gamma\epsilon]$. We prove first that the orbit of p_0 is Q_0. Indeed, given a generic point $p = [x : y : z : t] \in Q_0$ if $x \neq 0$ we have that

$$\left(\begin{pmatrix} x & 0 \\ z & 1 \end{pmatrix}, y/x \right) \cdot [1 : 1 : 0 : 0] = [x : y : z : zy/x] = [z : y : z : t]$$

If $x = 0$, as the point does not belong to L_1 and as $xt = yz$ we conclude that $y = 0$. Then the point $p = [0 : 0 : z : t]$ can be obtained as

$$\left(\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}, t/z \right) \cdot [1 : 1 : 0 : 0] = [0 : 0 : z : t]$$

Next we prove that the stabilizer of p_0 is H. Suppose that $g \cdot p_0 = p_0$, then $[\alpha : \alpha\epsilon : \gamma : \gamma\epsilon] = [1 : 1 : 0 : 0]$. Hence $\epsilon = 1,\ \gamma = 0$, i.e. $g \in B_2(k) \times 1 = H$ and the homogeneous space G/H is isomorphic to Q_0.

8. Exercises

1 Let G be an connected affine algebraic k–group and H a closed k–subgroup. Prove that given a finite number of invariant rational functions $f_i \in {}^{H}[k[G]],\ i = 1, \dots, n$ there exist $u_i,\ i = 0, 1, \dots, n,\ u_0 \neq 0$ and $\rho \in E_G(H)$ such that:
$f_i = u_i/u_0,\ x \cdot u_i = \rho(x)u_i$ for all $x \in H$ and all $i = 0, 1, \dots, n$.

2 Let G be an connected affine algebraic k–group where k is an algebraically closed field of characteristic p. Let $g \in [k[G]]$ be a rational function such that $g^{p^n} \in k[G]$ for some $n \geq 0$. Then $g \in k[G]$. The following steps are suggested: a) Prove that if $x_1, x_2, \dots, x_n \in G$ and $x_1 \cdot g, x_2 \cdot g, \dots, x_n \cdot g$ are linearly independent so are $x_1 \cdot g^{p^n}, x_2 \cdot g^{p^n}, \dots, x_n \cdot g^{p^n}$ and then conclude that the G–submodule of $[k[G]]$ generated by g is finite dimensional; b) Consider the ideal
$J = \{d \in k[G] : (x \cdot g)d \in k[G]\ \ \forall x \in G\}$. As $J \neq 0$ and is G–stable it has to be all of $k[G]$.

3 Concerning Corollary 1.6 prove that if G is an affine algebraic group acting lineraly on M, and for a fixed $m \in M$ we denote as $A : G \to M$ the orbit map –i.e. if $w \in G,\ A(w) = w \cdot m$– then $d_eA : \mathfrak{g} \to M$ is given as $d_eA(\sigma) = \sigma \cdot m$. Where $\mathfrak{g}$ denotes the Lie algebra of G and the action $(\sigma, m) \to \sigma \cdot m$ is the usual action of $\mathfrak{g}$ on M associated to the given action of G on M.

4 In the situation of Theorem 1.5 prove that in the hypothesis of condition 4. the function field of $O(x)$ and the field ${}^{G_x}[k[G]]$ are isomorphic. Conclude that the morphism π is separable–see App3–. (Hint: write an element of ${}^{G_x}[k[G]]$ as the quotient of two semi–invariant polynomials).

5 Assume that G is a finite solvable group that acts by automorphisms on a complete variety X. Prove that G has a fixed point in X.

6 Prove that $\mathrm{GL}_n(k)/\mathrm{T}_n(k)$ is isomorphic to the variety of all flags of k^n and hence that it is projective.

7 ([**26**]) Let k be a field of characteristic $p > 0$, and call G the group whose variety structure is k^4 endowed with the following product:
$(a, b, \alpha, \beta).(c, d, \gamma, \delta) = (a + c, b + d + \alpha\delta, \alpha + \gamma, \beta + \delta)$.

Show that the subgroup $H \subset G$ given as $H = \{(a, b, a^p, a^p) : a, b \in k\}$ is normal and the quotient is isomorphic to k^2 with its usual structure. Observe that G is unipotent and find an explicit immersion of G into $\mathrm{U}(5, k)$.

8 ([**19**]) Let $G = \mathrm{SL}_5(k)$ and $H = \{(a_{ij}) \in \mathrm{SL}_(5, k) : a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1, a_{ij} = 0, (i, j) \neq (1, 3), (i, j) \neq (1, 4), (i, j) \neq (2, 3), (i, j) \neq (2, 4)\}$. Consider $M = \bigwedge(k^5)$ with the natural G–action and call
$m_0 = e_1 \wedge e_3 + e_2 \wedge e_5 + e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_1 \wedge e_2 \wedge e_4 \wedge e_5$ where $\{e_1, \dots, e_5\}$ denotes the canonical basis of k^5. Prove that $H = G_{m_0}$ and that G/H is an affine variety. Describe the geometric structure of G/H explicitly.

9 Let G be a connected affine algebraic k–group that for any complete variety X where it acts regularly the action has at least one fixed point, then G is solvable.

10 Let G_1 and G_2 be affine algebraic k–groups and let H_1 and H_2 be closed k–subgroups. Call $G = G_1 \times G_2$ and $H = H_1 \times H_2$. Show that G/H is isomorphic

with $G_1/H_2 \times G_1/H_2$. Conclude in particular that
$\mathcal{O}_{G_1/H_1 \times G_2/H_2}(G_1/H_1 \times G_2/H_2) \cong {}^{H_1}k[G_1] \otimes {}^{H_2}k[G_2]$.

11 Show that $\mathrm{GL}_2(k)/B_2(k) \cong \mathbb{P}^1$. Here $\mathrm{GL}_2(k)$ denotes the general linear group of all invertible two by two square matrices with coefficients in the base field and $B_2(k)$ is the standard Borel subgroup consisting of the upper triangular matrices.

12 Show that $\mathbb{P}^1 \times k^*$ cannot be a quasi-affine or a projective variety.

13 Let $G = \mathrm{SL}_2(k)$ and H the subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0, b \in k \right\}$$

Prove directly that $G/H \cong \mathbb{P}^1$ and conclude that H is a Borel subgroup of G and that ${}^{H}k[G] = k$.

CHAPTER 2

Observable subgroups of affine algebraic groups

1. Introduction

In this chapter we define the concept of observable subgroup and prove some of their basic properties. The concept of observable subgroup was considered for the first time in [**1**]. The main concern of the authors was the study of the extension of the representations from a subgroup to the whole group. They defined a subgroup H of a group G to be observable when "every finite dimensional rational H–module can be imbedded as an H–submodule in a rational G–module"([**1**][page 131]). Hochschild and Mostow had studied the extension problem for Lie groups where not much could be proved in the general case, see [**25**],[**30**]. The situation for algebraic groups turned out to be much easier to handle. The main basic results of the theory were established in the above mentioned reference.

Afterwards the concept of observable subgroup has been extensively studied in particular in its relationship with the finite generation of invariants and Hilbert's 14th problem–see for example [**19**]–.

We will present a definition of observability that is not the standard one–but that can be considered to be implicit in [**1**]–. It concerns the possibility of finding inside an arbitrary H–stable ideal of $k[G]$ a non–zero H–fixed element. Its systematic use seems to set up a natural bridge between the algebra and the geometry of the extension problem. Other –similar– presentations of the basic material on observability can be found in [**19**][Chapter 1] or [**24**][Chapter XII].

In this chapter we also consider the induction functor from H–modules to G–modules. It is analogous to the classical induction functor for finite groups and as in the classical case it is adjoint to the restriction of the action from G to H. We characterize the observability in terms of this induction functor and finish the chapter with a proof of the usual geometric characterization of observability, i.e., we prove that H is observable in G if and only if the homogeneous space G/H is a quasi-affine variety. We also consider two interrelated conditions that strenghten the condition of observability: split and strong observability. These conditions have also strong consequences on the geometric structure of the quotient G/H as will be shown in Chapter 3.

2. Basic definitions

Let G be an affine algebraic k–group and H a closed k–subgroup. In Chapter 1, Corollary 1.2 we proved that for an arbitrary H–stable ideal of $I \subset k[G]$, there exists an element $f \in I$ that is a non–zero ρ–semiinvariant for a convenient extendible character ρ. If for all such type of ideal we can find a non-zero H–invariant (in other words if we can take ρ to be the trivial character), we say that the subgroup H is observable in G.

Definition 2.1. Let G be an affine algebraic k–group and H a closed k–subgroup. The subgroup H *is observable in* G if and only if all the H–stable non-zero ideals $I \subset k[G]$ verify that ${}^HI = I \cap {}^Hk[G] \neq 0$.

Observation 2.1. Let $G = \mathrm{SL}_2(k)$ and let us take H its standard Borel subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0, b \in k \right\}$$

The subgroup H is not observable in $\mathrm{SL}_2(k)$. Indeed, in accordance to Chapter 1 Exercise 13, in this situation ${}^Hk[G] \cong k$. Hence, if I is a non–zero H–stable proper ideal ${}^HI \subset k$ is forced to be zero.

A geometric explanation of the non–observability of H will be clear later. We will prove that the homogeneous space corresponding to $H \subset G$ has to be quasi-affine for H to be observable in G. In our particular case the mentioned exercise asserts that the homogeneous space is the projective line.

Observation 2.2. If H is unipotent the theorem of Kostant-Rosenlicht guarantees that an ideal I as above has a non-zero fixed point. Similary if H is the commutator subgroup of a certain algebraic subgroup of G it is obvious that the characters of H are trivial and thus that H is observable in G.

Observation 2.3. Assume that H is a finite subgroup of G, the following standard argument shows that in the case that G is connected H is observable in G. Let $0 \neq I \subset k[G]$ be an H–stable ideal of $k[G]$. The ring extension ${}^Hk[G] \subset k[G]$ is an integral extension –as H is a finite group–. If $f \in I$ consider a polynomial of minimal degree verified by f of the form $t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n \in {}^Hk[G][t]$. It is clear that the element $b_n \neq 0$ (otherwise f would verify a polynomial of smaller degree) belongs to the ideal I and it is H–invariant. Later we will show that in general a subgroup K is observable in G if and only if $K \cap G_1$ is observable in G_1. Another method of proof in the case that G is not connected is the following. We proved in the previous chapter that in this situation the homogeneous space G/H is an affine variety. Let $0 \neq I \subset k[G]$ be an H–stable ideal of $k[G]$ and call V the closed K–stable subset given by the zeroes of I, i.e., $V = \mathcal{Z}(I)$. Take now $\pi(V)$ the image of V by the cannonical projection $\pi : G \to G/H$. Being $\pi(V)$ closed and proper, there exists an element $f \in {}^Hk[G]$ such that $f(V) = 0$ and $f \neq 0$. Then a power f^n of f belongs to I and is a non zero H–fixed element. See also Exercise 1b.

Observation 2.4. Corollary 1.3 guarantees that the normal subgroups of G are observable.

In order to increase our control over the observable subgroups we will look more carefully at the extendible characters. Most of the proof of next Lemma will be omitted as it is more or less standardt.

Lemma 2.1. *Let G be an affine algebraic k–group and H a closed k–subgroup. Then*

(1) *If ρ is observable and $f \in k[G]$ is an extension of ρ, then for any $z \in G$, $f \cdot z$ is also an extension of ρ.*
(2) *If $\pi : k[G] \to k[H]$ is the canonical projection and ρ is an extendible character, there exists an extension $f \in k[G]$ of ρ such that $\pi(f) = \rho$.*
(3) *A character ρ is extendible if and only if there exists a rational G–module N and an injective morphism of H–modules $\iota : k\rho \to N|_H$. In other words ρ is extendible if and only if there exists a rational G–module N and a non–zero element $n \in N$*

such that $x \cdot n = \rho(x)n$ for all $x \in H$. Moreover the G–module N can be taken to be a finite dimensional G–submodule of $k[G]$.

(4) *The set $E_G(H) \subset \mathcal{X}(H)$ of extendible characters is a unital submonoid of $\mathcal{X}(H)$. In other words it is closed under multiplication and contains the character* $\mathbf{1}$.

(5) *For any $\gamma \in \mathcal{X}(H)$ there exists $\rho \in E_G(H)$ such that $\gamma\rho \in E_G(H)$.*

PROOF: We only prove numeral five.Consider $M = k\gamma$, by Theorem 1.1 there exists an extendible character ρ, a G–module N and an injective H–morphism $\iota : k\gamma \to (N|_H)_{\rho^{-1}}$. Call $n = \iota(\gamma)$, then for any $x \in H$ we have that $x \cdot n = x \cdot \iota(\gamma) = \rho(x)\iota(x \cdot \gamma) = \rho(x)\gamma(x)\iota(\gamma) = \rho(x)\gamma(x)n$. From 3 we conclude that $\rho\gamma \in E_G(H)$.

□

Next we present various algebraic characterizations of observability. Parts of the proof will be omitted.

THEOREM 2.1. *If G is an affine algebraic k–group and H a closed k–subgroup the following conditions are equivalent:*

(1) *The subgroup H is observable in G.*
(2) *$E_G(H) = \mathcal{X}(H)$, i.e. every rational character is extendible.*
(3) *For every $\rho \in E_G(H)$ there exists $q > 0$ such that $\rho^{-q} \in E_G(H)$.*
(4) *For every $\rho \in E_G(H)$, $\rho^{-1} \in E_G(H)$.*
(5) *For every character ρ we can find $q > 0$ such that $\rho^q \in E_G(H)$.*
(6) *For every finite dimensional rational H–module M there exists a finite dimensional rational G–module N and an injective morphism of H–modules $\tau : M \to N|_H$.*
(7) *For every finite dimensional rational H–module M there exists a finite dimensional rational G–module N and a surjective morphism of H–modules $\sigma : N|_H \to M$.*

In the case that condition 6) is verified and M is a simple H–module, then N can be taken to be a simple G–module.

PROOF: We prove that 1) implies 4). Assume $\rho \in E_G(H)$ and take $0 \neq f \in k[G]$ that extends ρ. Call I the principal ideal of $k[G]$ generated by f, i.e. $I = k[G]f$. As f is semi-invariant I is H–stable. Hence, there is an element $0 \neq g \in {}^H I$. Write $g = hf$ and compute the action of $x \in H$ on both sides of the equality. We have that $(x \cdot h)\rho(x)f = hf$ or $(x \cdot h - \rho(x)^{-1}h)f = 0$. If the group G is connected being $f \neq 0$ we conclude our result. If G is not connected the result follows from the corresponding result for the connected component of G at 1. The equivalence of 6) and 7) follows immediately by duality. Next we prove that 4) implies 6). If we take an arbitrary finite-dimensional rational H-module that we call M, using Theorem 1.1 we deduce the existence of an extendible character ρ, a finite dimensional rational G–module N and an injective map $\iota : M \to N$ such that $\iota(x \cdot m) = \rho^{-1}(x)x \cdot \iota(m)$ for all $x \in H$. By hypothesis the character ρ^{-1} is extendible, so if we take $f \in k[G]$ that extends ρ^{-1} and call N_0 the G–submodule of $k[G]$ generated by f, we can define an injective map $\tau : M \to N \otimes N_0$ given as $\tau(m) = \iota(m) \otimes f$. If we endow $N \otimes N_0$ with the diagonal G–module structure, the following computation shows that τ is H–equivariant. Take $x \in H$, then $\tau(x \cdot m) = \iota(x \cdot m) \otimes f = \rho^{-1}(x)x \cdot \iota(m) \otimes f = x \cdot \iota(m) \otimes \rho^{-1}(x)f = x \cdot \iota(m) \otimes x \cdot f = x \cdot \tau(m)$.

The assertion concerning the simplicity follows immediately from Theorem 1.1. □

The above theorem makes it very easy to prove that subgroups of finite index are observable.

OBSERVATION 2.5. Assume that H is a closed subgroup of G of finite index, then H is observable in G. Indeed, if we write the decomposition of G into left H–cosets: $G =$

$z_1H \cup z_2H \cup ... \cup z_rH$, $z_1 = e$ and $\gamma \in \mathcal{X}(H)$ is an arbitrary character; the function $f \in k[G]$ defined as $f|_{z_iH} = 0$ if $i = 2, 3, ..., r$ and $f|_H = \gamma$ is a non-zero γ semiinvariant.

3. Induction and restriction of representations

In a similar fashion than for finite groups one can perform the induction of representations from a closed subgroup to the whole algebraic group. In this section we define this induction as a functor and prove some of its basic properties –that are homologically weaker than for finite groups–.

The systematic consideration of induction procedures in the general situation of algebraic groups and arbitrary closed subgroups seems to have started in [**9**], [**20**] and [**35**].

As before, along this section G will be an affine algebraic k–group and H a closed k–subgroup.

Along this chapter we will often use Sweedler's notation.

Definition 2.2. The restriction functor $\operatorname{Res}_G^H : {}_G\mathcal{M} \to {}_H\mathcal{M}$ is the functor that at the level of objects sends M into $M|_H$ and at the level of morphisms is the identity.

Observation 2.6. In some cases, when the context does not allow confusions, if M is a G–module we omit the subscript in the notation of the H–module $M|_H$. For example if N is an H–module and M a G–module we may say that $\alpha : N \to M$ is a morphism of H–modules when what we really mean is that $\alpha : N \to M|_H$ is an H–morphism.

The restriction functor admits a right adjoint that we describe below. Let $M \in {}_H\mathcal{M}$ and endow $k[G] \otimes M$ with the left H–module structure given by the diagonal action, i.e. $x \cdot (f \otimes m) = x \cdot f \otimes x \cdot m$ and with a left G–module structure via $z \star (f \otimes m) = f \cdot z^{-1} \otimes m$. It is clear that with respect to both structures $k[G] \otimes M$ is a rational module. As both actions commute the fixed part with respect to the H–action –that as usual we denote as ${}^H(k[G] \otimes M)$– is a rational G–submodule of $k[G] \otimes M$.

Definition 2.3. We define the functor $\operatorname{Ind}_H^G : {}_H\mathcal{M} \to {}_G\mathcal{M}$ as follows. If $M \in {}_H\mathcal{M}$, $\operatorname{Ind}_H^G(M) = {}^H(k[G] \otimes M)$ equipped with the $\star$–structure of G–module as defined above. If $f : M \to M'$ is an H–morphism, $\operatorname{Ind}_H^G(f) = (id \otimes f)|_{{}^H(k[G] \otimes M)}$

Next lemma, that allows us to "concentrate" the action on one of the tensor factors of a rational G–module of the form $k[G] \otimes M$ –being M a rational G–module– plays an important technical role in some of the results of this section.

Lemma 2.2. `Concentration lemma` *Let M be a rational G–module and denote as M_0 the G–module that is supported on the same vector space M endowed with the trivial G–action. Let χ is the right $k[G]$–comodule structure on M associated to the given G–module structure, i.e., $\chi : M \to M \otimes k[G]$ with $\chi(m) = \sum m_0 \otimes m_1$ and $x \cdot m = \sum m_1(x) m_0$. Define $\theta : k[G] \otimes M \to k[G] \otimes M_0$ as $\theta(f \otimes m) = \sum f m_1 \otimes m_0$. Then θ is an isomorphism of G–modules with inverse $\eta(f \otimes m) = \sum f \mathcal{S}(m_1) \otimes m_0$. As usual $\mathcal{S}$ denotes the antipode of $k[G]$.*

Proof: Take $f \otimes m \in k[G] \otimes M$ and compute: $\eta(\theta(f \otimes m)) = \eta(\sum f m_1 \otimes m_0) = \sum f m_2 \mathcal{S}(m_1) \otimes m_0 = \sum f \varepsilon(m_1) \otimes m_0 = f \otimes m$, where ϵ is the counit in $k[G]$. In a similar fashion one checks that $\theta\eta = id$. As to the compatibility with the G–structures one has for $z \in G$ and $f \otimes m \in k[G] \otimes M$ that $\theta(z \cdot (f \otimes m)) = \theta(z \cdot f \otimes z \cdot m) = \sum (z \cdot f)(z \cdot m_1) \otimes m_0 = z \cdot \theta(f \otimes m)$. Recall that $\chi(z \cdot m) = \sum m_0 \otimes z \cdot m_1$. □

Next results follows easily from the concentration lemma.

COROLLARY 2.1. *The functor* Ind_G^G *is naturally equivalent with the identity functor.*

The G–module $\mathrm{Ind}_H^G(M)$ if $M \in {}_H\mathcal{M}$, can be defined as the G–module of H–equivariant polynomial maps from G to M or –in the same spirit– as the global sections of a certain sheaf on the homogeneous space G/H (see [**9**] for the first presentation and [**20**] for the second). One of the advantages of the definition we presented is that its generalization for general Hopf algebras is straightforward (see [**16**]).

OBSERVATION 2.7. The G–module $\mathrm{Ind}_H^G(M)$ is in a natural way a ${}^Hk[G]$–module by multiplication on the first tensorand. Indeed, if $g \in {}^Hk[G]$ and $\sum f_i \otimes m_i \in {}^H(k[G] \otimes M)$, then $\sum gf_i \otimes m_i \in {}^H(k[G] \otimes M)$. The induced module is a $({}^Hk[G], G)$–module in the sense of the following definition. See also Exercise 9) for a generalization.

DEFINITION 2.4. If G is an arbitrary group and A a k–algebra acted by G on the left by algebra automorphisms, an (A, G)–module is a k–space that is at the same time a G–module and an A–module in such a way that the actions are related by the formula: $z \cdot (am) = (z \cdot a)(z \cdot m)$ if $z \in G$, $a \in A$ and $m \in M$. In case that G is an algebraic group and all the G–actions involved are rational we say that M is a rational (A, G)–module.

DEFINITION 2.5. In the situation above, for any $M \in {}_H\mathcal{M}$ we define the k–linear map $E_M : \mathrm{Ind}_H^G(M) \to M$ as $E_M(\sum f_i \otimes m_i) = \sum f_i(e)m_i$. The map E_M will sometimes be called the evaluation map.

It is worth observing that if we endow M with the ${}^Hk[G]$–module structure given by $f \cdot m = f(e)m$, for $f \in {}^Hk[G]$, $m \in M$, then the map E_M is also a morphism of ${}^Hk[G]$–modules.

LEMMA 2.3. *The family of maps* $\{E_M : M \in {}_H\mathcal{M}\}$ *define a natural transformation between the functors* $\mathrm{Res}_G^H \circ \mathrm{Ind}_H^G$ *and* $Id : {}_H\mathcal{M} \to {}_H\mathcal{M}$.

Next theorem shows that the functors Res_G^H and Ind_H^G are adjoint to each other and the map E is the counit associated to the adjunction.

THEOREM 2.2. *In the situation above* Ind_H^G *is the right adjoint of the functor* Res_G^H.

PROOF: We want to check that for any triple $M \in {}_H\mathcal{M}$, $N \in {}_G\mathcal{M}$ and $\phi : \mathrm{Res}_G^H(N) \to M$ morphism of H–modules, there exists a unique morphism of G–modules $\widetilde{\phi} : N \to \mathrm{Ind}_H^G(M)$ such that the diagram that follows is commutative.

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "\widetilde{\phi}"'] & M \\ & \mathrm{Ind}_H^G(M) \arrow[u, "E_M"'] \end{tikzcd}$$

In order to define $\widetilde{\phi}$ we look at $\chi : N \to N \otimes k[G]$, the $k[G]$–comodule structure of N associated to the given G–module structure and define (in Sweedler's notation) $\widetilde{\phi}(n) = \sum \mathcal{S}(n_1) \otimes \phi(n_0)$ where $\mathcal{S}$ stands for the antipode map in $k[G]$. As we already observed, if $z \in G$ and $n \in N$ then $\chi(z \cdot n) = \sum n_0 \otimes z \cdot n_1$. Hence $\widetilde{\phi}(z \cdot n) = \sum \mathcal{S}(z \cdot n_1) \otimes \phi(n_0) = \sum \mathcal{S}(n_1) \cdot z^{-1} \otimes \phi(n_0) = z \star (\sum \mathcal{S}(n_1) \otimes \phi(n_0)) = z \star \widetilde{\phi}(n)$. Next we show that for all $n \in N$ $\widetilde{\phi}(n)$ is H–fixed, or equivalently that for all $x \in H$, $\sum x \cdot \mathcal{S}(n_1) \otimes x \cdot \phi(n_0) = \sum \mathcal{S}(n_1) \otimes \phi(n_0)$. To establish this equality we evaluate at $z \in G$. It is enough to prove that $\sum (x \cdot \mathcal{S}(n_1))(z)(x \cdot \phi(n_0)) = \sum \mathcal{S}(n_1)(z)\phi(n_0)$. Now, $\sum (x \cdot \mathcal{S}(n_1))(z)(x \cdot \phi(n_0)) = \sum n_1(x^{-1}z^{-1})x \cdot \phi(n_0) = \phi(x \cdot \sum n_1(x^{-1}z^{-1})n_0) = \phi(x \cdot ((x^{-1}z^{-1}) \cdot n)) = \phi(z^{-1} \cdot n) = \sum n_1(z^{-1})\phi(n_0) = \sum \mathcal{S}(n_1)(z)\phi(n_0)$.

Moreover $(E_M \circ \widetilde{\phi})(n) = \sum \mathcal{S}(n_1)(e)\phi(n_0) = \phi(\sum n_1(e)n_0) = \phi(n)$. The uniqueness of $\widetilde{\phi}$ is left as an exercise. $\square$

What is called the reciprocity law of induction is just a reformulation of the theorem just proved.

COROLLARY 2.2. `Reciprocity law of induction`. *In the situation of Theorem 2.2, there is an isomorphism, natural in $N \in {}_G\mathcal{M}$ and $M \in {}_H\mathcal{M}$ between* $\operatorname{Hom}_H(\operatorname{Res}^H_G(N), M)$ *and* $\operatorname{Hom}_G(N, \operatorname{Ind}^G_H(M))$.

Following the ideas of the concentration lemma one can prove the so called "tensor identity". We state it without proof.

THEOREM 2.3. `Tensor identity`. *Let G and H be as before and take $M \in {}_H\mathcal{M}$, $N \in {}_G\mathcal{M}$. Then there is an isomorphism* $\operatorname{Ind}^G_H(M) \otimes N \cong \operatorname{Ind}^G_H(M \otimes \operatorname{Res}^H_G(N))$ *that is G–equivariant and natural in M and N –the tensor products are equipped with the adequate diagonal structures–.*

OBSERVATION 2.8. If we consider the action of ${}^Hk[G]$ on $\operatorname{Ind}^G_H(M) \otimes N$ by multiplication on the first tensorand and on $\operatorname{Ind}^G_H(M \otimes N|_H)$ as in the general situation, the map Θ is a ${}^Hk[G]$–morphism.

OBSERVATION 2.9. If we take $M = k$ in the tensor identity we have that: ${}^Hk[G] \otimes N \cong \operatorname{Ind}^G_H(\operatorname{Res}^H_G(N))$ as G–modules. The action of G on the right side is the usual one and on the left side is the diagonal action with $y \star f = f \cdot y^{-1}$ for $y \in G, f \in {}^Hk[G]$. The above isomorphism can be veiwed as a generalization of the concentration lemma. Indeed, if we take $H = \{1\}$ in the result above we have that $k[G] \otimes N = \operatorname{Ind}^G_{\{1\}}((\operatorname{Res}^{\{1\}}_G(N)) = k[G] \otimes N_0$ and this isomorphism is the content of Lemma 2.2.

OBSERVATION 2.10. `The transfer principle.` Assume that N is an arbitrary rational G–module. The transfer principle is the assertion that that ${}^G({}^Hk[G] \otimes N)$ and ${}^HN$ are isomorphic. To prove this isomorphism we proceed as follows: consider a rational H–module M and a rational G–module N. The tensor identity guarantees the isomorphism of ${}^H(k[G] \otimes M) \otimes N$ and ${}^H(k[G] \otimes M \otimes N|_H)$ as G–modules when endowed with the diagonal structure on the first module and with the usual structure on the second. If we take the G–fixed part on the isomorphism above we obtain: ${}^G({}^H(k[G] \otimes M) \otimes N) \cong {}^G({}^H(k[G] \otimes M \otimes N|_H)) \cong {}^H({}^Gk[G] \otimes M \otimes N|_H) \cong {}^H(M \otimes N|_H)$. If we put $M = k$ we obtain the transfer principle. For future reference let us recall that the G–action on ${}^Hk[G] \otimes N$ that we use to take the fixed part is the diagonal one, the G–action on ${}^Hk[G]$ is given as follows: if $f \in {}^Hk[G]$, if $w \in G$ $w \to f = f \cdot w^{-1}$.

The name "transfer principle" for the above isomorphism was suggested by A. Borel (see [**3**]). It was used in particular cases by the classical invariant theorists and was called "the adjunction principle". It can be of great use to study problems of finite generation of invariants. See [**19**] for interesting historical considerations about the subject and for some applications.

4. Induction and observability

As soon as the induction functor was defined its relationship with observability was noticed (see for example [**9**]). The following characterization of observability is easy to prove: the natural transformation $E : \operatorname{Res}^G_H \circ \operatorname{Ind}^G_H \to Id$ is surjective if and only if H

is observable in G. Then we apply this characterization to study the behaviour of the observability property along towers of groups and subgroups.

The proof of the theorem that follows will be omitted –see Theorem 2.1 for a hint on the proof–.

THEOREM 2.4. *Let G be an affine algebraic k–group and H a closed k–subgroup. The subgroup H is observable in G if and only if for all $M \in {}_H\mathcal{M}$ the map $E_M : \operatorname{Ind}_H^G(M) \to M$ is surjective.*

COROLLARY 2.3. *Let $K \subset H \subset G$ be a tower of closed k–subgroups of the affine algebraic k–group G. Assume that K is observable in H and that H is observable in G, then K is observable in G. Conversely if K is observable in G it is also observable in H.*

PROOF: The first part follows immediately from the transitivity of induction (Exercise 3) and Theorem 2.4. As to the second part if we take $\gamma \in \mathcal{X}(K)$ and choose a γ semi-invariant $f \in k[G]$ with $f(1) = 1$, the polynomial $h = f|_H$ obtained by restriction is a non zero K-semiinvariant of weight γ contained in $k[H]$.

$\square$

In the case of a tower as above it is false in general that the observability of K in G implies the observability of H in G (take $\{e\} \subset H \subset SL_2(k)$ as in Observation 2.1). Next theorem guarantees that the observability within a group G is preserved by extensions of subgroups of finite index.

THEOREM 2.5. *Assume that $K \subset H \subset G$ is a tower of k–closed subgroups of the affine algebraic k–group G. Assume that K is of finite index in H, then K is observable in G if and only if H is observable in G.*

PROOF: If H is observable in G so is K. This follows from the transitivity and from the fact that any finite index subgroup of a given group is observable–see Observation 2.5–. To prove the converse we proceed as follows. Assume first that G is connected. Let I be a non zero H–stable ideal of $k[G]$. We can find $0 \neq f \in I$ such that $y \cdot f = f$ for all $y \in K$, i.e. $0 \neq f \in {}^K I \subset {}^K k[G]$. The inclusion of rings ${}^H k[G] \subset {}^K k[G]$ is integral being K of finite index in H. In this situation it is well known–and proved in Observation 2.3 that the ideal ${}^K I \cap {}^H k[G] = {}^H I$ is non-zero. In the general situation the argument is slightly more elaborate. First we use the transfer principle to prove that ${}^K I = {}^H({}^K k[H] \otimes I)$. As K is observable in G we can find a non–zero element $\xi \in {}^H({}^K k[H] \otimes I)$. Now, ${}^K k[H]$ admits a finite basis that we call $\{\delta_1, \cdots, \delta_r\}$. Writing $\xi = \sum \delta_i \otimes f_i$ with $f_i \in I$ and using the fact that ξ is H–fixed, we see that the action by an element $h \in H$ has to permute the set $\{f_1, \cdots, f_r\}$. If we call Σ_i for $1 \leq i \leq r$ the elementary symmetric functions in r variables, we deduce that $\Sigma_i(f_1, \cdots, f_r) \in {}^H I$ for all $1 \leq i \leq r$ and one of them cannot be zero –otherwise the elements f_i would be nilpotent and hence zero–. $\square$

The content of next theorem is that observability is a property at the connected level.

THEOREM 2.6. *Let G be an affine algebraic k–group and H a closed k–subgroup. Then the following conditions are equivalent:*

1. *The subgroup H is observable in G.*
2. *The subgroup $H \cap G_1$ is observable in G_1.*
3. *The subgroup H_1 is observable in G_1.*

As usual G_1 indicates the connnected component of G that contains 1 and similarly for H_1.

PROOF: Consider the following diagram of inclusions

$$\begin{tikzcd} H_1 \arrow[dr, hook] & & \\ & H \cap G_1 \arrow[r, hook] \arrow[d, hook] & H \arrow[d, hook] \\ & G_1 \arrow[r, hook] & G \end{tikzcd}$$

It is clear that $H \cap G_1$ has finite index in H and being normal contains H_1. The results of Theorem 2.5 guarantee that H_1 is observable in G_1 if and only if $H \cap G_1$ is observable in G_1.

Assume now that $H \cap G_1$ is observable in G_1, as G_1 is observable in G –see Observation 2.3– we conclude the same for $H \cap G_1$ inside of G. Using Theorem 2.5 we deduce that H is observable in G.

Conversely, if H is observable in G so is $H \cap G_1$. Then by Corollary 2.3, $H \cap G_1$ is observable in G_1. □

5. The geometric characterization of observability

In this section we prove that H is observable in G if and only if the quasi-projective homogeneous space G/H is a quasi–affine variety.

The first ingredient of the proof is to translate the observability condition –defined in terms of ideals of $k[G]$– into a condition guarantying the existence of enough global sections on the sheaf of regular functions on the homogeneous space G/H.

Once this task is performed the proof of the observability of H in G from the quasi-affiness of G/H follows from easy general algebraico–geometrical considerations.

The proof that if H is observable in G then G/H is quasi–affine –i.e. the converse of the above assertion– will be developped along the following lines. First we show that if H is observable in G, then any H–invariant rational function on G can be written as the quotient of two H–invariant polynomials. Once this is established the proof follows from the results of Chapter 1 Section 6–especifically from Theorem 1.7–.

THEOREM 2.7. *Let G be an affine algebraic k–group and H a closed k–subgroup. The subgroup H is observable in G if and only if for any $C \subset G/H$ closed and proper, there exists an element $0 \neq f \in {}^H k[G]$ such that $f(C) = 0$.*

PROOF: Assume H is observable in G and take $C \subset G/H$ a closed and proper subset. Then $\pi^{-1}(C) \subset G$ is a proper closed subset of G whose ideal I is non-zero and H–stable. Any element $0 \neq f \in {}^H k[G] \cap I$–that exists because of the observability hypothesis– is a non-zero global section of the structure sheaf of G/H that takes the value zero on C. Conversely, assume that $I \subset k[G]$ is an H–stable non-zero ideal and call $V = \mathcal{Z}(I)$. It is clear that V is saturated and hence that $\pi(V)$ is closed in G/H–see Theorem 1.6–. As $\pi(V) \neq G/H$, we can find $0 \neq f \in {}^H k[G]$ such that $f(\pi(V)) = 0$, i.e. $f(V) = 0$. Then for some n the polinomial $0 \neq g = f^n$ belongs to I and it is clearly H–fixed. We conclude that H is observable in G. □

THEOREM 2.8. *Let X be a quasi-affine variety defined over an arbitrary closed field k. Let C be an arbitrary proper and closed subset of X then, there exists an element $0 \neq f \in \mathcal{O}_X(X)$ such that $f(C) = 0$.*

PROOF: Assume that X is imbedded as a dense an open subset in an affine variety Y. Let C be as in the statement of the theorem and call $\overline{C}$ the closure of C in Y. As $\overline{C}$ is a proper and closed subset of the affine variety Y –observe that the intersection of $\overline{C}$ with X coincides with C–, there exists a polynomial $g \in k[Y]$ such that $g(\overline{C}) = 0$. If we call $f = g|_X$ it is clear that $f \in \mathcal{O}_X(X)$, $f(C) = 0$ and $f \neq 0$ –recall that X is dense in Y–.

□

OBSERVATION 2.11. Theorem 2.7 admits a converse. As it will not be used in the future only the general pattern of its proof will be presented in the form of an exercise (see Exercise 7). It is relevant to observe that this converse uses deeper results in algebraic geometry than the ones used–and proved– along the text.

Let X be a normal quasi-projective variety defined over an algebraically closed field k. The variety X is quasi-affine if and only if for every proper and closed subset C of X there exists and element $0 \neq f \in \mathcal{O}_X(X)$ such that $f(C) = 0$.

Next condition is an immediate consequence of theorems just proved.

COROLLARY 2.4. *Let G be an affine algebraic k–group and H a closed k–subgroup. Suppose that the homogeneous space G/H is quasi–affine then H is observable in G.*

To prove the converse we need an easy lema in order to apply the results of Chapter 1 Section 6.

LEMMA 2.4. *Let G be an irreducible affine algebraic k–group and H a closed k–subgroup. Suppose that H is observable in G, then ${}^H[k[G]] = [{}^Hk[G]]$.*

PROOF: Clearly $[{}^Hk[G]] \subset {}^H[k[G]]$. If $0 \neq g \in {}^H[k[G]]$ call $I_g = k[G]g \cap k[G]$ the ideal of the numerators of g. This ideal is H–stable and not zero. Using the observability hypothesis we conclude that there is an H–fixed element $0 \neq f_1 \in I_g$. Writing $f_1 = f_2g$ and acting on both sides of the equality with a generic element $x \in H$ we have that: $f_2g = f_1 = x \cdot f_1 = x \cdot (f_2g) = (x \cdot f_2)(x \cdot g) = (x \cdot f_2)g$. Then $(f_2 - x \cdot f_2)g = 0$ and as G is irreducible we conclude that $f_2 \in {}^Hk[G]$. □

THEOREM 2.9. *Let G be an affine algebraic k–group and H a closed k–subgroup. Suppose that H is observable in G, then the homogeneous space G/H is a quasi-afine variety.*

PROOF: By Theorem 2.6 and Exercise 6 we can assume that G is irreducible. In this situation the observability of H in G implies (Lemma 2.4) that ${}^H[k[G]] = [{}^Hk[G]]$. Using Theorem 1.7 we conclude that G/H is quasi-affine. □

THEOREM 2.10. *Let G be an affine algebraic k–group and H a closed k–subgroup, then the following three conditions are equivalent:*

1. *The subgroup H is observable in G.*
2. *The homogeneous space G/H is a quasi-affine variety.*
3. *For an arbitrary proper and closed subset $C \subset G/H$, there exists an element $0 \neq f \in {}^Hk[G]$ such that $f(C) = 0$.*

If moreover G is connected the above conditions are equivalent to:

4. *$H = \{x \in G : x \cdot f = f,\ \forall f \in {}^Hk[G]\}$.*
5. *${}^H[k[G]] = [{}^Hk[G]]$.*

PROOF: The equivalence of 1) and 2) is the content of Corollary 2.4 and Theorem 2.9, the equivalence of 1) and 3) is Theorem 2.4. Assuming that G is connected the fact that 1) implies 5) was proved in Lemma 2.4. The fact that 5) implies 2) is exactly the content of

Theorem 1.7. In the same connectivity hypothesis the equivalence of 4) and 5) was proved in Theorem 1.4. □

It is important to observe–see Exercise 5– that the equivalence of conditions 2), 3) and 5) of Theorem 2.7 is just a particularization of a general result about quasi–projective varieties.

Observation 2.12. In Observation 2.1 the corresponding homogeneous space is isomorphic to $\mathbb{P}^1(k)$ and this explains geometrically the non-observability of H inside of G.

The methods of this section are slightly different from the original arguments that appeared in [**1**]. The ideal theoretical characterization of observability–that is implicit in the above reference– , once it is unmasked makes some of the arguments more transparent.

More recent presentations that follow the same line of argumentation than [**1**] are for example: [**19**] and [**24**].

6. Exercises

1 a Let k be an algebraically closed field and consider inside of $\mathrm{GL}_2(k)$ the subgroup
$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} : 0 \neq a \in k, b \in k \right\}$$
Prove that H is observable in of $\mathrm{GL}_2(k)$.
b Generalize the method of the proof of Observation 2.3 in order to guarantee that if H is a closed subgroup of G such that the variety G/H is affine, then H is observable.

2 In the situation of Theorem 2.2 prove the uniqueness of the map $\widetilde{\phi}$.

3 Let $K \subset H \subset G$ be a tower of closed k–subgroups of the affine algebraic k–group G. Prove that the functors Ind_K^G and $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H : {}_K\mathcal{M} \to {}_H\mathcal{M} \to {}_G\mathcal{M}$ are naturally isomorphic. Prove also that for any $M \in {}_K\mathcal{M}$ the evaluation maps $E_{M,K}^G$ and $E_{M,K}^H \circ E_{\mathrm{Ind}_K^H(M),H}^G$ are equal up to the natural isomorphism mentioned above. Notice that in the above considerations–as there is danger of confusion– we introduced in the notations corresponding to the evaluation map, the given group and subgroup.

4 Prove that if H is a closed subgroup of the affine algebraic k–group G and $M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in ${}_H\mathcal{M}$, the corresponding sequence in ${}_G\mathcal{M}$, $\mathrm{Ind}_H^G(M_1) \to \mathrm{Ind}_H^G(M_2) \to \mathrm{Ind}_H^G(M_3) \to 0$ is also exact.

5 (`M. Brion`) The purpose of this exercise is to give the general lines of a proof of the converse of Theorem 2.7. Let X be a normal quasi-projective variety with the property that for an arbitrary proper and closed subvariety $C \subset X$ there exists a non–zero regular function that is zero on C. Then, X is quasi-affine.
(a) Reduce the assertion to the case that X is irreducible.
(b) Assuming X irreducible call $k(X)$ the field of rational functions on X. In the hypothesis of the exercise prove that $k(X) = [\mathcal{O}_X(X)]$. (Hint: Let $0 \neq f \in k(X)$ and call C a closed subvariety of X such that f is defined on the complement of C. Choose $0 \neq g \in \mathcal{O}_X(X)$ with $g(C) = 0$. For a large enough n we have that $g^n f \in \mathcal{O}_X(X)$.)
(c) If $\{f_1, \cdots, f_n\}$ is a set of field generators of $k(X)$, write $f_i = g_i/h_i, i = 1, \cdots, n$ and prove that the map $\phi : X \to k^{2n}$ given as $\phi(x) = (h_1(x), \cdots, h_n(x), g_1(x), \cdots, g_n(x))$ is injective.

(d) Call R the integral closure in $k(X)$ of $k[h_1, \cdots, h_n, g_1, \cdots, g_n]$. Use Zariski's Main theorem to prove that the map $X \to \mathrm{S}pec(R)$ is an open immersion. Conclude that X is quasi-affine.

6 Prove that G/H is quasi-affine if and only if G_1/H_1 is quasi-affine.

7 Describe explicitly the homogeneous space G/H where $G = \mathrm{SL}_2(k)$ and

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in k \right\}$$

Conclude that G/H is a quasi-affine variety, and hence H is observable in G. Observe that this has to be so as H is unipotent.

8 Find an example of an affine algebraic k–group G and two closed subgroups H and K such that both are observable in G but the group generated by H and K is not observable.

9 Assume that G is an affine algebraic k–group and H a closed k–subgroup. Let A be a rational H–module algebra and M an (A,H)–module.

(a) Prove that $\mathrm{Ind}_H^G(A)$ is in a natural way a rational G–module algebra and that the evaluation map $E_A : \mathrm{Ind}_H^G(A) \to A$ is a morphism of H–module algebras.

(b) Prove that $\mathrm{Ind}_H^G(M)$ is in a natural way a rational $(\mathrm{Ind}_H^G(A), G)$–module and that the natural map $E_M : \mathrm{Ind}_H^G(M) \to M$ is a morphism of $(\mathrm{Ind}_H^G(A), H)$–modules where we endow M with the structure of $\mathrm{Ind}_H^G(A)$–module given through $E_A : \mathrm{Ind}_H^G(A) \to A$.

(c) Notice that the situation described in Observation 2.7 is a particular case of the above considerations.

CHAPTER 3

Affine homogeneous spaces

1. Introduction

In this chapter we use the concept of observable subgroup in order to study the affiness of a homogeneous space of the form G/H for an arbitrary affine k–group G and a closed k–subgroup H.

We prove the following results: if H is an exact subgroup of G or if it is geometrically reductive then it is observable. Here by exact we mean that the induction functor Ind_H^G is exact. Once this is done we proceed further ahead and prove that in both cases –H geometrically reductive or exact in G– the homogeneous space G/H is affine. To establish the affiness we proceed as follows, once we know that it is quasi-affine it is easy to see that it can be covered by affine patches of the form $(G/H)_f$ where f is an H–invariant polynomial. We finish the proof by putting all these affine patches together in order to guarantee that G/H is affine.

In the proofs sketched above the observability is used as an intermediary platform that allows us to prove the quasi-affiness of the corresponding homogeneous spaces. Once this is established, the rest follows more or less easily.

The affiness of G/H in the case that H is (geometrically) reductive has been known for quite a long time and can be considered as a classical result. In fact, most of the results we prove for G/H –for H reductive– are valid for quotients of actions of reductive groups on arbitrary affine varieties (see [**31**]). The case we present here –i.e the case of homogeneous spaces– illustrates the adaptability and the power of the concept of observability as part of the basic toolkit in invariant theory.

The theorem that relates the affiness of G/H with the exactness of the induction functor was proved more or less simultaneously and independently in [**9**], [**35**] and [**20**]. The proof we present here appeared in [**17**] in 1985, and a different purely algebraic proof was published the same year in [**13**]. Other proofs are known, the more general –valid for arbitrary Hopf algebras– seems to be the one presented in [**38**] in 1990.

2. Geometric reductivity and observability

In this section we prove that a closed and geometrically reductive subgroup of an affine algebraic group is always observable. Even though the necessary tools for our proof were already available in Chapter 2 it seemed better to delay the presentation of this result until we could prove not only the quasi-affiness but the affiness or the homogeneous space in this situation. Moreover, for reasons that will be clear along the presentation it is convenient to have a unified treatment of the geometrically reductive case and the case that the subgroup is exact.

First we introduce the concept of geometric reductivity.

In positive characteristic, the complete reducibility of the representations was proved to be a very exceptional phenomenon so that it was necessary to device a more general concept in order to be able to develop the theory of algebraic invariants with a reasonable degree of generality. With that objective in mind the concept of geometric reductivity was defined in [**31**] where it was conjectured to be equivalent to reductivity, i.e. to the triviality of the unipotent radical. It was soon verified that the notion of geometric reductivity was strong enough as to guarantee the needed results about invariants (see [**31**] or [**33**]). In [**34**] it was proved that a geometrically reductive group is reductive and in [**21**] the converse was established – that by then was called the Mumford conjecture–.

In what follows if M is an arbitrary vector space $S^q(M)$ will denote the homogeneous component of degree q of the symmetric algebra built on M and if M is equipped with the action of a group G , $S^q(M)$ will be considered with the induced action.

Definition 3.1. An affine algebraic group H defined over an algebraically closed field k is said to be *geometrically reductive* if for every rational H–module M and every non-zero H–morphism $\lambda : M \to k$ there exists a positive integer q and an element $\xi \in {}^H S^q(M)$ such that $S^q(\lambda)(\xi) = 1$.

In the definition above $S^q(\lambda)$ denotes the map $S^q(\lambda) : S^q(M) \to k$ defined from λ in the usual fashion.

Definition 3.2. In the case that in Definition 3.1 for all $M \in {}_H\mathcal{M}$ and for all λ, the exponent q can be taken to be 1, we say that the group H is linearly reductive.

Observation 3.1. It is not hard to prove that in the case of characteristic p, the exponent q –if it exists– can be taken to be a power of p, and in the case of characteristic zero it can be taken to be 1.

The following theorem is a useful characterization of geometric reductivity. The proof will be omitted.

Theorem 3.1. *Let H be an arbitrary affine algebraic group defined over k, then the following conditions are equivalent*

(1) *If $N \subset M$ are rational H–modules with N of codimension one in M, there exists a natural number n and a G–submodule $T \subset S^{p^n}(M)$ such that $S^{p^n}(M) = NS^{p^n-1}(M) \oplus T$.*
(2) *The group H is geometrically reductive.*
(3) *For all surjective arrows $\phi : R_1 \to R_2$ in the category of rational H–module algebras and for all $r_2 \in {}^H R_2$, there exists a $n \geq 0$ and an element $r_1 \in {}^H R_1$ such that $\phi(r_1) = r_2^{p^n}$.*

Theorem 3.2. *Let G be an affine algebraic group defined over an algebraically closed field k and let H be a closed k–subgroup. If H is geometrically reductive, then it is observable in G.*

Proof: Assume that $\rho \in \mathcal{X}(H)$ is an arbitrary character and choose a polynomial $f \in k[G]$ such that $\pi(f) = \rho$ –as usual π denotes the canonical projection into $k[H]$–. Call $M = \langle H \cdot f \rangle$ the finite dimensional H–module generated by f and consider the map $\lambda = \epsilon \circ \pi|_M : M \to k$ where ϵ denotes–as usual– the evaluation map at the unit of G –i.e. the counit of the Hopf algebra $k[G]$–. In explicit terms if $a_i \in k$ and $x_i \in H$ then $\lambda(\sum a_i(x_i \cdot f)) = \sum a_i\rho(x_i)$. As $\lambda(f) = 1$ the map λ is surjective. Also $\lambda(x \cdot (\sum a_i(x_i \cdot$

$f))) = \lambda(\sum a_i(xx_i \cdot f)) = \rho(x)\sum a_i\rho(x_i)$. Hence if we endow M with the structure of H–module $x \rightharpoonup g = \rho^{-1}(x)x \cdot g$ and call $M_{\rho^{-1}}$ the corresponding object, the map λ becomes an H–morphism $\lambda : M_{\rho^{-1}} \to k$. By the reductivity hypothesis, we can find an element $\xi = \sum a_{i_1}...a_{i_q}(x_{i_1} \cdot f) \otimes ... \otimes (x_{i_q} \cdot f) \in S^q(M|_{\rho^{-1}})$ such that : a) $x \rightharpoonup \xi = \xi$, b) $S^q(\lambda)(\xi) = 1$. Condition a) means that $\sum a_{i_1}...a_{i_q}(x_{i_1} \cdot f) \otimes ... \otimes (x_{i_q} \cdot f) = \xi = x \rightharpoonup \xi = \rho^{-q}(x)x \cdot \xi = \rho^{-q}(x)\sum a_{i_1}..a_{i_q}(xx_{i_1} \cdot f) \otimes ... \otimes (xx_{i_q} \cdot f)$. Applying the multiplication map to this equality and calling $g = \sum a_{i_1}...a_{i_q}(x_{i_1} \cdot f)...(x_{i_q} \cdot f) \in k[G]$, we obtain that $x \cdot g = \rho^q(x)g$. Condition b) means that $1 = \sum a_{i_1}...a_{i_q}\rho(x_{i_1})...\rho(x_{i_q})$ and as $\rho(x) = f(x)$ for all $x \in H$, we have that $g(1) = \sum a_{i_1}...a_{i_q}f(x_{i_1})...f(x_{i_q}) = \sum a_{i_1}...a_{i_q}\rho(x_{i_1})...\rho(x_{i_q}) = 1$. Hence g is an extension of ρ^q and Theorem 2.1 part 5, guarantees our result. □

3. Exact subgroups

Definition 3.3. Let G be an affine algebraic k–group and H a closed subgroup. We say that H is exact in G if the functor $\operatorname{Ind}_H^G : {}_H\mathcal{M} \to {}_G\mathcal{M}$ is exact.

Observation 3.2. The functor Ind_H^G is the composition of the functor tensor product by $k[G]$–over the field k– with a fixed point functor. As an arbitrary fixed point functor is always left exact and the tensor product is performed over a field, the only relevant part of the above definition concerns the surjectivity. Hence H is exact in G if and only if for every surjective morphism of H–modules $\alpha : M \to L$, the corresponding morphism $\operatorname{Ind}_H^G(\alpha) = (id \otimes \alpha)|_{\operatorname{Ind}_H^G(M)} : \operatorname{Ind}_H^G(M) \to \operatorname{Ind}_H^G(L)$ is also surjective.

Observation 3.3. If H is an affine algebraic group a cohomology theory for the category of rational H–modules can be developed–see for example [**22**] for the original paper or [**10**] and [**27**] for more recent presentations–. If $H^n(G, -)$ denotes the n–th cohomology functor, it is easy to prove that H is exact in G if and only if for every rational H–module M, $H^1(H, k[G] \otimes M) = 0$.

Observation 3.4. We can transform the problem of the exactness of the functor Ind_H^G in the problem of the exactness of a fixed point functor in another category. Denote as ${}_{(k[G],H)}\mathcal{M}$ the category of all left $(k[G], H)$–modules and as usual ${}_{{}^Hk[G]}\mathcal{M}$ the category of all left ${}^Hk[G]$–modules. Let $\mathcal{F} : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^Hk[G]}\mathcal{M}$ be the H–fixed part functor. Then H is exact in G if and only if the functor $\mathcal{F}$ is exact–we ask the reader to prove this assertion in Exercise 1–.

Theorem 3.3. *Let G be an affine algebraic k–group and H a closed subgroup. If H is exact in G then it is observable in G.*

Proof: We proceed in a similar fashion than in Theorem 3.2. Let $\rho \in \mathcal{X}(H)$ and choose $f \in k[G]$ such that $\pi(f) = \rho$. Consider $M = \langle H \cdot f\rangle$ the finite dimensional H–module generated by f and the surjective H-morphism $\lambda = \epsilon \circ \pi|_M : M|_{\rho^{-1}} \to k$ that in explicit terms can be given as $\lambda(\sum a_i(x_i \cdot f)) = \sum a_i\rho(x_i)$. From the exactness of the induction functor we deduce that the map $\operatorname{Ind}_H^G(\lambda) : {}^H(k[G] \otimes M|_{\rho^{-1}}) \to {}^Hk[G]$ is surjective. The map induced by λ sends $\sum f_i \otimes (x_i \cdot f)$ into $\sum \rho(x_i)f_i$. Using the surjectivity of the induced map we can guarantee the existence of an H–fixed element $\sum f_i \otimes (x_i \cdot f)$ such that $\sum \rho(x_i)f_i = \mathbf{1}$. As $\sum f_i \otimes (x_i \cdot f)$ is fixed by H we have that $\sum f_i \otimes x_i \cdot f = \sum x \cdot f_i \otimes x \rightharpoonup (x_i \cdot f) = \sum x \cdot f_i \otimes \rho^{-1}(x)(xx_i \cdot f) = \rho^{-1}(x)(\sum x \cdot f_i \otimes (xx_i \cdot f)$. Then the polynomial $g = \sum f_i(x_i \cdot f)$ verifies that $x \cdot g = \sum(x \cdot f_i)(xx_i \cdot f) = \rho(x)\sum f_i(x_i \cdot f) = \rho(x)g$. Also

$g(1) = \sum f_i(1)(x_i \cdot f)(1) = \sum f_i(1)\rho(x_i) = 1$. Hence the character ρ is extendible and then by Theorem 2.1 the group H is observable in G. □

The above proof is a simplification of the one that appeared in [**17**]. The same kind of argument was later taken in [**14**] and in [**19**]. Another proof of the same result appears in Exercise 9.

4. From quasi-affine to affine homogeneous spaces

The following easy theorem from general algebraic geometry is crucial in order to guarantee the passage from the quasi–affine to the affine case.

THEOREM 3.4. *Let X be a quasi–affine algebraic variety defined over k and such that for any proper ideal J of the ring of regular functions on X –$J \subset \mathcal{O}_X(X)$– proper ideal of the ring of regular functions on X there exists a point $x \in X$ that is annihilated by the ideal, i.e. such that for all $f \in J$, $f(x) = 0$. In this situation X is affine.*

PROOF: As X is quasi–affine there exists an affine variety Y where X is contained as an open subvariety. Substituting Y by the closure of X we can assume that X is dense in Y. Consider the injective morphism of algebras $k[Y] \to \mathcal{O}_X(X)$ given by the restriction of functions and define C as the complement of X in Y, i.e. $C = Y \setminus X$. Call $I \subset k[Y]$ the ideal of C on $k[Y]$. If C is empty there is nothing to prove. Otherwise I is a proper ideal of $k[Y]$. Call J the ideal induced by I on $\mathcal{O}_X(X)$, i.e. $J = I\mathcal{O}_X(X)$. Clearly J cannot have a zero on X because if $x \in X$ is such that $f(x) = 0$ for all $f \in J$, then $f(x) = 0$ for all $f \in I$ and as Y is affine and C closed we conclude that $x \in C$. This is impossible because $x \in X$. Hence $I\mathcal{O}_X(X) = \mathcal{O}_X(X)$ and we can find a finite number of global sections $f_i \in I$ for $i = 1, ..., r$ such that the ideal $\langle f_1, ..., f_r\rangle_{\mathcal{O}_X(X)} = \mathcal{O}_X(X)$. In this situation the principal open subsets of X, $X_{f_i} = \{x \in X : f_i(x) \neq 0\}$ are affine varieties–X_{f_i} coincides for all $i = 1, ..., r$ with Y_{f_i} that is affine because Y is affine–. Using an standard result on general algebraic geometry –see Exercise 3–, we conclude that X is affine. □

OBSERVATION 3.5. The above theorem is a converse of Hilbert's Nullstellensatz. One can say that in the class of quasi-affine varieties the Nulllstellensatz characterizes the affine ones.

In the case of homogeneous spaces next lemma is a more manageable formulation of Theorem 3.4

LEMMA 3.1. *Let G be an affine algebraic group k–group and H a closed k–subgroup that is observable. Assume that for every proper ideal $J \subset {}^H k[G]$ the ideal $Jk[G]$ is proper in $k[G]$. Then G/H is affine.*

PROOF: We know that G/H is quasi-affine and that $\mathcal{O}_{G/H}(G/H) = {}^H k[G]$. Take a proper ideal $J \subset {}^H k[G]$. We want to prove that it has a zero in G/H in order to apply Theorem 3.4 to guarantee that G/H is affine. By hypothesis the ideal $Jk[G] \subset k[G]$ is proper and then we can find an element $w \in G$ that is a common zero to all the polynomials in $Jk[G]$. In that case $\pi(x)$ is a common zero to all the polynomials in J. □

Next lemma appears in [**34**]. The proof can be performed by induction on the minimal number of elements of the ideal J necessary to generate the unit ideal of R.

LEMMA 3.2. *Let H be a geometrically reductive affine algebraic group defined over a field k and let R be an arbitrary commutative H–module algebra. If J is an ideal of ${}^H R$ such that $JR = R$ then $J = {}^H R$.*

LEMMA 3.3. *Assume that G is an affine algebraic k–group and H a closed and exact k–subgroup of G. Let $J \subset {}^H k[G]$ be an ideal such that $Jk[G] = k[G]$, then $J = {}^H k[G]$.*

PROOF: Let $1 = j_1 f_1 + ... + j_n f_n$ with $j_i \in J$ and $f_i \in k[G]$. Consider the map $\Phi : \bigoplus_{i=1}^n k[G] \to k[G]$ defined as $\Phi(g_1, ..., g_n) = \sum g_i j_i$. It is easy to verify that Φ is a surjective $(k[G], H)$–module morphism. By the exactness of H in G we conclude that it is also surjective when restricted to the H–fixed parts. In other words, we can guarantee the existence of elements $g_1, ..., g_n \in {}^H k[G]$ such that $\sum j_i g_i = 1$, i.e. $J = {}^H k[G]$. $\square$

Putting together the results of Theorems 3.2, 3.3 and of Lemmas 3.1, 3.2 and 3.3, we conclude the following results.

THEOREM 3.5. *Let G be an affine algebraic k–group and H a closed k–subgroup. Suppose that H is geometrically reductive, then the homogeneous space G/H is an affine variety.*

THEOREM 3.6. *Let G be an affine algebraic k–group and H a closed k–subgroup. Suppose that H is exact in G, then the homogeneous space G/H is an affine variety.*

Theorem 3.5 is a particular case of general results about actions of (geometrically) reductive groups on affine varieties. This important subject is developped in [**31**]. The proof we present here is elementary and is –in our opinion– a very good illustration of the use of the concept of observable subgroup.

Theorem 3.6 –as we mentioned before– has been proved by different authors using different methods of proofs and in some cases with a larger degree of generality. See for example: [**9**], [**13**], [**21**], [**35**], [**38**]. The proof we present above appeared in [**17**].

5. Exactness , Reynolds operators, total integrals

In this section we relate the concept of exactness with some other relevant objects that traditionally have played an important role in invariant theory.

DEFINITION 3.4. Let G be an affine algebraic k–group and H a closed k–subgroup. Denote as ${}_{(k[G],H)}\mathcal{M}$ the category of rational $(k[G], H)$–modules and ${}_{{}^H k[G]}\mathcal{M}$ the category of ${}^H k[G]$–modules. Let us call $I : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^H k[G]}\mathcal{M}$ the functor consisting of the restriction of scalars and $F : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^H k[G]}\mathcal{M}$ the functor consisting of taking the H–fixed part. A Reynolds operator for the pair (H, G) or for the category ${}_{(k[G],H)}\mathcal{M}$ is a natural transformation $\mathcal{R} : I \to F : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^H k[G]}\mathcal{M}$ such that $\mathcal{R} \circ \mathcal{I} = id_F : F \to F : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^H k[G]}\mathcal{M}$. Here $\mathcal{I} : F \to I : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^H k[G]}\mathcal{M}$ is the natural transformation given by the inclusion ${}^H M \subset M$. In other words a Reynolds operator is a natural transformation that splits the canonical inclusion of ${}^H M \subset M$ in the category of ${}^H k[G]$–modules.

More explicitely we have for each object $M \in {}_{(k[G],H)}\mathcal{M}$ a morphism $\mathcal{R}_M : M \to {}^H M$ of ${}^H k[G]$–modules that verifies that $\mathcal{R}_M(m) = m$ for all $m \in {}^H M$ and for any morphism of $(k[G], H)$–modules $\alpha : M \to N$ there is a commutative square of the form:

$$\begin{tikzcd} M \arrow[r, "\mathcal{R}_M"] \arrow[d, "\alpha"'] & {}^H M \arrow[d, "\alpha|_{{}^H M}"] \\ N \arrow[r, "\mathcal{R}_N"] & {}^H N \end{tikzcd}$$

In particular if $f \in {}^H k[G]$ and $m \in M$ we have that $\mathcal{R}_M(fm) = f\mathcal{R}_M(m)$.

The exactness of a subgroup $H \subset G$ and the existence of a Reynolds operator are equivalent. Part of this equivalence is the content of the lemma that follows.

Lemma 3.4. *Let G be an affine algebraic k–group and H a closed k–subgroup. If a Reynolds operator exists for the category ${}_{(k[G],H)}\mathcal{M}$, then H is exact in G.*

Proof: Let $f : M \to N$ be a surjective morphism in the category ${}_{(k[G],H)}\mathcal{M}$ and consider the corresponding diagram:

$$\begin{tikzcd} M \arrow[r, "\mathcal{R}_M"] \arrow[d, "f"'] & {}^H M \arrow[d, "f|_{{}^H M}"] \\ N \arrow[r, "\mathcal{R}_N"] & {}^H N \end{tikzcd}$$

As the Reynolds operators are projectors and f is assumed to be surjective, it follows immediately that $f|_{{}^H M}$ is surjective and then Exercise 3 guarantees our result. □

Another characterization of exactness is presented in the theorem that follows that appeared in [**9**]. Our proof is basically the same than the original one.

Theorem 3.7. *Let G be an affine algebraic k–group and H a closed k–subgroup. If H is exact in G, $k[G]$ is an injective object in ${}_H\mathcal{M}$.*

Proof: Suppose H is exact in G and assume that we have a diagram as below in the category ${}_H\mathcal{M}$ with M and N finite dimensional

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\phi"'] & N \arrow[dl, dotted, "\hat{\phi}"] \\ k[G] & \end{tikzcd}$$

We want to complete it with an H–morphism $\hat{\phi}$ so as to make it commutative. Consider $X = \operatorname{Hom}_k(M, k[G])$ and $Y = \operatorname{Hom}_k(N, k[G])$ endowed with the usual structure of H–module and with a multiplication by elements of $k[G]$ defined pointwise by multiplication on the codomain. In this manner X and Y become elements of the category ${}_{(k[G],H)}\mathcal{M}$. As M is an H–submodule of N the restriction map is a morphism $\iota : Y \to X$ in the category ${}_{(k[G],H)}\mathcal{M}$. From the surjectivity of ι and the exactness hypothesis, we conclude that $\iota|_{{}^H Y} : {}^H Y \to {}^H X$ is also surjective. Any element $\hat{\phi} \in {}^H Y$ mapped into ϕ can be taken as the extension of ϕ we are looking for.

The case that the original diagram consists in not necessarily finite dimensional rational H–modules can be obtained from the finite dimensional situation using standardt arguments. □

A more complete algebraic characterization of exactness is presented in the next theorem. First we need a couple of definitions.

Definition 3.5. [**9**] Let G be an affine algebraic k–group and H a closed k–subgroup. H is said to be strongly observable in G if for all $M \in {}_H\mathcal{M}$ there exists an $N \in {}_G\mathcal{M}$ and an injective morphism $\iota : M \to N|_H$ such that $\iota({}^H M) \subset {}^G N$.

The concept of strongly observable subgroup of a given group was defined in [**9**]. In the mentioned article, it was also established the equivalence of strong observability with the exactness of the induction functor as well as the equivalence with the affiness of the homogeneous space G/H.

Definition 3.6. Let G be an affine algebraic k–group and H a closed k–subgroup. An H–module morphism $\sigma : k[H] \to k[G]$ that sends the polynomial $\mathbf{1_H}$ into $\mathbf{1_G}$ is called a total integral for H with values in $k[G]$.

Observation 3.6. Notice that one could in the same manner define total integral for H with values in R where R is an arbitrary unital H–module algebra.

Theorem 3.8. *Let G be an affine algebraic k–group and H a closed k–subgroup. The following conditions are equivalent.*

1. *H is exact in G.*
2. *$k[G]$ is injective as a rational H–module.*
3. *There exists a total integral $\sigma : k[H] \to k[G]$ for H with values in $k[G]$.*
4. *There exists a Reynolds operator for the category ${}_{(k[G],H)}\mathcal{M}$.*
5. *H is strongly observable in G.*

Proof: We use Sweedler's notation in a systematic way in this proof. The fact that conditions 2), 3) and 5) are equivalent is the content of the theorems that will proved inmediately after the present one (see Theorem 3.9 and Theorem 3.10). That 1) implies 2) was proved in Theorem 3.7.

Next we prove 2) implies 4). Given $\sigma : k[H] \to k[G]$ as above, define for any $M \in {}_{(k[G],H)}\mathcal{M}$ the map $\mathcal{R}_M(m) = \sum \sigma(S_H(m_1))m_0$ if $\chi(m) = \sum m_0 \otimes m_1$ denotes the right $k[H]$–comodule structure on M associated to the left action of H. If y is an arbitrary element of H then: $\sum y \cdot m_0 \otimes m_1 = \sum m_0 \otimes m_1 \cdot y$. To verify this equality we evaluate at $x \in H$, then the left hand side equals $\sum (y \cdot m_0)m_1(x) = y \cdot \sum m_1(x)m_0 = y \cdot (x \cdot m)$ and the right hand side takes the value: $\sum (m_1 \cdot y)(x)m_0 = \sum m_1(yx)m_0 = (yx) \cdot m$. Other way of writing the above equality is $\sum y \cdot m_0 \otimes m_1 \cdot y^{-1} = \sum m_0 \otimes m_1$. Applying S_H, σ and then multiplying we get: $\sum y \cdot \sigma(S_H(m_1))y \cdot m_0 = \sum \sigma(S_H(m_1))m_0$ or in other words $y \cdot \mathcal{R}_M(m) = \mathcal{R}_M(m)$ for all $y \in H$. So that $\mathcal{R}_M(m) \in {}^HM$. If $m \in {}^HM$, $\chi(m) = m \otimes 1$ and then $\mathcal{R}_M(m) = m$ –notice that here we abbreviated $\mathbf{1_H} = 1$–. Assume now that $f : M \to N$ is a map in the category of $(k[G], H)$-modules, then $f(\mathcal{R}_M(m)) = f(\sum \sigma(S_H(m_1))m_0) = \sum \sigma(S_H(m_1))f(m_0)$. As $\chi(f(m)) = \sum f(m_0) \otimes m_1$, we have that $\mathcal{R}_N(f(m)) = \sum \sigma(S_H(m_1))f(m_0)$ and this part of the proof is finished. Finally the assertion that 4) implies 1), was proved in Lemma 3.4 ☐

Theorem 3.9. *([**11**], [**12**], [**13**], [**9**], [**15**]) Let H be an affine algebraic k–group and R a commutative rational H–module algebra. Then H admits a total integral with values in R if and only if R is injective as an H–module.*

Proof: Assume that R is injective as an H–module and use this injectivity hypothesis to complete the diagram

$$\begin{tikzcd} k \arrow[r] \arrow[dr] & k[H] \arrow[d, "\sigma"] \\ & R \end{tikzcd}$$

to produce a morphism of H–modules $\sigma : k[H] \to R$ that sends 1 into 1.

Conversely, call $\chi : R \to R \otimes k[H]$ the comodule structure map associated to the rational action of H in R and assume that $\sigma : k[H] \to R$ is a total integral. Define the map $\Lambda : R \otimes k[H] \to R$ given by the formula $\Lambda(r \otimes \alpha) = \sum r_0\sigma(S(r_1)\alpha)$. Then $(\Lambda\chi)(r) = \sum r_0\sigma(S(r_1)r_2) = r\sigma(1) = r$. If $r \in R$ and $x \in H$ we have that $\sum x \cdot r_0 \otimes r_1 \cdot x^{-1} = \sum r_0 \otimes r_1$; indeed evaluating both sides at $y \in H$ we have that the right hand side equals

$\sum r_0 r_1(y) = y \cdot r$ and the left hand side equals $\sum x \cdot r_0(r_1 \cdot x^{-1})(y) = \sum x \cdot r_0 r_1(x^{-1}y) = x \cdot \sum r_0 r_1(x^{-1}y) = x \cdot x^{-1} y \cdot r = y \cdot r$. Then for all $x \in H$, $\Lambda(r \otimes x \cdot \alpha) = \sum r_0 \sigma(S(r_1)(x \cdot \alpha)) = \sum (x \cdot r_0)\sigma(S(r_1 \cdot x^{-1})(x \cdot \alpha)) = \sum (x \cdot r_0)\sigma(x \cdot (S(r_1)\alpha)) = x \cdot \sum r_0 \sigma(S(r_1)\alpha) = x \cdot \Lambda(r \otimes \alpha)$. Call $R_0 \otimes k[H]$ the vector space $R \otimes k[H]$ endowed with the H–module structure consisting in the trivial action on the first tensor factor and the usual right action on the second. One easily verifies that the map $\chi : R \to R_0 \otimes k[H]$ is an H–morphism and we just proved that it splits the H–morphism Λ. Hence R is a direct H–module summand of $R_0 \otimes k[H]$ and as such is injective as a rational H–module. ☐

THEOREM 3.10. *Let G be an affine algebraic k–group and H a closed k–subgroup. Then H is strongly observable in G if and only if there exists a total integral for H with values in $k[G]$.*

PROOF: Let σ be a total integral. We show first that H is observable in G. Let $\pi : k[G] \to k[H]$ be the canonical projection and take γ an arbitrary character of H. Fix an element $f \in k[G]$ such that $\pi(f) = \gamma$ and consider the element $g = \sum \sigma(S(\pi(f_2))\gamma)f_1 \in k[G]$. The polynomial g verifies: a) $x \cdot g = \gamma(x)g$ for all $x \in H$, b) $\pi(g) = \gamma$. To prove conditions a) and b) first we write again that for an arbitrary $z \in G$ the equality $\sum z \cdot f_1 \otimes f_2 \cdot z^{-1} = \sum f_1 \otimes f_2$. Then, if $x \in H$ we have that $x \cdot g = \sum x \cdot [\sigma(S(\pi(f_2))\gamma)](x \cdot f_1) = \gamma(x) \sum \sigma(x \cdot S(\pi(f_2))\gamma)(x \cdot f_1) = \gamma(x) \sum \sigma(S(\pi(f_2 \cdot x^{-1}))\gamma) x \cdot f_1 = \gamma(x) \sum \sigma(S(\pi(f_2))\gamma) f_1 = \gamma(x) g$. As to condition b) we observe first that $\pi(f) = \gamma$ implies that $\sum \pi(f_1) \otimes \pi(f_2) = \gamma \otimes \gamma$. Then $\pi(g) = \sum \pi\sigma(S(\pi(f_2))\gamma)\pi(f_1) = \pi\sigma(\gamma^{-1}\gamma)\gamma = \pi(1)\gamma = \gamma$.

To prove that the observability is strong we proceed as follows. Let N be a rational H–module and consider $\mathrm{Soc}(N) = \sum S_i$, where the S_i are all the simple submodules of N. For each S_i consider the H–submodule ${}^H S_i \subset S_i$. If ${}^H S_i = 0$ we use the observability of H to construct a simple G–module T_i and an injective morphism of H–modules $\eta_i : S_i \to T_i$. It will verify that $\eta({}^H S_i) = \eta(0) = 0 \subset {}^G T_i$. In the case that ${}^H S_i = S_i$, i.e. in the case that H acts trivially on S_i we have that the simplicity hypothesis implies that $S_i = k$. Take T_i to coincide with S_i as a k–space equipped with the trivial G–module structure and $\eta_i = id$. Hence we have found for every S_i a simple G–module T_i and an H–morphism $\eta_i : S_i \to T_i$ such that $\eta_i({}^H S_i) \subset {}^G T_i$.

Putting all the maps η_i as well as all the T_i together we have a rational completely reducible G–module M and an injective morphism of H–modules $\eta : \mathrm{Soc}(N) \to M$ such that $\eta({}^H\mathrm{Soc}(N)) \subset {}^G M$. Next consider the morphism $\chi\eta : \mathrm{Soc}(N) \to M \otimes k[G]$–being χ the $k[G]$–comodule structure on M. Being the map $\chi : M \to M \otimes k[G]$ is a morphism of G–modules when we endow the target space with the action only on the second tensor factor it is clear that $\chi({}^G M) = {}^G(M \otimes k[G])$. Recalling that $M \otimes k[G]$ with the structure mentioned above is injective as an H–module after renaming the modules and the maps, we can state what we have proved as follows: for an arbitrary rational H–module N there exists a G–module M that when considered as an H–module is injective and an injective morphism of H–modules $\eta : \mathrm{Soc}(N) \to M$ such that $\eta({}^H\mathrm{Soc}(N)) \subset {}^G M$.

Our injectivity hypothesis allows us to extend the morphism η to an H–morphism $\theta : N \to M|_H$. The injectivity of η implies the injectivity of θ and it is clear that ${}^H\mathrm{Soc}(N) = {}^H N$. Hence $\theta({}^H N) = \eta({}^H\mathrm{Soc}(N)) \subset {}^G M$.

Assume that H is strongly observable in G. Consider the H–module $k[H]$ and a G–module M such that there is an injection $j : k[H] \to M$ that is a morphism of H–modules and verifies that $j({}^H k[H]) = j(k) \subset {}^G M$, i.e. that $j(1) = m \in {}^G M$. Consider an element

$\alpha \in M^*$ such that $\alpha(m) = 1$ and define $\sigma : k[H] \to k[G]$ as $\sigma(f) = \alpha|j(f)$. We have that $\sigma(x \cdot f) = \alpha|j(x \cdot f) = \alpha|x \cdot j(f) = x \cdot \alpha|j(f) = x \cdot \sigma(f)$. Also $\sigma(1) = \alpha|m$ and $\alpha|m(z) = \alpha(z \cdot m) = \alpha(m) = 1$ for all $z \in G$. □

6. Affine homogeneous spaces and exactness

We want to prove the converse of Theorem 3.6 and show that if the homogeneous space G/H is affine then H is exact in G. This result was proved for the first time in [**9**] and [**35**]. Later other proofs appeared in [**13**], [**21**], [**38**], etc. Our arguments will follow closely the ones presented in [**9**] which are attributed by the authors to H. Kraft.

Along this section G and H will be as usual an affine algebraic k–group and a closed k–subgroup respectively.

Definition 3.7. Let G and H be as before, we define the Galois morphism $\mathcal{G}_{G,H}$: $k[G] \otimes_{{}^H k[G]} k[G] \to k[G] \otimes k[H]$ as the map induced on the quotient by $\mathcal{H}_{G,H} : k[G] \otimes k[G] \to k[G] \otimes k[H]$ given as $\mathcal{H}_{G,H}(f \otimes g) = \sum f g_1 \otimes g_2|_H$. If $\Delta : k[G] \to k[G] \otimes k[G]$ is the comultiplication of $k[G]$ we denote as usual $\Delta(g) = \sum g_1 \otimes g_2$.

The fact that $\mathcal{H}_{G,H}$ factors through the quotient $k[G] \otimes_{{}^H k[G]} k[G]$ of $k[G] \otimes k[G]$ is very easy to verify.

Observation 3.7. The Galois map $\mathcal{G}_{G,H}$ can be defined in the case that we take instead of $k[G]$ an arbitrary algebra A and instead of $k[H]$ an arbitrary Hopf algebra that coacts on A. For a justification of the name Galois morphism the reader can look in [**29**]. Here we just mention the following. Let H be a finite group acting by automorphisms on a given field E and call ${}^H E = F$. Consider the map –that plays in this context the role of $\mathcal{G}_{G,H}$ – $\mathcal{G} : E \otimes_F E \to E \otimes k[H]$, $\mathcal{G}(a \otimes b) = \sum_{x \in H} a(x \cdot b) \otimes \delta_x$, where δ_x is the characteristic function at the point $x \in H$. It can be verified that the finite field extension $E \subset F$ is Galois if and only if the map $\mathcal{G}$ is bijective.

Observation 3.8. The map $\mathcal{G}_{G,H} : k[G] \otimes_{{}^H k[G]} k[G] \to k[G] \otimes k[H]$ is a morphism of H–modules provided we equip the domain and codomain with H–structures that are trivial on the first tensorands and are the natural ones on the second tensorands. See Exercise 4.

First we prove that in case that the homogeneous space G/H is an affine variety, the map $\mathcal{G}_{G,H}$ is bijective.

Lemma 3.5. *Let G and H be as above and assume also that G/H is an affine variety. If $p \in k[G]$ is a polynomial such that $p|_H = 0$ then $\sum S(p_1) \otimes_{{}^H k[G]} p_2 = 0$.*

Proof: Recall that $k[G] \otimes_{{}^H k[G]} k[G]$ is the algebra of polynomial functions on the affine variety $G \times_{G/H} G$. To verify that $\sum S(p_1) \otimes_{{}^H k[G]} p_2 = 0$ we have to prove that for $(w, z) \in G \times_{G/H} G$, $\sum S(p_1)(w) p_2(z) = 0$. The computation that follows proves the required annihilation $\sum S(p_1)(w)p_2(z) = \sum p_1(w^{-1})p_2(z) = p(w^{-1}z) = 0$ as $w^{-1}z \in H$. □

Theorem 3.11. *Let G and H be as above and suppose that G/H is affine. Then the Galois map $\mathcal{G}_{G,H} : k[G] \otimes_{{}^H k[G]} k[G] \to k[G] \otimes k[H]$ is bijective.*

Proof: Along the proof we will omit the subscript ${}^H k[G]$ in the notation of the elements of $k[G] \otimes_{{}^H k[G]} k[G]$. To an arbitrary element $f \otimes p \in k[G] \otimes k[H]$ we associate the element $\sum f S(g_1) \otimes g_2 \in k[G] \otimes_{{}^H k[G]} k[G]$ where $g \in k[G]$ is a polynomial that extends p as a function and $\Delta(g) = \sum g_1 \otimes g_2$. Lemma 3.5 guarantees that the element constructed above is independent of g. Define $\mathcal{K}_{G,H} : k[G] \otimes k[H] \to k[G] \otimes_{{}^H k[G]} k[G]$ by the formula $\mathcal{K}_{G,H}(f \otimes p) = \sum f S(g_1) \otimes g_2$. We prove that the map $\mathcal{K}_{G,H}$ is the inverse of the Galois map. Indeed,

$(\mathcal{G}_{G,H}\mathcal{K}_{G,H})(f\otimes p) = \sum fS(g_1)g_2\otimes g_3|_H = f\otimes g|_H = f\otimes p$. Also $(\mathcal{K}_{G,H}\mathcal{G}_{G,H})(f\otimes g) = \mathcal{K}_{G,H}(\sum fg_1\otimes g_2|_H) = \sum fg_1S(g_2)\otimes g_3 = f\otimes g$. $\square$

Next result that belongs to the realm of commutative algebra is a direct consequence of Noether normalization theorem.

LEMMA 3.6. *Let A be a commutative integral domain and let $B\subset A$ be a subring that is finitely generated as a B–algebra. There exists an element $c\in B$ such that $B[c^{-1}]\subset A[c^{-1}]$ is free.*

DEFINITION 3.8. A commutative ring extension $B\subset A$ is said to be faithfully flat if for all sequences of B–modules: $\mathcal{E}: 0\to M\to N\to T\to 0$, $\mathcal{E}$ is exact if and only if $\mathcal{E}\otimes_B A: 0\to M\otimes_B A\to N\otimes_B A\to T\otimes_B A\to 0$ is exact.

Note that if the extension $B\subset A$ is free –i.e. if A is free as a B–module– then it is faithfully flat.

LEMMA 3.7. *Let A be an integral domain and let $B\subset A$ be a finitely generated commutative ring extension. Suppose we can find $b_1,\dots,b_n\in B$ such that: (i) the elements $b_1,\dots,b_n$ generate the unit ideal of B (ii) $A[b_i^{-1}]$ is faithfully flat as a $B[b_i^{-1}]$–module. Then A is faithfully flat as a B–module.*

PROOF: We use here Exercise 3. First suppose that M is a B–module such that $M\otimes_B A=0$. Then $M\otimes_B A\otimes_A A[b_i^{-1}]=0$ or equivalentely $M\otimes_B A[b_i^{-1}]=0$. Then $M\otimes_B B[b_i^{-1}]\otimes_{B[b_i^{-1}]}A[b_i^{-1}]=0$ and from the hypothesis we conclude that $M\otimes_B B[b_i^{-1}]=0$. Hence, for an arbitrary $m\in M$ there exists an exponent p such that for all $1\leq ileqn$, $b_i^p m=0$. As the ideal generated by $\{b_1^p,\dots,b_n^p\}$ is also the unit ideal, we conclude that $m=0$ and then $M=0$. Next assume that $\alpha: M\to N$ is an injective morphism of B–modules. Then $id\otimes\alpha: B[b_i^{-1}]\otimes_B M\to B[b_i^{-1}]\otimes_B N$ is injective and so is $id\otimes id\otimes\alpha: A[b_i^{-1}]\otimes_{B[b_i^{-1}]}B[b_i^{-1}]\otimes_B M\to A[b_i^{-1}]\otimes_{B[b_i^{-1}]}B[b_i^{-1}]\otimes_B N$. Then the morphism $id\otimes\alpha: A[b_i^{-1}]\otimes_B M\to A[b_i^{-1}]\otimes_B N$ is injective. Consider an element $\sum a_k\otimes m_k\in A\otimes_B M$ such that $0=\sum a_k\otimes\alpha(m_k)\in A\otimes_B N$. By the injectivity of $id\otimes\alpha: A[b_i^{-1}]\otimes_B M\to A[b_i^{-1}]\otimes_B N$ we conclude that $0=\sum a_k\otimes m_k\in A[b_i^{-1}]\otimes_B M$ for all $i=1,\dots,n$. It follows from standardt results on tensor products that $0=\sum a_k\otimes m_k\in A\otimes_B M$. $\square$

COROLLARY 3.1. *Assume that G and H are as before and that the homogeneous space G/H is an affine variety, then $k[G]$ is faithfully flat as a ${}^Hk[G]$–module.*

PROOF: In accordance with Exercise 6, if G/H is affine so is $G_1/(G_1\cap H)$. By the same exercise H is exact in G if and only if $G_1\cap H$ is exact in G_1. Then we can assume that the group G is connected. Consider the extension ${}^Hk[G]\subset k[G]$ Using Lemma 3.6 we conclude that there is an element $0\neq f\in {}^Hk[G]$ such that $k[G]_f$ is free as a ${}^Hk[G]_f$ module. If we take an arbitrary element $z\in G$ we have that: $k[G]_{f\cdot z}=k[G]_f\cdot z$ and ${}^Hk[G]_{f\cdot z}={}^Hk[G]_f\cdot z$. Hence for all $z\in G$, $k[G]_{f\cdot z}$ is free as a ${}^Hk[G]_{f\cdot z}$–module. The family of polynomials $\{f\cdot z: z\in G\}$ cannot have a common zero in G/H because G acts transitively on G/H. As we are assuming that G/H is affine we conclude that they generate the unit ideal of ${}^Hk[G]$. Using Lemma 3.7 we conclude that $k[G]$ is faithfully flat as a ${}^Hk[G]$–module. $\square$

THEOREM 3.12. *Assume that G and H are as before and that the homogeneous space G/H is an affine variety, then the functor $\operatorname{Ind}_H^G: {}_H\mathcal{M}\to {}_G\mathcal{M}$ is exact.*

PROOF: Consider a surjective morphism $\alpha: M_1\to M_2$ of rational H–modules we want to prove the surjectivity of the map $id\otimes\alpha: {}^H(k[G]\otimes M_1)\to {}^H(k[G]\otimes M_2)$. By Corollary 3.1 this map is surjective if and only if the map $id\otimes_{{}^Hk[G]}id\otimes\alpha: k[G]\otimes_{{}^Hk[G]}{}^H(k[G]\otimes M_1)\to$

$k[G] \otimes_{{}^H k[G]} {}^H(k[G] \otimes M_2)$ is surjectvie. If we let H act trivially in the first tensor factor of the expressions below, the map above can be written as: $id \otimes_{{}^H k[G]} id \otimes \alpha : {}^H(k[G] \otimes_{{}^H k[G]} k[G] \otimes M_1) \to {}^H(k[G] \otimes_{{}^H k[G]} k[G] \otimes M_2)$. If we use the Galois morphism $\mathcal{G}_{G,H}$ and recall its H–equivariant properties, it is clear that the surjectivity of this last map is equivalent to the surjectivity of $id \otimes id \otimes \alpha : {}^H(k[G] \otimes k[H] \otimes M_1) \to {}^H(k[G] \otimes k[H] \otimes M_2)$. Notice that we are omitting the restriction symbol for many of the maps. Using the fact that H is acting trivially in the first tensor factor and that–as we already proved in Corollary 2.4– ${}^H(k[H] \otimes M)$ is functorially isomorphic to M for any H–module M, we conclude that the surjectivity of $id \otimes id \otimes \alpha : {}^H(k[G] \otimes k[H] \otimes M_1) \to {}^H(k[G] \otimes k[H] \otimes M_2)$ is equivalent to the surjectivity of $id \otimes \alpha : k[G] \otimes M_1 \to k[G] \otimes M_2$ which is an obvious consequence of the surjectivity of α. $\square$

7. Affine homogeneous spaces and reductivity

In this section we prove that if G is a geometrically reductive affine algebraic group and H a closed subgroup of G such that the homogeneous space G/H is affine, then H is geometrically reductive. This result was first proved Matsushima in [**28**] and then by Borel and Harish-Chandra in the case of the field of complex numbers, see [**6**]. Later Bialynicki-Birula proved it in characteristic zero (see [**2**]). Richardson in [**36**], Haboush in [**20**] and Cline, Parashall and Scott in the basic article [**9**] presented proofs for arbitrary characteristic. The proof that we present here –assuming the algebraic version of reductivity, i.e. assuming geometric reductivity– relies in the concept of exactness and in this sense is similar to the one in [**20**]. It appeared for the first time in [**15**].

Theorem 3.13. *Let G be an affine geometrically reductive algebraic k–group and H a closed k–subgroup such that the homogeneous space G/H is an affine variety. Then H is geometrically reductive.*

Proof: Let M be an arbitrary rational H–module and take $\lambda : M \to k$ a surjective H–morphism. Considering the induced representations we obtain a commutative diagram in the category of H–modules as follows

$$\begin{tikzcd} {}^H(k[G] \otimes M) \arrow[r, "id\otimes\lambda"] \arrow[d, "E_M"] & {}^H k[G] \arrow[d, "E"] \\ M \arrow[r, "\lambda"] & k \end{tikzcd}$$

where $E(f) = f(1)$.

By the exactness of H in G we can find an element $\xi = \sum f_i \otimes m_i \in {}^H(k[G] \otimes M)$ that verifies that $\sum \lambda(m_i) f_i = \mathbf{1}$. Consider the G–submodule $\langle G \star \xi \rangle$ of ${}^H(k[G] \otimes M)$ generated by ξ and recall that the action is given as $z \star (\sum g_j \otimes n_j) = \sum g_j \cdot z^{-1} \otimes n_j$. We have that $(id \otimes \lambda)(z \star \xi) = \sum \lambda(m_i)(f_i \cdot z^{-1}) = (\sum \lambda(m_i) f_i) \cdot z^{-1} = \mathbf{1}$. If we call Λ the restriction of $id \otimes \lambda$ to $\langle G \star \xi \rangle$, the map Λ can be considered as a G–map with codomain k, i.e. $\Lambda : \langle G \star \xi \rangle \to k$. Hence the above commutative square can be restricted to a diagram as the one below.

$$\begin{tikzcd} \langle G \star \xi \rangle \arrow[r, "\Lambda"] \arrow[d, "E_M"] & k \\ M \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Being Λ surjective and G is geometrically reductive we can find $q > 0$ and $\eta \in {}^GS^q(\langle G\star\xi\rangle)$ such that $S^q(\Lambda)(\eta) = 1$. Calling $\nu = S^q(E_M)(\eta)$ we have that: $S^q(\lambda)(\nu) = S^q(\lambda)(S^q(E_M)(\eta)) = S^q(\Lambda)(\eta) = 1$. From the fact that $\eta \in {}^GS^q(\langle G \star \xi\rangle) \subset {}^HS^q(\langle G \star \xi\rangle)$ we deduce that $\nu \in {}^HS^q(M)$. □

A version of the above therem concerning linear reductivity is left as an exercise to the reader, see Exercise 8.

Observation 3.9. The reader has probably noticed the similarity between our treatment of the situation of an exact subgroup and the case of a geometrically reductive subgroup. The deep reason for that situation is that a unified theory can be developped ([**18**]).

As an illustration of the use of the ideas concerning integrals and reductivity we prove the following partial converse to the transitivity result of Exercise 2.

Theorem 3.14. *Let $K \subset H \subset G$ be a sequence of closed k–subgroups of the affine algebraic k–group G. Prove that if K is exact in G, normal in H and the quotient K/H is linearly reductive, then H is exact in G.*

Proof: As K is exact in G we can construct a total integral $\sigma : k[K] \to k[G]$. Inducing up to H we obtain another morphism $\tau : k[H] \to {}^Kk[H] \otimes k[G]$ that also sends one into one. The linear reductivity hypothesis on H/K guarantees the existence of an H–equivariant morphism $\kappa : {}^Kk[H] \to k$ sending one into one. Them map $\lambda = (\kappa \otimes id)\tau : k[H] \to k[G]$ is a total integral. Hence H is exact in G. □

8. Exercises

1 Let G be an affine algebraic k–group and H a closed subgroup. Denote as ${}_{(k[G],H)}\mathcal{M}$ the category of all left $(k[G], H)$–modules and as usual ${}_{{}^Hk[G]}\mathcal{M}$ the category of all left ${}^Hk[G]$–modules. Let $\mathcal{F} : {}_{(k[G],H)}\mathcal{M} \to {}_{{}^Hk[G]}\mathcal{M}$ the H–fixed part functor. Prove that H is exact in G if and only if the functor $\mathcal{F}$ is exact.

2 (a) Let $K \subset H \subset G$ be a sequence of closed k–subgroups of the affine algebraic k–group G. Prove that if K is exact in H and H exact in G , then K is exact in G.
  (b) Prove that if H is subgroup of G normal and of finite index, it is exact in G.

3 Let X be an affine algebraic variety defined over an algebraically closed field k and suppose that $f_1, .., f_r$ are global sections of the structure sheaf of X such that: a) the functions $f_1, .., f_r$ generate the unit ideal of the ring of global sections; b) The principal open subsets X_{f_i} are affine for $i = 1, .., r$. Then X is affine.

4 Prove that the map $\mathcal{G}_{G,H} : k[G] \otimes_{{}^Hk[G]} k[G] \to k[G] \otimes k[H]$ is a morphism of H–modules, i.e. prove that
$\mathcal{G}_{G,H}(f \otimes_{{}^Hk[G]} x \cdot g) = \sum fg_1 \otimes x \cdot (g_2)|_H = x \cdot \mathcal{G}_{G,H}(f \otimes g)$ if $x \in H$ and $f, g \in k[G]$.

5 (a) Let $B \subset A$ be a commutative ring extension. We say that A is flat as a B–module if for any injective morphism $\alpha : M \to N$ of B–modules, the morphism $id \otimes_B \alpha : A \otimes_B M \to A \otimes_B N$ is also injective. Prove that $B \subset A$ is faithfully flat if it is flat and for any B–module M, $M \otimes_B A = 0$ implies $M = 0$.
  (b) Show that if A is a commutative integral domain and $a \in A$ is an arbitrary element , then $A \subset A_[a^{-1}]$ is flat.

6 Let G be an affine algebraic k–group and H a k–closed subgroup.

(a) Using total integrals prove that G/H affine implies that $G_1/(G_1 \cap H)$ is also affine.
(b) Prove that H is exact in G if and only if $G_1 \cap H$ is exact in G_1.

7 Characterize the universally exact groups. A group is said to be universally exact if whenever is embedded in a larger group as a closed subgroup, then it is exact inside the larger group.

8 Suppose that G is a linearly reductive affine algebraic k–group and H a closed k–subgroup such that the homogeneous space G/H is an affine variety, prove that H is linearly reductive. Prove this result following the method of Theorem 3.13 and provide another proof using total integrals.

9 Let G be an affine algebraic k–group and H a closed k–subgroup. Let N be an arbitrary rational H–module and consider $\pi \otimes id : k[G] \otimes N \to k[H] \otimes N$. If we endow the left hand side with its usual structure of $(k[G], H)$–module and the left hand side with the structure of $(k[G], H)$–module induced by π, it follows that $\pi \otimes id$ is a morphism of the corresponding category. Conclude using the very definition of exactness that if H is exact in G, then the map E_M is surjective.

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